O N E

Introduction

ANSWERS TO REVIEW QUESTIONS

1. Guided missiles, automatic gain control in radio receivers, satellite tracking antenna

2. Yes - power gain, remote control, parameter conversion; No - Expense, complexity

3. Motor, low pass filter, inertia supported between two bearings

4. Closed-loop systems compensate for disturbances by measuring the response, comparing it to the input response (the desired output), and then correcting the output response.

5. Under the condition that the feedback element is other than unity

6. Actuating signal

7. Multiple subsystems can time share the controller. Any adjustments to the controller can be implemented with simply software changes.

8. Stability, transient response, and steady-state error

9. Steady-state, transient

10. It follows a growing transient response until the steady-state response is no longer visible. The system will either destroy itself, reach an equilibrium state because of saturation in driving amplifiers, or hit limit stops.

11. Transient response

12. True

13. Transfer function, state-space, differential equations

14. Transfer function - the Laplace transform of the differential equation

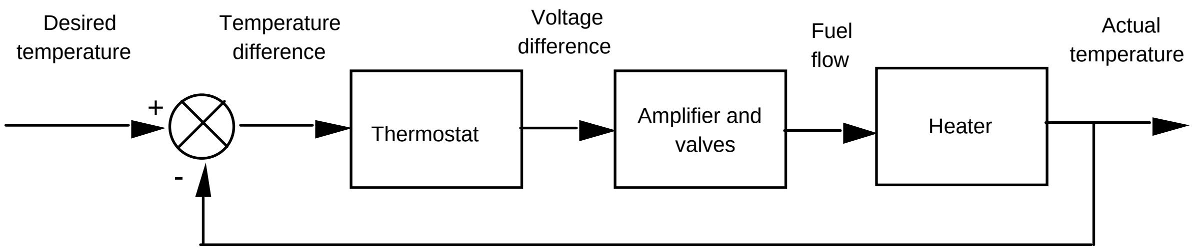
State-space - representation of an nth order differential equation as n simultaneous first-order differential equations

Differential equation - Modeling a system with its differential equation

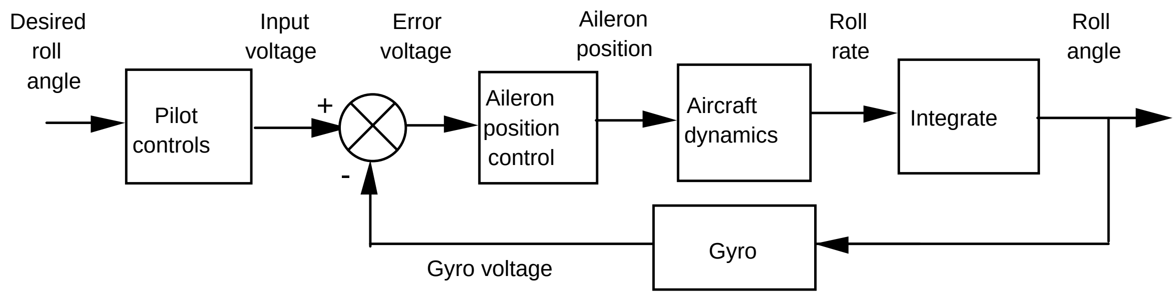
SOLUTIONS TO PROBLEMS

1. Five turns yields 50 v. Therefore $K = \dfrac{50 \text{ volts}}{5 \text{ x } 2\pi \text{ rad}} = 1.59$

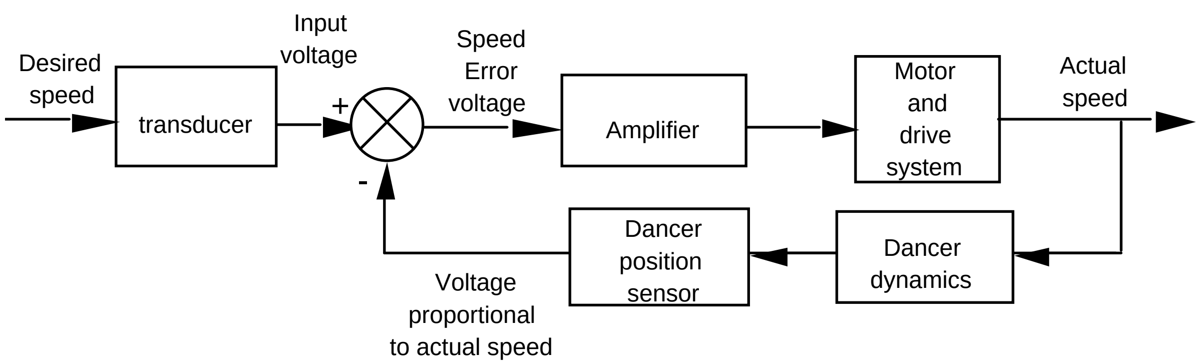
2.


Desired
temperature
Temperature
difference
Voltage
difference
Fuel
flow
Actual
temperature
+
-
Thermostat
Amplifier and
valves
Heater

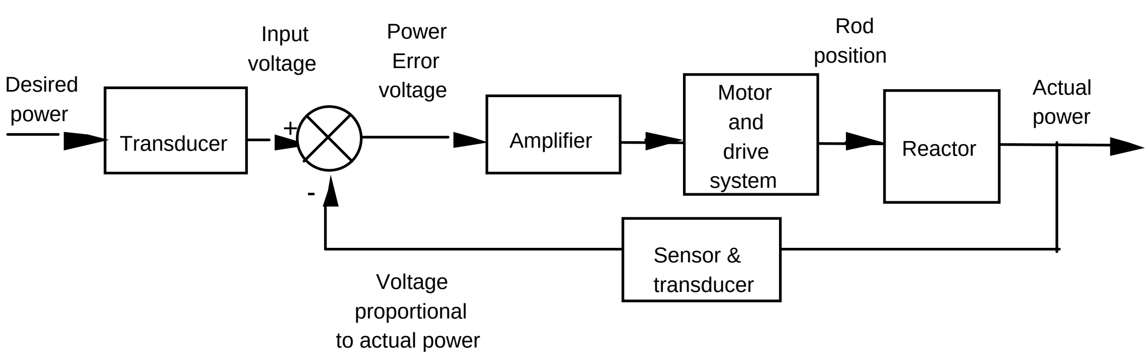
3.


Desired
roll
angle
Input
voltage
Error
voltage
Aileron
position
Roll
rate
Roll
angle
Pilot
controls
+
-
Aileron
position
control
Aircraft
dynamics
Integrate
Gyro
Gyro voltage

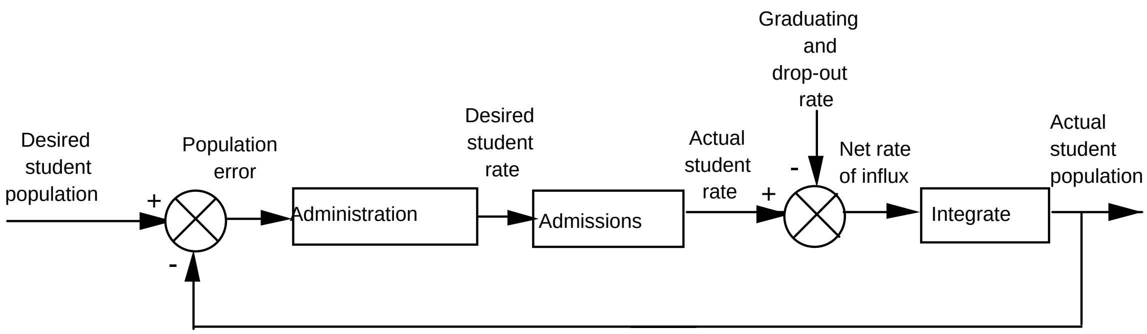
4.


Desired
speed
Input
voltage
Speed
Error
voltage
Actual
speed
transducer
+
-
Amplifier
Motor
and
drive
system
Dancer
position
sensor
Dancer
dynamics
Voltage
proportional
to actual speed

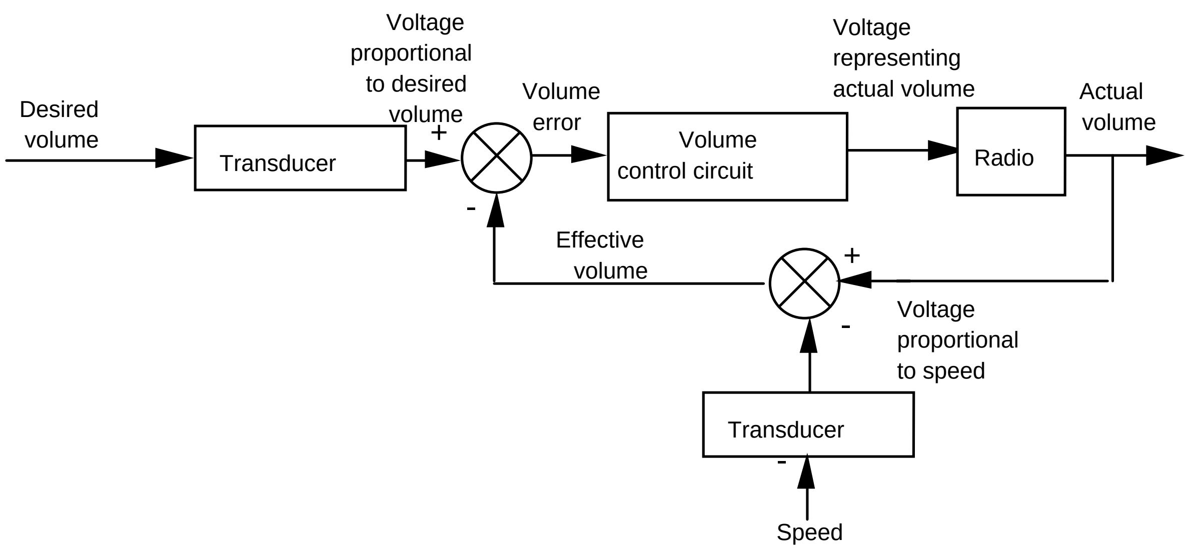
5.


Desired
power
Input
voltage
Power
Error
voltage
Rod
position
Actual
power
Transducer
+
-
Amplifier
Motor
and
drive
system
Reactor
Sensor &
transducer
Voltage
proportional
to actual power

6.

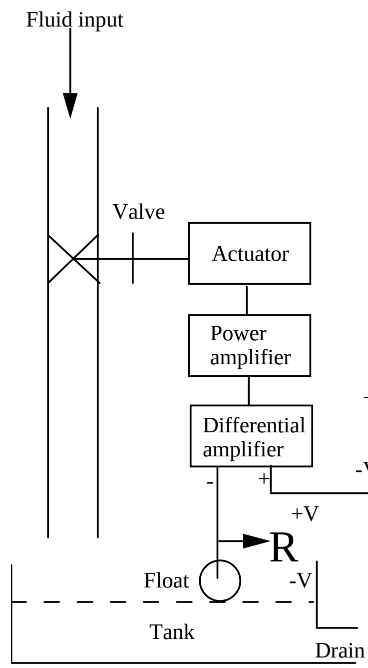

Desired student population
Population error
Administration
Desired student rate
Admissions
Actual student rate
Graduating and drop-out rate
Net rate of influx
Integrate
Actual student population
+
-
+
-

7.

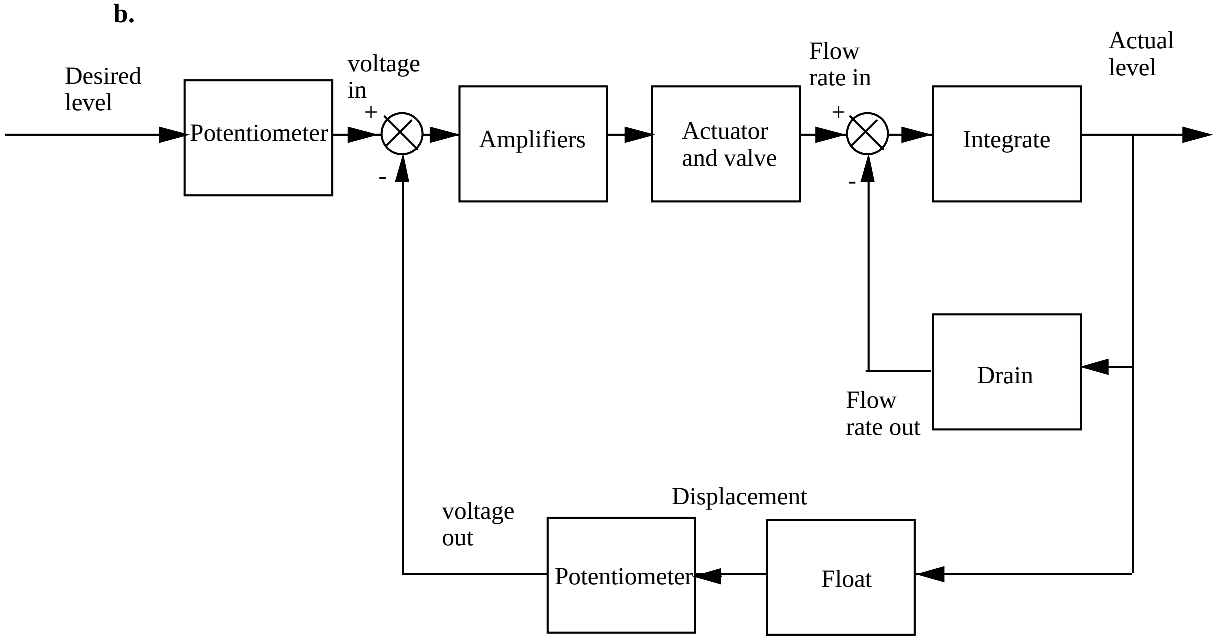

Desired volume
Transducer
Voltage proportional to desired volume
Volume error
Volume control circuit
Voltage representing actual volume
Radio
Actual volume
Effective volume
Voltage proportional to speed
Transducer
Speed
+
-
+
-
-

8.

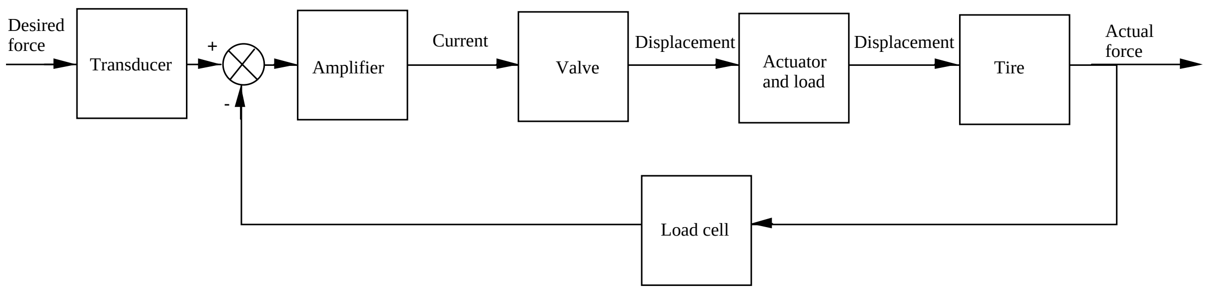
a.


Fluid input
Valve
Actuator
Power amplifier
Differential amplifier
+V
R
-V
Desired level
-
+
+V
R
-V
Float
Tank
Drain

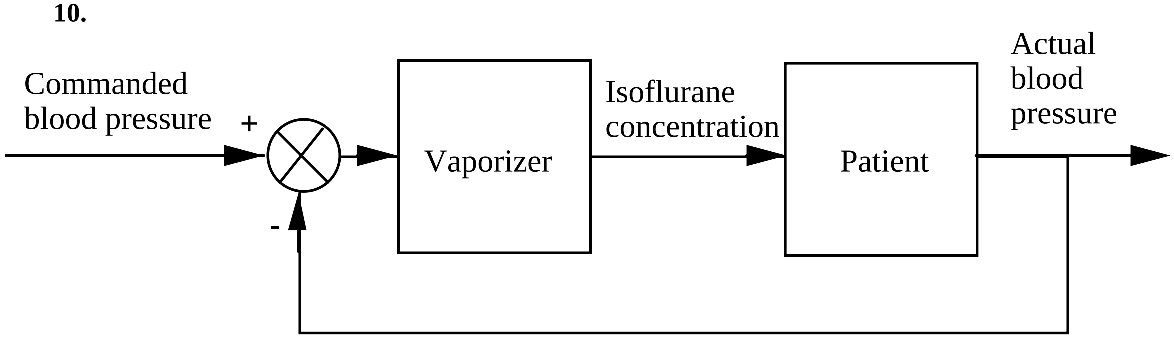
b.


Desired level
Potentiometer
voltage in
+
-
Amplifiers
Actuator and valve
Flow rate in
+
-
Integrate
Actual level
Drain
Flow rate out
Displacement
voltage out
Potentiometer
Float

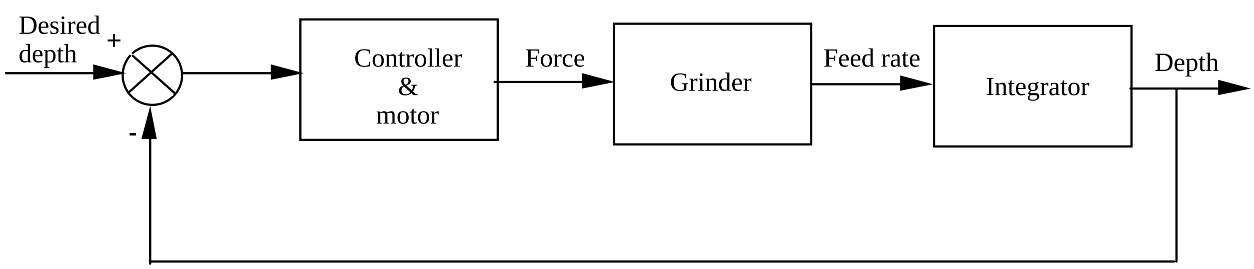
9.



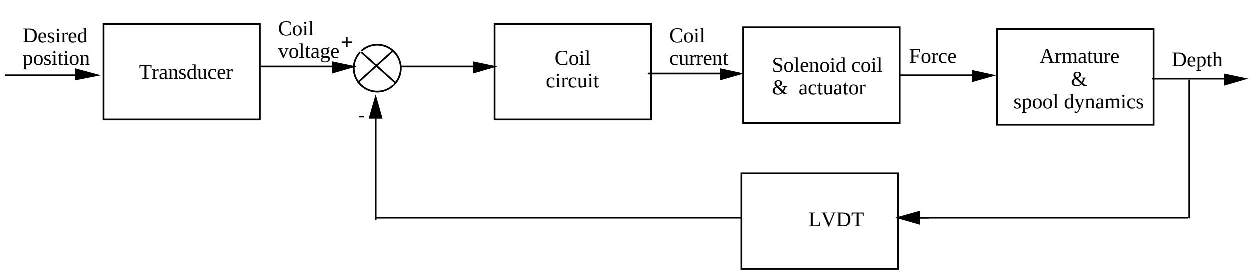
10.



11.



12.



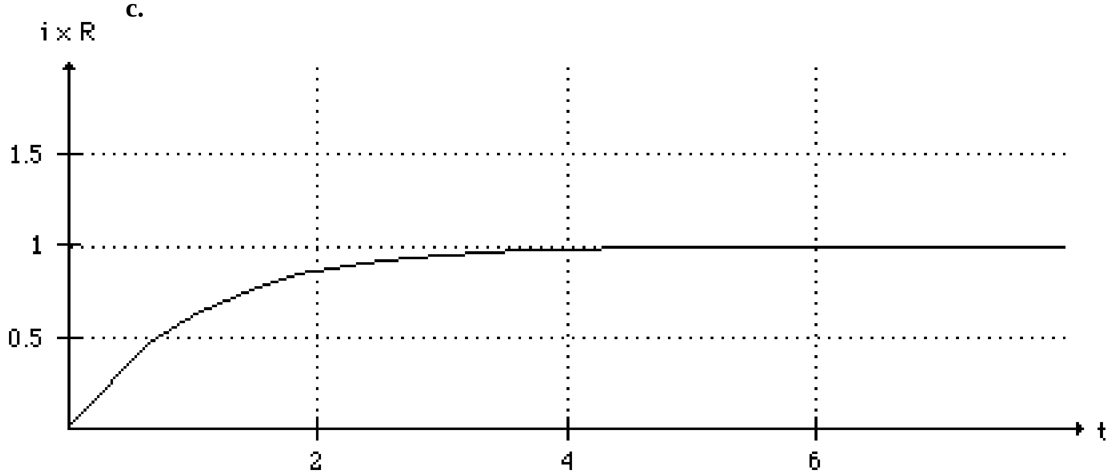
13.

a. $L\frac{di}{dt} + Ri = u(t)$

b. Assume a steady-state solution i_{SS} = B. Substituting this into the differential equation yields RB = 1,

from which B = $\frac{1}{R}$. The characteristic equation is LM + R = 0, from which M = -$\frac{R}{L}$. Thus, the total solution is $i(t) = Ae^{-(R/L)t} + \frac{1}{R}$. Solving for the arbitrary constants, i(0) = A + $\frac{1}{R}$ = 0. Thus, A = -$\frac{1}{R}$. The final solution is i(t) = $\frac{1}{R}$ -- $\frac{1}{R}$ $e^{-(R/L)t}$ = $\frac{1}{R}(1-e^{-(R/L)t})$.

c.



14.

a. Writing the loop equation, $Ri + L\frac{di}{dt} + \frac{1}{C}\int idt + v_C(0) = v(t)$

b. Differentiating and substituting values, $\frac{d^2i}{dt^2} + 2\frac{di}{dt} + 30i = 0$

Writing the characteristic equation and factoring,

$$M^2 + 2M + 30 = \left(M + 1 + \sqrt{29}\ i\right)\left(M + 1 - \sqrt{29}\ i\right).$$

The general form of the solution and its derivative is

$$i = e^{-t}\cos\left(\sqrt{29}\ t\right) A + \left(B \sin\left(\sqrt{29}\ t\right)\right) e^{-t}$$

$$\frac{di}{dt} = \left(-A + \sqrt{29}\ B\right) e^{-t}\cos\left(\sqrt{29}\ t\right) - \left(\sqrt{29}\ A + B\right) e^{-t}\sin\left(\sqrt{29}\ t\right)$$

Using $i(0) = 0; \frac{di}{dt}(0) = \frac{v_L(0)}{L} = \frac{1}{L} = 2$

$i(0) = A = 0$

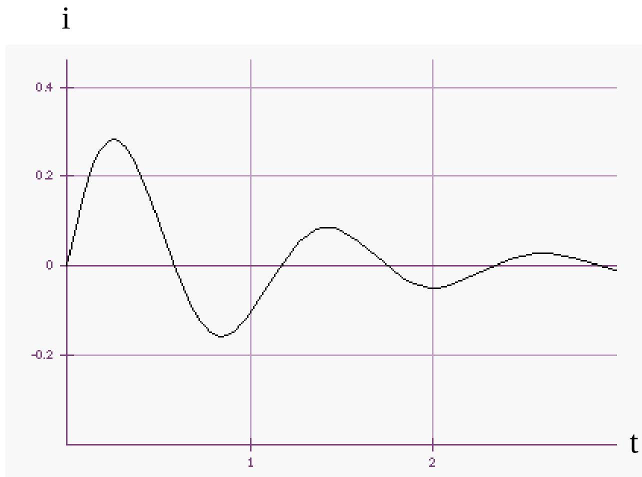
$\frac{di}{dt}(0) = -A + \sqrt{29}B = 2$

Thus, $A = 0$ and $B = \frac{2}{\sqrt{29}}$.

The solution is

$$i = \frac{2}{29}\sqrt{29}\, e^{-t}\sin\left(\sqrt{29}\, t\right)$$

c.



15.

a. Assume a particular solution of

Substitute into the differential equation and obtain

$$(7C + 2D)\cos(2t) + (-2C + 7D)\sin(2t) = 5\cos(2t)$$

Equating like coefficients,

$$7C + 2D = 5$$

$$-2C + 7D = 0$$

From which, $C = \frac{35}{53}$ and $D = \frac{10}{53}$.

The characteristic polynomial is

$$M + 7 = 0$$

Thus, the total solution is

$$x(t) = A e^{-7t} + \left(\frac{35}{53}\cos[2t] + \frac{10}{53}\sin[2t]\right)$$

Solving for the arbitrary constants, $x(0) = A + \frac{35}{53} = 0$. Therefore, $A = -\frac{35}{53}$. The final solution is

$$x(t) = \left(-\frac{35}{53}\right)e^{-7t} + \left(\frac{35}{53}\cos[2t] + \frac{10}{53}\sin[2t]\right)$$

b. Assume a particular solution of

$$x_p = A\sin 3t + B\cos 3t$$

Substitute into the differential equation and obtain

$$(18A - B)\cos(3t) - (A + 18B)\sin(3t) = 5\sin(3t)$$

Therefore, 18A – B = 0 and –(A + 18B) = 5. Solving for A and B we obtain

$$x_p = (-1/65)\sin 3t + (-18/65)\cos 3t$$

The characteristic polynomial is

$$M^2 + 6M + 8 = (M + 4)(M + 2)$$

Thus, the total solution is

$$x = C e^{-4t} + D e^{-2t} + \left(-\frac{18}{65}\cos(3t) - \frac{1}{65}\sin(3t) \right)$$

Solving for the arbitrary constants, $x(0) = C + D - \frac{18}{65} = 0$.

Also, the derivative of the solution is

$$\frac{dx}{dt} = -\frac{3}{65}\cos(3t) + \frac{54}{65}\sin(3t) - 4Ce^{-4t} - 2De^{-2t}$$

Solving for the arbitrary constants, $\dot{x}(0)$ $-\frac{3}{65} - 4C - 2D = 0$, or $C = -\frac{3}{10}$ and $D = \frac{15}{26}$.

The final solution is

$$x = -\frac{18}{65}\cos(3t) - \frac{1}{65}\sin(3t) - \frac{3}{10}e^{-4t} + \frac{15}{26}e^{-2t}$$

c. Assume a particular solution of

$$x_p = A$$

Substitute into the differential equation and obtain 25A = 10, or A = 2/5.

The characteristic polynomial is

$$M^2 + 8M + 25 = (M + 4 + 3i)(M + 4 - 3i)$$

Thus, the total solution is

$$x = \frac{2}{5} + e^{-4t}(B\sin(3t) + C\cos(3t))$$

Solving for the arbitrary constants, x(0) = C + 2/5 = 0. Therefore, C = -2/5. Also, the derivative of the solution is

$$\frac{dx}{dt} = ((3B - 4C)\cos(3t) - (4B + 3C)\sin(3t))e^{-4t}$$

Solving for the arbitrary constants, $\dot{x}(0) = 3B - 4C = 0$. Therefore, B = -8/15. The final solution is

$$x(t) = \frac{2}{5} - e^{-4t}\left(\frac{8}{15}\sin(3t) + \frac{2}{5}\cos(3t)\right)$$

16.

a. Assume a particular solution of

$$x_p(t) = C\cos(2t) + D\sin(2t)$$

Substitute into the differential equation and obtain

$$-2(C - 2D)\cos(2t) - 4\left(C + \frac{1}{2}D\right)\sin(2t) = \sin(2t)$$

Equating like coefficients,

$$-2(C - 2D) = 0$$
$$-4\left(C + \frac{1}{2}D\right) = 1$$

From which, $C = -\frac{1}{5}$ and $D = -\frac{1}{10}$.

The characteristic polynomial is

$$M^2 + 2M + 2 = (M + 1 + i)(M + 1 - i)$$

Thus, the total solution is

$$x = -\frac{1}{5}\cos(2t) - \frac{1}{10}\sin(2t) + e^{-t}(A\cos[t] + B\sin[t])$$

Solving for the arbitrary constants, $x(0) = A - \frac{1}{5} = 2$. Therefore, $A = \frac{11}{5}$. Also, the derivative of the solution is

$$\frac{dx}{dt} = -\frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) + (-A + B)e^{-t}\cos(t) - (A + B)e^{-t}\sin(t)$$

Solving for the arbitrary constants, $\dot{x}(0) = -A + B - 0.2 = -3$. Therefore, $B = -\frac{3}{5}$. The final solution is

$$x(t) = -\frac{1}{5}\cos(2t) - \frac{1}{10}\sin(2t) + e^{-t}\left(\frac{11}{5}\cos(t) - \frac{3}{5}\sin(t)\right)$$

b. Assume a particular solution of

$$x_p = Ce^{-2t} + Dt + E$$

Substitute into the differential equation and obtain

$$Ce^{-2t} + Dt + 2D + E = 5e^{-2t} + t$$

Equating like coefficients, C = 5, D = 1, and 2D + E = 0.

From which, C = 5, D = 1, and E = - 2.

The characteristic polynomial is

$$M^2 + 2M + 1 = (M + 1)^2$$

Thus, the total solution is

$$x(t) = Ae^{-t} + Be^{-t}t + 5e^{-2t} + t - 2$$

Solving for the arbitrary constants, x(0) = A + 5 - 2 = 2 Therefore, A = -1. Also, the derivative of the solution is

$$\frac{dx}{dt} = (-A + B)e^{-t} - Bte^{-t} - 10e^{-2t} + 1$$

Solving for the arbitrary constants, $\dot{x}(0)$ = B - 8 = 1. Therefore, B = 9. The final solution is

$$x(t) = -e^{-t} + 9te^{-t} + 5e^{-2t} + t - 2$$

c. Assume a particular solution of

$$x_p = Ct^2 + Dt + E$$

Substitute into the differential equation and obtain

$$4Ct^2 + 4Dt + 2C + 4E = t^2$$

Equating like coefficients, C = $\frac{1}{4}$, D = 0, and 2C + 4E = 0.

From which, C = $\frac{1}{4}$, D = 0, and E = - $\frac{1}{8}$.

The characteristic polynomial is

$$M^2 + 4 = (M + 2i)(M - 2i)$$

Thus, the total solution is

$$x(t) = A\cos(2t) + B\sin(2t) + \frac{1}{4}t^2 - \frac{1}{8}$$

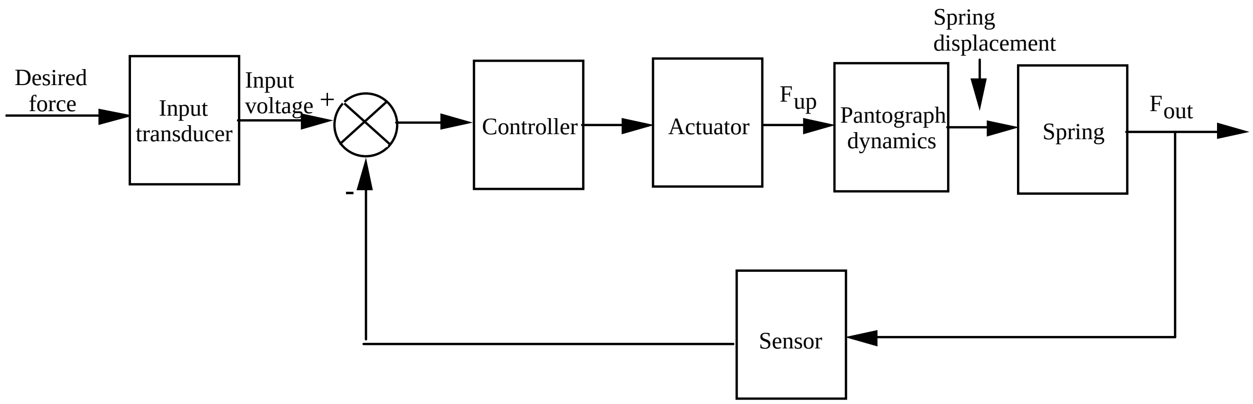
Solving for the arbitrary constants, x(0) = A - $\frac{1}{8}$ = 1 Therefore, A = $\frac{9}{8}$. Also, the derivative of the solution is

$$\frac{dx}{dt} = 2B\cos(2t) - 2A\sin(2t) + \frac{1}{2}t$$

Solving for the arbitrary constants, $\dot{x}(0)$ = 2B = 2. Therefore, B = 1. The final solution is

$$x(t) = \frac{9}{8}\cos(2t) + \sin(2t) + \frac{1}{4}t^2 - \frac{1}{8}$$

17.


Desired force
Input transducer
Input voltage
+
-
Controller
Actuator
F_{up}
Pantograph dynamics
Spring displacement
Spring
F_{out}
Sensor

T W O

Modeling in the Frequency Domain

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Transfer Functions

Finding each transfer function:

Pot: $\frac{V_i(s)}{\theta_i(s)} = \frac{10}{\pi}$;

Pre-Amp: $\frac{V_p(s)}{V_i(s)} = K$;

Power Amp: $\frac{E_a(s)}{V_p(s)} = \frac{150}{s+150}$

Motor: $J_m = 0.05 + 5\left(\frac{50}{250}\right)^2 = 0.25$

$D_m = 0.01 + 3\left(\frac{50}{250}\right)^2 = 0.13$

$\frac{K_t}{R_a} = \frac{1}{5}$

$\frac{K_t K_b}{R_a} = \frac{1}{5}$

Therefore: $\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a J_m}}{s(s+\frac{1}{J_m}(D_m+\frac{K_t K_b}{R_a}))} = \frac{0.8}{s(s+1.32)}$

And: $\frac{\theta_o(s)}{E_a(s)} = \frac{1}{5}\frac{\theta_m(s)}{E_a(s)} = \frac{0.16}{s(s+1.32)}$

Transfer Function of a Nonlinear Electrical Network

Writing the differential equation, $\frac{d(i_0 + \delta i)}{dt} + 2(i_0 + \delta i)^2 - 5 = v(t)$. Linearizing i^2 about i_0,

$(i_0+\delta i)^2 - i_0^2 = 2i\Big|_{i=i_0} \delta i = 2i_0 \delta i$. Thus, $(i_0+\delta i)^2 = i_0^2 + 2i_0 \delta i$.

Substituting into the differential equation yields, $\frac{d\delta i}{dt} + 2i_0^2 + 4i_0\delta i - 5 = v(t)$. But, the resistor voltage equals the battery voltage at equilibrium when the supply voltage is zero since the voltage across the inductor is zero at dc. Hence, $2i_0^2 = 5$, or $i_0 = 1.58$. Substituting into the linearized differential equation, $\frac{d\delta i}{dt} + 6.32\delta i = v(t)$. Converting to a transfer function, $\frac{\delta i(s)}{V(s)} = \frac{1}{s+6.32}$. Using the linearized i about i_0, and the fact that $v_r(t)$ is 5 volts at equilibrium, the linearized $v_r(t)$ is $v_r(t) = 2i^2 = 2(i_0+\delta i)^2 = 2(i_0^2+2i_0\delta i) = 5+6.32\delta i$. For excursions away from equilibrium, $v_r(t) - 5 = 6.32\delta i = \delta v_r(t)$. Therefore, multiplying the transfer function by 6.32, yields, $\frac{\delta V_r(s)}{V(s)} = \frac{6.32}{s+6.32}$ as the transfer function about $v(t) = 0$.

ANSWERS TO REVIEW QUESTIONS

1. Transfer function
2. Linear time-invariant
3. Laplace
4. $G(s) = C(s)/R(s)$, where $c(t)$ is the output and $r(t)$ is the input.
5. Initial conditions are zero
6. Equations of motion
7. Free body diagram
8. There are direct analogies between the electrical variables and components and the mechanical variables and components.
9. Mechanical advantage for rotating systems
10. Armature inertia, armature damping, load inertia, load damping
11. Multiply the transfer function by the gear ratio relating armature position to load position.
12. (1) Recognize the nonlinear component, (2) Write the nonlinear differential equation, (3) Select the equilibrium solution, (4) Linearize the nonlinear differential equation, (5) Take the Laplace transform of the linearized differential equation, (6) Find the transfer function.

SOLUTIONS TO PROBLEMS

1.

a. $$F(s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

b. $$F(s) = \int_0^\infty te^{-st}dt = \frac{e^{-st}}{s^2}(-st-1)\Big|_0^\infty = \frac{-(st+1)}{s^2e^{st}}\Bigg|_0^\infty$$

Using L'Hopital's Rule

$$F(s)\Big|_{t\to\infty} = \frac{-s}{s^3 e^{st}}\Bigg|_{t\to\infty} = 0.\text{ Therefore, } F(s) = \frac{1}{s^2}.$$

c. $F(s) = \int_0^\infty \sin\omega t\, e^{-st}dt = \frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\Big|_0^\infty = \frac{\omega}{s^2+\omega^2}$

d. $F(s) = \int_0^\infty \cos\omega t\, e^{-st}dt = \frac{e^{-st}}{s^2+\omega^2}(-s\cos\omega t + \omega\sin\omega t)\Big|_0^\infty = \frac{s}{s^2+\omega^2}$

2.

a. Using the frequency shift theorem and the Laplace transform of sin ωt, $F(s) = \frac{\omega}{(s+a)^2+\omega^2}$.

b. Using the frequency shift theorem and the Laplace transform of cos ωt, $F(s) = \frac{(s+a)}{(s+a)^2+\omega^2}$.

c. Using the integration theorem, and successively integrating u(t) three times, $\int dt = t$; $\int t dt = \frac{t^2}{2}$;

$\int \frac{t^2}{2}dt = \frac{t^3}{6}$, the Laplace transform of $t^3u(t)$, $F(s) = \frac{6}{s^4}$.

3.

a. The Laplace transform of the differential equation, assuming zero initial conditions,

is, $(s+7)X(s) = \frac{5s}{s^2+2^2}$. Solving for X(s) and expanding by partial fractions,

$$\frac{5s}{(s+7)(s^2+4)} = -\frac{35}{53}\frac{1}{s+7} + \frac{5}{53}\frac{7s+4}{s^2+4}$$

Or,

$$\frac{5s}{(s+7)(s^2+4)} = -\frac{35}{53}\frac{1}{s+7} + \frac{5}{53}\frac{7s+2\sqrt{4}}{s^2+4}$$

Taking the inverse Laplace transform, $x(t) = -\frac{35}{53}e^{-7t} + (\frac{35}{53}\cos 2t + \frac{10}{53}\sin 2t)$.

b. The Laplace transform of the differential equation, assuming zero initial conditions, is,

$(s^2+6s+8)X(s) = \frac{15}{s^2+9}$.

Solving for X(s)

$$X(s) = \frac{15}{(s^2+9)(s^2+6s+8)}$$

and expanding by partial fractions,

$$X(s) = -\frac{3}{65}\frac{6s + \frac{1}{\sqrt{9}}\sqrt{9}}{s^2+9} - \frac{3}{10}\frac{1}{s+4} + \frac{15}{26}\frac{1}{s+2}$$

Taking the inverse Laplace transform,

$$x(t) = -\frac{18}{65}\cos(3t) - \frac{1}{65}\sin(3t) - \frac{3}{10}e^{-4t} + \frac{15}{26}e^{-2t}$$

c. The Laplace transform of the differential equation is, assuming zero initial conditions, $(s^2+8s+25)x(s) = \frac{10}{s}$. Solving for X(s)

$$X(s) = \frac{10}{s\left(s^2 + 8s + 25\right)}$$

and expanding by partial fractions,

$$X(s) = \frac{2}{5}\frac{1}{s} - \frac{2}{5}\frac{1\,(s+4) + \frac{4}{\sqrt{9}}\sqrt{9}}{s+4^2+9}$$

Taking the inverse Laplace transform,

$$x(t) = \frac{2}{5} - e^{-4t}\left(\frac{8}{15}\sin(3t) + \frac{2}{5}\cos(3t)\right)$$

4.

a. Taking the Laplace transform with initial conditions, $s^2X(s)-2s+3+2sX(s)-4+2X(s) = \frac{2}{s^2+2^2}$.

Solving for X(s),

$$X(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+2s+2)}.$$

Expanding by partial fractions

$$X(s) = -\left(\frac{1}{5}\right)\frac{s + \frac{1}{\sqrt{4}}\sqrt{4}}{s^2+4} + \left(\frac{1}{5}\right)\frac{11(s+1) - \frac{3}{\sqrt{1}}\sqrt{1}}{(s+1)^2+1}$$

Therefore, $x(t) = -0.2\cos 2t - 0.1\sin 2t + e^{-t}(2.2\cos t - 0.6\sin t)$.

b. Taking the Laplace transform with initial conditions, $s^2X(s)-2s-1+2sX(s)-4+X(s) = \frac{5}{s+2} + \frac{1}{s^2}$.

Solving for X(s),

$$X(s) = \frac{2s^4 + 9s^3 + 15s^2 + s + 2}{(s+2)(s+1)^2 s^2}$$

$$X(s) = 5\frac{1}{s+2} - \frac{1}{s+1} + 9\frac{1}{(s+1)^2} - 2\frac{1}{s} + \frac{1}{s^2}$$

Therefore, $x(t) = 5e^{-2t} - e^{-t} + 9te^{-t} - 2 + t$.

c. Taking the Laplace transform with initial conditions, $s^2X(s)-s-2+4X(s) = \frac{2}{s^3}$. Solving for X(s),

$$X(s) = \frac{s^4 + 2s^3 + 2}{(s^2+4)s^3}$$

$$X(s) = \frac{1}{8}\frac{9s+8.2}{s^2+4} - \frac{1}{8}\frac{1}{s} + \frac{1}{2}\frac{1}{s^3}$$

Therefore, $x(t) = \frac{9}{8}\cos 2t + \sin 2t - \frac{1}{8} + \frac{1}{4}t^2$.

5.

Program:

```
syms t
f=5*t^2*cos(3*t+45);
pretty(f)
F=laplace(f);
F=simple(F);
pretty(F)
'b'
f=5*t*exp(-2*t)*sin(4*t+60);
pretty(f)
F=laplace(f);
F=simple(F);
pretty(F)
```

Computer response:

```
ans =

a


                                 2
                              5 t  cos(3 t + 45)

               3                                   2
              s  cos(45) - 27 s cos(45) - 9 s  sin(45) + 27 sin(45)
           10 -----------------------------------------------------
                                        2     3
                                      (s  + 9)

ans =

b


                            5 t exp(-2 t) sin(4 t + 60)

               sin(60)          ((s + 2) sin(60) + 4 cos(60)) (s + 2)
          -5 ------------- + 10 -------------------------------------
                      2                          2      2
                (s + 2)  + 16              ((s + 2)  + 16)
```

6.

Program:

```
syms s
'a'
G=(s^2+3*s+7)*(s+2)/[(s+3)*(s+4)*(s^2+2*s+100)];
pretty(G)
g=ilaplace(G);
pretty(g)
'b'
G=(s^3+4*s^2+6*s+5)/[(s+8)*(s^2+8*s+3)*(s^2+5*s+7)];
pretty(G)
g=ilaplace(G);
pretty(g)
```

Computer response:

```
ans =

a
                               2
```

```
                              (s  + 3 s + 7) (s + 2)
                         --------------------------------
                                               2
                         (s + 3) (s + 4) (s  + 2 s + 100)

                        11              4681            1/2          1/2
  - 7/103 exp(-3 t) + -- exp(-4 t) - ----- exp(-t) 11    sin(3 11    t)
                        54              61182

            4807                  1/2
         + ---- exp(-t) cos(3 11    t)
            5562

ans =

b
                                3      2
                              s  + 4 s  + 6 s + 5
                       -------------------------------------
                                2              2
                       (s + 8) (s  + 8 s + 3) (s  + 5 s + 7)

      299              1367                     1/2
  - --- exp(-8 t) + ---- exp(-4 t) cosh(13    t)
      93               417

            4895              1/2         1/2
         - ---- exp(-4 t) 13    sinh(13    t)
            5421

             232               1/2            1/2
         - ----- exp(- 5/2 t) 3    sin(1/2 3    t)
            12927

            272                          1/2
         - ---- exp(- 5/2 t) cos(1/2 3    t)
            4309
```

7.

The Laplace transform of the differential equation, assuming zero initial conditions, is,

$(s^3+3s^2+5s+1)Y(s) = (s^3+4s^2+6s+8)X(s)$.

Solving for the transfer function, $\frac{Y(s)}{X(s)} = \frac{s^3+4s^2+6s+8}{s^3+3s^2+5s+1}$.

8.

a. Cross multiplying, $(s^2+2s+7)X(s) = F(s)$.

Taking the inverse Laplace transform, $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 7x = f(t)$.

b. Cross multiplying after expanding the denominator, $(s^2+15s+56)X(s) = 10F(s)$.

Taking the inverse Laplace transform, $\frac{d^2x}{dt^2} + 15\frac{dx}{dt} + 56x = 10f(t)$.

c. Cross multiplying, $(s^3+8s^2+9s+15)X(s) = (s+2)F(s)$.

Taking the inverse Laplace transform, $\frac{d^3x}{dt^3} + 8\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 15x = \frac{df(t)}{dt} + 2f(t)$.

9.

The transfer function is $\frac{C(s)}{R(s)} = \frac{s^5+2s^4+4s^3+s^2+3}{s^6+7s^5+3s^4+2s^3+s^2+3}$.

Cross multiplying, $(s^6+7s^5+3s^4+2s^3+s^2+3)C(s) = (s^5+2s^4+4s^3+s^2+3)R(s)$.

Taking the inverse Laplace transform assuming zero initial conditions,

$$\frac{d^6c}{dt^6}+7\frac{d^5c}{dt^5}+3\frac{d^4c}{dt^4}+2\frac{d^3c}{dt^3}+\frac{d^2c}{dt^2}+3c=\frac{d^5r}{dt^5}+2\frac{d^4r}{dt^4}+4\frac{d^3r}{dt^3}+\frac{d^2r}{dt^2}+3r.$$

10.

The transfer function is $\frac{C(s)}{R(s)}=\frac{s^4+2s^3+5s^2+s+1}{s^5+3s^4+2s^3+4s^2+5s+2}$.

Cross multiplying, $(s^5+3s^4+2s^3+4s^2+5s+2)C(s) = (s^4+2s^3+5s^2+s+1)R(s)$.

Taking the inverse Laplace transform assuming zero initial conditions,

$$\frac{d^5c}{dc^5}+3\frac{d^4c}{dt^4}+2\frac{d^3c}{dt^3}+4\frac{d^2c}{dt^2}+5\frac{dc}{dt}+2c=\frac{d^4r}{dt^4}+2\frac{d^3r}{dt^3}+5\frac{d^2r}{dt^2}+\frac{dr}{dt}+r.$$

Substituting $r(t) = t^3$, $\frac{d^5c}{dc^5}+3\frac{d^4c}{dt^4}+2\frac{d^3c}{dt^3}+4\frac{d^2c}{dt^2}+5\frac{dc}{dt}+2c$

$= 18\delta(t) + (36 + 90t + 9t^2 + 3t^3)\,u(t)$.

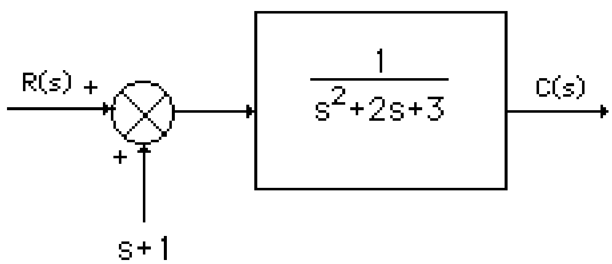
11.

Taking the Laplace transform of the differential equation, $s^2X(s)-s+1+2sX(s)-2+3x(s)=R(s)$.

Collecting terms, $(s^2+2s+3)X(s) = R(s)+s+1$.

Solving for $X(s)$, $X(s)=\frac{R(s)}{s^2+2s+3}+\frac{s+1}{s^2+2s+3}$.

The block diagram is then,



12.

Program:

```
'Factored'
Gzpk=zpk([-15 -26 -72],[0 -55 roots([1 5 30])' roots([1 27 52])'],5)
'Polynomial'
Gp=tf(Gzpk)
```

Computer response:

```
ans =

Factored

Zero/pole/gain:
            5 (s+15) (s+26) (s+72)
--------------------------------------------
s (s+55) (s+24.91) (s+2.087) (s^2 + 5s + 30)

ans =
```

```
Polynomial

Transfer function:
                 5 s^3 + 565 s^2 + 16710 s + 140400
--------------------------------------------------------------------
s^6 + 87 s^5 + 1977 s^4 + 1.301e004 s^3 + 6.041e004 s^2 + 8.58e004 s
```

13.

Program:

```
'Polynomial'
Gtf=tf([1 25 20 15 42],[1 13 9 37 35 50])
'Factored'
Gzpk=zpk(Gtf)
```

Computer response:

```
ans =

Polynomial

Transfer function:
    s^4 + 25 s^3 + 20 s^2 + 15 s + 42
-----------------------------------------
s^5 + 13 s^4 + 9 s^3 + 37 s^2 + 35 s + 50

ans =

Factored

Zero/pole/gain:
      (s+24.2) (s+1.35) (s^2 - 0.5462s + 1.286)
------------------------------------------------------
(s+12.5) (s^2 + 1.463s + 1.493) (s^2 - 0.964s + 2.679)
```

14.

Program:

```
numg=[-10 -60];
deng=[0 -40 -30 (roots([1 7 100]))' (roots([1 6 90]))'];
[numg,deng]=zp2tf(numg',deng',1e4);
Gtf=tf(numg,deng)
G=zpk(Gtf)
[r,p,k]=residue(numg,deng)
```

Computer response:

```
Transfer function:
                        10000 s^2 + 700000 s + 6e006
-----------------------------------------------------------------------------
s^7 + 83 s^6 + 2342 s^5 + 33070 s^4 + 3.735e005 s^3 + 2.106e006 s^2

                                                              + 1.08e007 s

Zero/pole/gain:
              10000 (s+60) (s+10)
------------------------------------------------
s (s+40) (s+30) (s^2 + 6s + 90) (s^2 + 7s + 100)

r =

  -0.0073
   0.0313
   2.0431 - 2.0385i
   2.0431 + 2.0385i
  -2.3329 + 2.0690i
  -2.3329 - 2.0690i
   0.5556

p =
```

```
 -40.0000
 -30.0000
  -3.5000 + 9.3675i
  -3.5000 - 9.3675i
  -3.0000 + 9.0000i
  -3.0000 - 9.0000i
        0

k =

     []
```

15.

Program:

```
syms s
'(a)'
Ga=45*[(s^2+37*s+74)*(s^3+28*s^2+32*s+16)]...
/[(s+39)*(s+47)*(s^2+2*s+100)*(s^3+27*s^2+18*s+15)];
'Ga symbolic'
pretty(Ga)
[numga,denga]=numden(Ga);
numga=sym2poly(numga);
denga=sym2poly(denga);
'Ga polynimial'
Ga=tf(numga,denga)
'Ga factored'
Ga=zpk(Ga)
'(b)'
Ga=56*[(s+14)*(s^3+49*s^2+62*s+53)]...
/[(s^2+88*s+33)*(s^2+56*s+77)*(s^3+81*s^2+76*s+65)];
'Ga symbolic'
pretty(Ga)
[numga,denga]=numden(Ga);
numga=sym2poly(numga);
denga=sym2poly(denga);
'Ga polynimial'
Ga=tf(numga,denga)
'Ga factored'
Ga=zpk(Ga)
```

Computer response:

```
ans =

(a)

ans =

Ga symbolic

                     2                 3       2
                   (s  + 37 s + 74) (s  + 28 s  + 32 s + 16)
        45 ----------------------------------------------------------
                                  2                 3       2
              (s + 39) (s + 47) (s  + 2 s + 100) (s  + 27 s  + 18 s + 15)

ans =

Ga polynimial

Transfer function:

         45 s^5 + 2925 s^4 + 51390 s^3 + 147240 s^2 + 133200 s + 53280
--------------------------------------------------------------------------------
```

```
s^7 + 115 s^6 + 4499 s^5 + 70700 s^4 + 553692 s^3 + 5.201e006 s^2 + 3.483e006 s

                                                                      + 2.75e006

ans =

Ga factored

Zero/pole/gain:
     45 (s+34.88) (s+26.83) (s+2.122) (s^2 + 1.17s + 0.5964)
-----------------------------------------------------------------
(s+47) (s+39) (s+26.34) (s^2 + 0.6618s + 0.5695) (s^2 + 2s + 100)


ans =

(b)

ans =

Ga symbolic

                                   3       2
                       (s + 14) (s   + 49 s   + 62 s + 53)
         56 ----------------------------------------------------------
              2               2                 3       2
            (s   + 88 s + 33) (s   + 56 s + 77) (s   + 81 s   + 76 s + 65)

ans =

Ga polynimial

Transfer function:

                56 s^4 + 3528 s^3 + 41888 s^2 + 51576 s + 41552
--------------------------------------------------------------------------------
s^7 + 225 s^6 + 16778 s^5 + 427711 s^4 + 1.093e006 s^3 + 1.189e006 s^2

                                                             + 753676 s + 165165

ans =

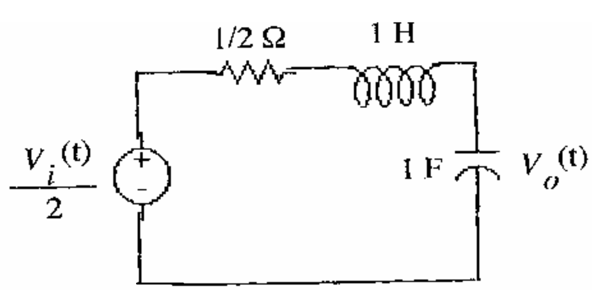
Ga factored

Zero/pole/gain:
                56 (s+47.72) (s+14) (s^2 + 1.276s + 1.111)
---------------------------------------------------------------------------
(s+87.62) (s+80.06) (s+54.59) (s+1.411) (s+0.3766) (s^2 + 0.9391s + 0.8119)
```

16.

a. Writing the node equations, $\frac{V_o - V_i}{s} + \frac{V_o}{s} + V_o = 0$. Solve for $\frac{V_o}{V_i} = \frac{1}{s+2}$.

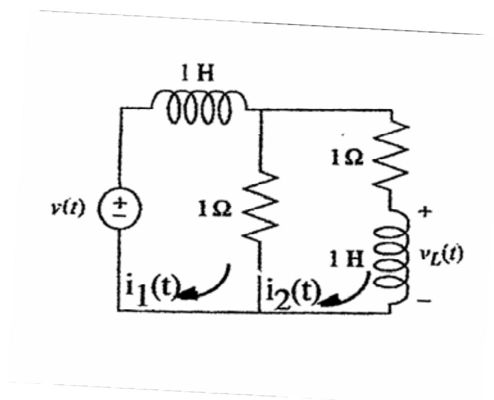
b. Thevenizing,



Using voltage division, $V_o(s) = \dfrac{V_i(s)}{2} \dfrac{\frac{1}{s}}{\frac{1}{2} + s + \frac{1}{s}}$. Thus, $\dfrac{V_o(s)}{V_i(s)} = \dfrac{1}{2s^2 + s + 2}$

17.

a.



Writing mesh equations

$$(s+1)I_1(s) - I_2(s) = V_i(s)$$

$$-I_1(s) + (s+2)I_2(s) = 0$$

But, $I_1(s) = (s+2)I_2(s)$. Substituting this in the first equation yields,

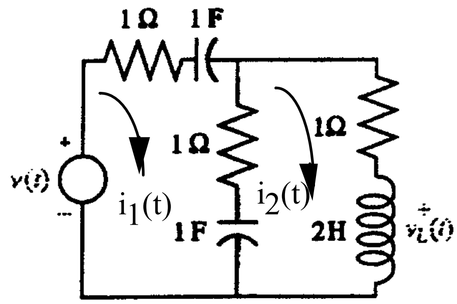
$$(s+1)(s+2)I_2(s) - I_2(s) = V_i(s)$$

or

$$I_2(s)/V_i(s) = 1/(s^2 + 3s + 1)$$

But, $V_L(s) = sI_2(s)$. Therefore, $V_L(s)/V_i(s) = s/(s^2 + 3s + 1)$.

b.



$$\left(2+\frac{2}{s}\right)I_1(s)-\left(1+\frac{1}{s}\right)I_2(s)=V(s)$$

$$-\left(1+\frac{1}{s}\right)I_1(s)+\left(2+\frac{1}{s}+2s\right)I_2(s)=0$$

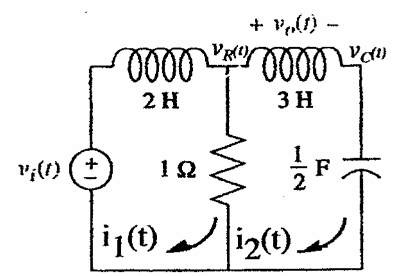
Solving for $I_2(s)$:

$$I_2(s)=\frac{\begin{vmatrix}\frac{2(s+1)}{s} & V(s)\\ -\frac{s+1}{s} & 0\end{vmatrix}}{\begin{vmatrix}\frac{2(s+1)}{s} & -\frac{s+1}{s}\\ -\frac{s+1}{s} & \frac{2s^2+2s+1}{s}\end{vmatrix}}=\frac{V(s)s}{4s^2+3s+1}$$

Therefore, $\dfrac{V_L(s)}{V(s)}=2s\dfrac{I_2(s)}{V(s)}=\dfrac{2s^2}{4s^2+3s+1}$

18.

a.



Writing mesh equations,

$$(2s + 1)I_1(s) - I_2(s) = V_i(s)$$

$$-I_1(s) + (3s + 1 + 2/s)I_2(s) = 0$$

Solving for $I_2(s)$,

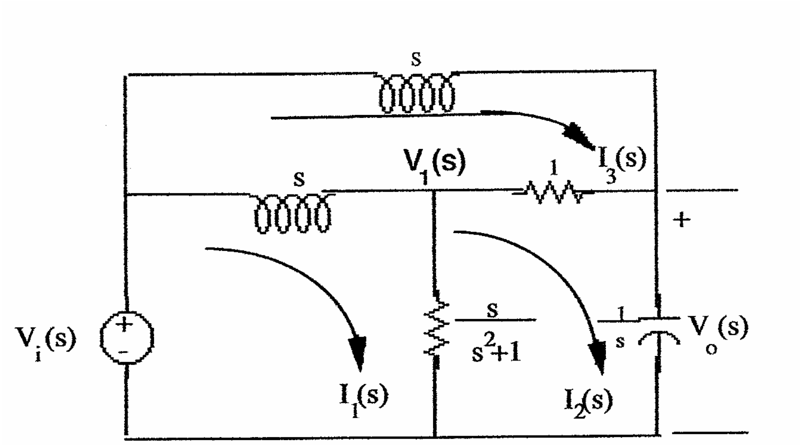
$$I_2(s) = \frac{\begin{vmatrix} 2s+1 & V_i(s) \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} 2s+1 & -1 \\ -1 & \dfrac{3s^2+s+2}{s} \end{vmatrix}}$$

Solving for $I_2(s)/V_i(s)$,

$$\frac{I_2(s)}{V_i(s)} = \frac{s}{6s^3+5s^2+4s+2}$$

But $V_o(s) = I_2(s)3s$. Therefore , $G(s) = 3s^2/(6s^3 + 5s^2 + 4s + 2)$.

b. Transforming the network yields,



Writing the loop equations,

$$(s+\frac{s}{s^2+1})I_1(s) - \frac{s}{s^2+1}I_2(s) - sI_3(s) = V_i(s)$$
$$-\frac{s}{s^2+1}I_1(s) + (\frac{s}{s^2+1}+1+\frac{1}{s})I_2(s) - I_3(s) = 0$$
$$-sI_1(s) - I_2(s) + (2s+1)I_3(s) = 0$$

Solving for $I_2(s)$,

$$I_2(s) = \frac{s(s^2+2s+2)}{s^4+2s^3+3s^2+3s+2}V_i(s)$$

But, $V_o(s) = \frac{I_2(s)}{s} = \frac{(s^2+2s+2)}{s^4+2s^3+3s^2+3s+2}V_i(s)$. Therefore,

$$\frac{V_o(s)}{V_i(s)} = \frac{s^2+2s+2}{s^4+2s^3+3s^2+3s+2}$$

19.

a. Writing the nodal equations yields,

$$\frac{V_R(s)-V_i(s)}{2s}+\frac{V_R(s)}{1}+\frac{V_R(s)-V_C(s)}{3s}=0$$

$$-\frac{1}{3s}V_R(s)+\left(\frac{1}{2}s+\frac{1}{3s}\right)V_C(s)=0$$

Rewriting and simplifying,

$$\frac{6s+5}{6s}V_R(s)-\frac{1}{3s}V_C(s)=\frac{1}{2s}V_i(s)$$

$$-\frac{1}{3s}V_R(s)+\left(\frac{3s^2+2}{6s}\right)V_C(s)=0$$

Solving for $V_R(s)$ and $V_C(s)$,

$$V_R(s)=\frac{\begin{vmatrix}\frac{1}{2s}V_i(s) & -\frac{1}{3s}\\ 0 & \frac{3s^2+2}{6s}\end{vmatrix}}{\begin{vmatrix}\frac{6s+5}{6s} & -\frac{1}{3s}\\ -\frac{1}{3s} & \frac{3s^2+2}{6s}\end{vmatrix}};\ V_C(s)=\frac{\begin{vmatrix}\frac{6s+5}{6s} & \frac{1}{2s}V_i(s)\\ -\frac{1}{3s} & 0\end{vmatrix}}{\begin{vmatrix}\frac{6s+5}{6s} & -\frac{1}{3s}\\ -\frac{1}{3s} & \frac{3s^2+2}{6s}\end{vmatrix}}$$

Solving for $V_o(s)/V_i(s)$

$$\frac{V_o(s)}{V_i(s)}=\frac{V_R(s)-V_C(s)}{V_i(s)}=\frac{3s^2}{6s^3+5s^2+4s+2}$$

b. Writing the nodal equations yields,

$$\frac{(V_1(s)-V_i(s))}{s}+\frac{(s^2+1)}{s}V_1(s)+(V_1(s)-V_o(s))=0$$

$$(V_o(s)-V_1(s))+sV_o(s)+\frac{(V_o(s)-V_i(s))}{s}=0$$

Rewriting and simplifying,

$$(s+\frac{2}{s}+1)V_1(s)-V_o(s)=\frac{1}{s}V_i(s)$$

$$V_1(s)+(s+\frac{1}{s}+1)V_o(s)=\frac{1}{s}V_i(s)$$

Solving for $V_o(s)$

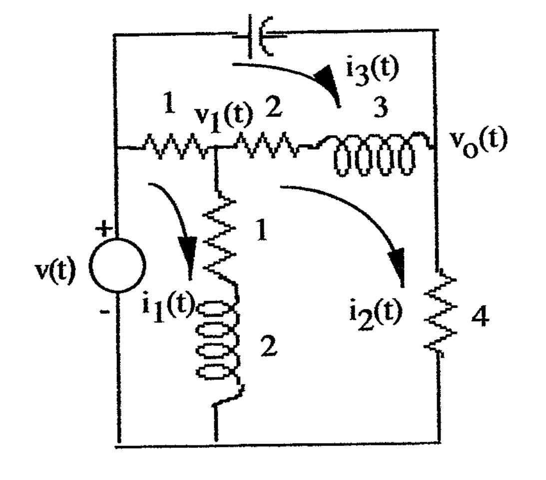
$$V_o(s) = \frac{(s^2+2s+2)}{s^4+2s^3+3s^2+3s+2} V_i(s).$$

Hence,

$$\frac{V_o(s)}{V_i(s)} = \frac{(s^2+2s+2)}{s^4+2s^3+3s^2+3s+2}$$

20.

a.



Mesh:

$$(2+2s)I_1(s) - (1+2s)I_2(s) - I_3(s) = V(s)$$

$$-(1+2s)I_1(s) + (7+5s)I_2(s) - (2+3s)I_3(s) = 0$$

$$-I_1(s) - (2+3s)I_2(s) + \left(3+3s+\frac{5}{s}\right)I_3(s) = 0$$

Nodal:

$$V_1(s) - V(s) + \frac{V_1(s)}{(1+2s)} + \frac{(V_1(s) - V_o(s))}{2+3s} = 0$$

$$\frac{(V_o(s) - V_1(s))}{2+3s} + \frac{V_o(s)}{4} + \frac{(V_o(s) - V(s))}{\frac{5}{s}} = 0$$

or

$$\frac{6s^2+12s+5}{6s^2+7s+2} V_1(s) - \frac{1}{3s+2} V_o(s) = V(s)$$

$$-\frac{1}{3s+2} V_1(s) + \frac{1}{20}\frac{12s^2+23s+30}{3s+2} V_o(s) = \frac{s}{5} V(s)$$

b.

Program:

```
syms s V                          %Construct symbolic object for frequency
```

```
                              %variable 's' and V.
'Mesh Equations'
A2=[(2+2*s) V -1
-(1+2*s) 0 -(2+3*s)
-1 0 (3+3*s+(5/s))]           %Form Ak = A2.
A=[(2+2*s) -(1+2*s) -1
-(1+2*s) (7+5*s) -(2+3*s)
-1 -(2+3*s) (3+3*s+(5/s))]    %Form A.
I2=det(A2)/det(A);            %Use Cramer's Rule to solve for I2.
G1=I2/V;                      %Form transfer function, G1(s) = I2(s)/V(s).
G=4*G1;                       %Form transfer function, G(s) = V4(s)/V(s).
'G(s) via Mesh Equations'     %Display label.
pretty(G)                     %Pretty print G(s)


'Nodal Equations'
A2=[(6*s^2+12*s+5)/(6*s^2+7*s+2) V
-1/(3*s+2) s*(V/5)]           %Form Ak = A2.
A=[(6*s^2+12*s+5)/(6*s^2+7*s+2) -1/(3*s+2)
-1/(3*s+2) (1/20)*(12*s^2+23*s+30)/(3*s+2)]
                              %Form A.
I2=simple(det(A2))/simple(det(A));
                              %Use Cramer's Rule to solve for I2.
G1=I2/V;                      %Form transfer function, G1(s) = I2(s)/V(s).
'G(s) via Nodal Equations'    %Display label.
pretty(G)                     %Pretty print G(s)
```

Computer response:

```
ans =

Mesh Equations

A2 =

[     2+2*s,         V,        -1]
[    -1-2*s,         0,    -2-3*s]
[        -1,         0, 3+3*s+5/s]

A =

[     2+2*s,    -1-2*s,        -1]
[    -1-2*s,     7+5*s,    -2-3*s]
[        -1,    -2-3*s, 3+3*s+5/s]

ans =

G(s) via Mesh Equations



                             2          3
                  15 s + 12 s  + 5 + 6 s
                4 -------------------------
                             2           3
                 120 s + 78 s  + 65 + 24 s

ans =

Nodal Equations

A2 =

[ (6*s^2+12*s+5)/(2+7*s+6*s^2),                           V]
[                  -1/(2+3*s),                      1/5*s*V]

A =
```

```
[  (6*s^2+12*s+5)/(2+7*s+6*s^2),                       -1/(2+3*s)]
[                   -1/(2+3*s), (3/5*s^2+23/20*s+3/2)/(2+3*s)]

ans =

G(s) via Nodal Equations

                                 2           3
                        15 s + 12 s  + 5 + 6 s
                      4 --------------------------
                            3       2
                        24 s  + 78 s  + 120 s + 65
```

21.

a.

$$Z_1(s) = 5x10^5 + \frac{10^6}{s}$$

$$Z_2(s) = 10^5 + \frac{10^6}{s}$$

Therefore,

$$-\frac{Z_2(s)}{Z_1(s)} = -\frac{1}{5}\frac{s+10}{s+2}$$

b.

$$Z_1(s) = 100000 + \frac{1}{1\times10^{-6}s} = 100000\frac{s+10}{s}$$

$$Z_2(s) = 100000 + \frac{1}{1\times10^{-6}s + \frac{1}{100000}} = 100000\frac{s+20}{s+10}$$

Therefore,

$$G(s) = -\frac{Z_2}{Z_1} = -\frac{(s+20)s}{(s+10)^2}$$

22.

a.

$$Z_1(s) = 200000 + \frac{1}{1\times10^{-6}s}$$

$$Z_2(s) = 100000 + \frac{1}{1\times10^{-6}s}$$

Therefore,

$$G(s) = \frac{Z_1+Z_2}{Z_1} = \frac{3}{2}\frac{s+\frac{20}{3}}{s+5}.$$

b.

$$Z_1(s) = 2x10^5 + \frac{\frac{5x10^{11}}{s}}{5x10^5 + \frac{10^6}{s}}$$

$$Z_2(s) = 5x10^5 + \frac{\frac{10^{11}}{s}}{10^5 + \frac{10^6}{s}}$$

Therefore,

$$\frac{Z_1(s) + Z_2(s)}{Z_1(s)} = \frac{7}{2}\frac{(s+3.18)(s+11.68)}{(s+7)(s+10)}$$

23.

Writing the equations of motion, where $x_2(t)$ is the displacement of the right member of springr,

$$(s^2+s+1)X_1(s) \; -X_2(s) = 0$$

$$-X_1(s) +X_2(s) = F(s)$$

Adding the equations,

$$(s^2+s)X_1(s) \; = F(s)$$

From which, $\frac{X_1(s)}{F(s)} = \frac{1}{s(s+1)}$.

24.

Writing the equations of motion,

$$(s^2+s+1)X_1(s) - (s+1)X_2(s) = F(s)$$

$$-(s+1)X_1(s) + (s^2+s+1)X_2(s) = 0$$

Solving for $X_2(s)$,

$$X_2(s) = \frac{\begin{vmatrix} (s^2+s+1) & F(s) \\ -(s+1) & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+s+1) & -(s+1) \\ -(s+1) & (s^2+s+1) \end{vmatrix}} = \frac{(s+1)F(s)}{s^2(s^2+2s+2)}$$

From which,

$$\frac{X_2(s)}{F(s)} = \frac{(s+1)}{s^2(s^2+2s+2)}.$$

25.

Let $X_1(s)$ be the displacement of the left member of the spring and $X_3(s)$ be the displacement of the mass.

Writing the equations of motion

$$2x_1(s) - 2x_2(s) = F(s)$$
$$-2X_1(s) + (5s+2)X_2(s) - 5sX_3(s) = 0$$
$$-5sX_2(s) + (10s^2 + 7s)X_3(s) = 0$$

Solving for $X_2(s)$,

$$X_2(s) = \frac{\begin{vmatrix} 5s^2+10 & F(s) \\ -10 & 0 \end{vmatrix}}{\begin{vmatrix} 5s^2+10 & -10 \\ -10 & \frac{1}{5}s+10 \end{vmatrix}} = \frac{10F(s)}{s(s^2+50s+2)}$$

Thus, $\dfrac{X_2(s)}{F(s)} = \dfrac{1}{10}\dfrac{(10s+7)}{s(5s+1)}$

26.

$$\left(s^2+3s+2\right)X_1(s) - (s+1)X_2(s) = 0$$
$$-(s+1)X_1(s) + \left(s^2+2s+1\right)X_2(s) = F(s)$$

Solving for $X_1(s)$; $X_1 = \dfrac{\left|\begin{pmatrix} 0 & -[s+1] \\ F & s^2+2s+1 \end{pmatrix}\right|}{\left|\begin{pmatrix} s^2+3s+2 & -[s+1] \\ -[s+1] & s^2+2s+1 \end{pmatrix}\right|} = \dfrac{F(s)}{s^3+4s^2+4s+1}$. Thus, $\dfrac{X_1}{F(s)} = \dfrac{1}{s^3+4s^2+4s+1}$

27.

Writing the equations of motion,

$$(s^2+s+1)X_1(s) - sX_2(s) = 0$$
$$-sX_1(s) + (s^2+2s+1)X_2(s) - X_3(s) = F(s)$$
$$-X_2(s) + (s^2+s+1)X_3(s) = 0$$

Solving for $X_3(s)$,

$$X_3(s) = \frac{\begin{vmatrix} (s^2+s+1) & -s & 0 \\ -s & (s^2+2s+1) & F(s) \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+s+1) & -s & 0 \\ -s & (s^2+2s+1) & -1 \\ 0 & -1 & (s^2+s+1) \end{vmatrix}} = \frac{F(s)}{s(s^3+3s^2+3s+3)}$$

From which, $\frac{X_3(s)}{F(s)} = \frac{1}{s(s^3+3s^2+3s+3)}$.

28.

a.

$$(s^2+2s+1)X_1(s) - 2sX_2(s) - X_3(s) = F(s)$$
$$-2sX_1(s) + (s^2+4s)X_2(s) - sX_3(s) = 0$$
$$-X_1(s) - sX_2(s) + (s+1)X_3(s) = 0$$

Solving for $X_2(s)$,

$$X_2(s) = \frac{\begin{vmatrix} (s^2+2s+1) & F(s) & -1 \\ -2s & 0 & -s \\ -1 & 0 & s+1 \end{vmatrix}}{\Delta} = \frac{-F(s)\begin{vmatrix} -2s & -s \\ -1 & s+1 \end{vmatrix}}{\Delta}$$

or,

$$\frac{X_2(s)}{F(s)} = \frac{2s+3}{s(s^3+6s^2+9s+3)}$$

b.

$$(4s^2+s+4)X_1(s) - (s+1)X_2(s) - 3X_3(s) = 0$$
$$-(s+1)X_1(s) + (2s^2+5s+1)X_2(s) - 4sX_3(s) = F(s)$$
$$-3X_1(s) - 4sX_2(s) + (4s+3)X_3(s) = 0$$

Solving for $X_3(s)$,

$$X_3(s) = \frac{\begin{vmatrix} (4s^2+s+4) & -(s+1) & 0 \\ -(s+1) & (2s^2+5s+1) & F(s) \\ -3 & -4s & 0 \end{vmatrix}}{\Delta} = \frac{-F(s)\begin{vmatrix} (4s^2+s+4) & -(s+1) \\ -3 & -4s \end{vmatrix}}{\Delta}$$

or

$$\frac{X_3(s)}{F(s)} = \frac{16s^3+4s^2+19s+3}{32s^5+48s^4+114s^3+18s^2}$$

29.

Writing the equations of motion,

$$(s^2+2s+2)X_1(s) - X_2(s) - sX_3(s) = 0$$
$$-X_1(s) + (s^2+s+1)X_2(s) - sX_3(s) = F(s)$$
$$-sX_1(s) - sX_2(s) + (s^2+2s+1) = 0$$

30.

a.

Writing the equations of motion,

$$(s^2+9s+8)\theta_1(s)-(2s+8)\theta_2(s)=0$$
$$-(2s+8)\theta_1(s)+(s^2+2s+11)\theta_2(s)=T(s)$$

b.

Defining

$\theta_1(s)$ = rotation of J_1

$\theta_2(s)$ = rotation between K_1 and D_1

$\theta_3(s)$ = rotation of J_3

$\theta_4(s)$ = rotation of right-hand side of K_2

the equations of motion are

$$(J_1s^2+K_1)\theta_1(s)-K_1\theta_2(s)=T(s)$$
$$-K_1\theta_1(s)+(D_1s+K_1)\theta_2(s)-D_1s\theta_3(s)=0$$
$$-D_1s\theta_2(s)+(J_2s^2+D_1s+K_2)\theta_3(s)-K_2\theta_4(s)=0$$
$$-K_2\theta_3(s)+(D_2s+(K_2+K_3))\theta_4(s)=0$$

31.

Writing the equations of motion,

$$(s^2+2s+1)\theta_1(s)-(s+1)\theta_2(s)=T(s)$$
$$-(s+1)\theta_1(s)+(2s+1)\theta_2(s)=0$$

Solving for $\theta_2(s)$

$$\theta_2(s)=\frac{\begin{vmatrix}(s^2+2s+1) & T(s)\\ -(s+1) & 0\end{vmatrix}}{\begin{vmatrix}(s^2+2s+1) & -(s+1)\\ -(s+1) & (2s+1)\end{vmatrix}}=\frac{T(s)}{2s(s+1)}$$

Hence,

$$\frac{\theta_2(s)}{T(s)}=\frac{1}{2s(s+1)}$$

32.

Reflecting impedances to θ_3,

$$(J_{eq}s^2+D_{eq}s)\theta_3(s) = T(s)\left(\frac{N_4N_2}{N_3N_1}\right)$$

Thus,

$$\frac{\theta_3(s)}{T(s)} = \frac{\frac{N_4N_2}{N_3N_1}}{J_{eq}s^2 + D_{eq}s}$$

where

$$J_{eq} = J_4+J_5+(J_2+J_3)\left(\frac{N_4}{N_3}\right)^2 + J_1\left(\frac{N_4N_2}{N_3N_1}\right)^2, \text{ and}$$

$$D_{eq} = (D_4 + D_5) + (D_2 + D_3)\left(\frac{N_4}{N_3}\right)^2 + D_1\left(\frac{N_4N_2}{N_3N_1}\right)^2$$

33.

Reflecting all impedances to $\theta_2(s)$,

$$\left\{\left[J_2+J_1\left(\frac{N_2}{N_1}\right)^2+J_3\left(\frac{N3}{N4}\right)^2\right]s^2 + \left[f_2+f_1\left(\frac{N2}{N1}\right)^2+f_3\left(\frac{N3}{N4}\right)^2\right]s + \left[K\left(\frac{N3}{N4}\right)^2\right]\right\}\theta_2(s) = T(s)\frac{N2}{N1}$$

Substituting values,

$$\left\{\left[1+2(3)^2+16\left(\frac{1}{4}\right)^2\right]s^2 + \left[2+1(3)^2+32\left(\frac{1}{4}\right)^2\right]s + 64\left(\frac{1}{4}\right)^2\right\}\theta_2(s) = T(s)(3)$$

Thus,

$$\frac{\theta_2(s)}{T(s)} = \frac{3}{20s^2+13s+4}$$

34.

Reflecting impedances to θ_2,

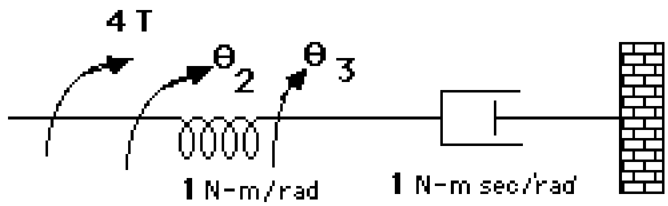
$$\left[200 + 3\left(\frac{50}{5}\right)^2 + 200\left(\frac{5}{25}x\frac{50}{5}\right)^2\right]s^2 + \left[1000\left(\frac{5}{25}x\frac{50}{5}\right)^2\right]s + \left[250 + 3\left(\frac{50}{5}\right)^2\right] = \left(\frac{50}{5}\right)T(s)$$

Thus,

$$\frac{\theta_2(s)}{T(s)} = \frac{10}{1300s^2 + 4000s + 550}$$

35.

Reflecting impedances and applied torque to respective sides of the spring yields the following equivalent circuit:



Writing the equations of motion,

$$\theta_2(s) - \theta_3(s) = 4T(s)$$

$$-\theta_2(s) + (s+1)\theta_3(s) = 0$$

Solving for $\theta_3(s)$,

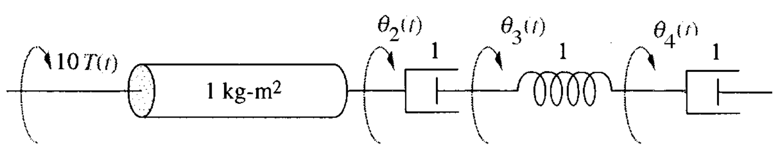
$$\theta_3(s) = \frac{\begin{vmatrix} 1 & 4T(s) \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -1 & (s+1) \end{vmatrix}} = \frac{4T(s)}{s}$$

Hence, $\frac{\theta_3(s)}{T(s)} = \frac{4}{s}$. But, $\theta_4(s) = \frac{1}{5}\ \theta_3(s)$. Thus, $\frac{\theta_4(s)}{T(s)} = \frac{4/5}{s}$.

36.

Reflecting impedances and applied torque to respective sides of the viscous damper yields the following

equivalent circuit:



Writing the equations of motion,

$$(s^2 + s)\theta_2(s) - s\theta_3(s) = 10T(s)$$
$$-s\theta_2(s) + (s+1)\theta_3(s) - \theta_4(s) = 0$$
$$-\theta_3(s) + (s+1)\theta_4(s) = 0$$

Solving for $\theta_4(s)$,

$$\theta_4(s)=\frac{\begin{vmatrix} s(s+1) & -s & 10T(s) \\ -s & (s+1) & 0 \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} s(s+1) & -s & 0 \\ -s & (s+1) & -1 \\ 0 & -1 & (s+1) \end{vmatrix}}=\frac{s10T(s)}{\begin{vmatrix} s(s+1) & -s & 0 \\ -s & (s+1) & -1 \\ 0 & -1 & (s+1) \end{vmatrix}}$$

Thus,

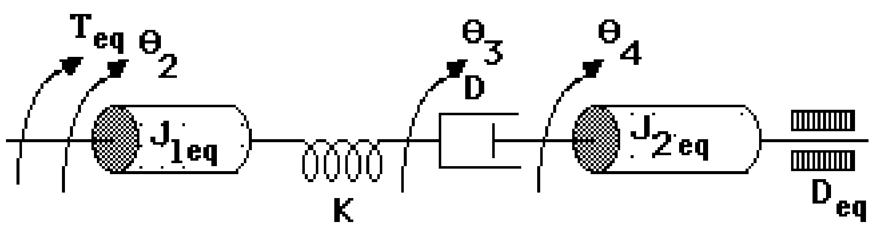
$$\frac{\theta_4(s)}{T(s)}=\frac{10}{s(s+1)^2}$$

But, $\theta_L(s)=5\theta_4(s)$. Hence,

$$\frac{\theta_4(s)}{T(s)}=\frac{50}{s(s+1)^2}$$

37.

Reflect all impedances on the right to the viscous damper and reflect all impedances and torques on the left to the spring and obtain the following equivalent circuit:



Writing the equations of motion,

$$(J_{1eq}s^2+K)\theta_2(s) - K\theta_3(s) = T_{eq}(s)$$

$$-K\theta_2(s)+(Ds+K)\theta_3(s) - Ds\theta_4(s) = 0$$

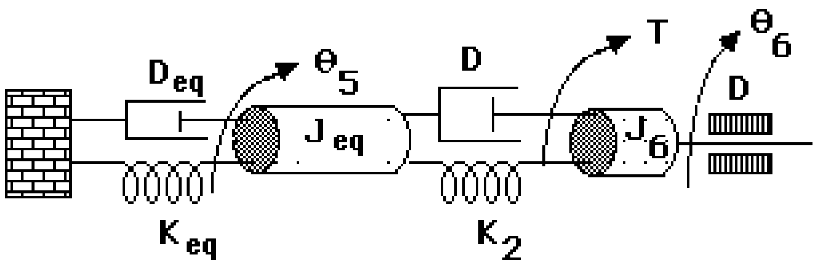
$$-Ds\theta_3(s) +[J_{2eq}s^2 +(D+D_{eq})s]\theta_4(s) = 0$$

where: $J_{1eq} = J_2+(J_a+J_1)\left(\frac{N_2}{N_1}\right)^2$; $J_{2eq} = J_3+(J_L+J_4)\left(\frac{N_3}{N_4}\right)^2$; $D_{eq} = D_L\left(\frac{N_3}{N_4}\right)^2$; $\theta_2(s) = \theta_1(s)$

$\frac{N_1}{N_2}$.

38.

Reflect impedances to the left of J_5 to J_5 and obtain the following equivalent circuit:



Writing the equations of motion,

$$[J_{eq}s^2+(D_{eq}+D)s+(K_2+K_{eq})]\theta_5(s) \qquad -[Ds+K_2]\theta_6(s) = 0$$

$$-[K_2+Ds]\theta_5(s) + [J_6s^2+2Ds+K_2]\theta_6(s) = T(s)$$

From the first equation, $\frac{\theta_6(s)}{\theta_5(s)} = \frac{J_{eq}s^2+(D_{eq}+D)s+ (K_2+K_{eq})}{Ds+K_2}$. But, $\frac{\theta_5(s)}{\theta_1(s)} = \frac{N_1N_3}{N_2N_4}$. Therefore,

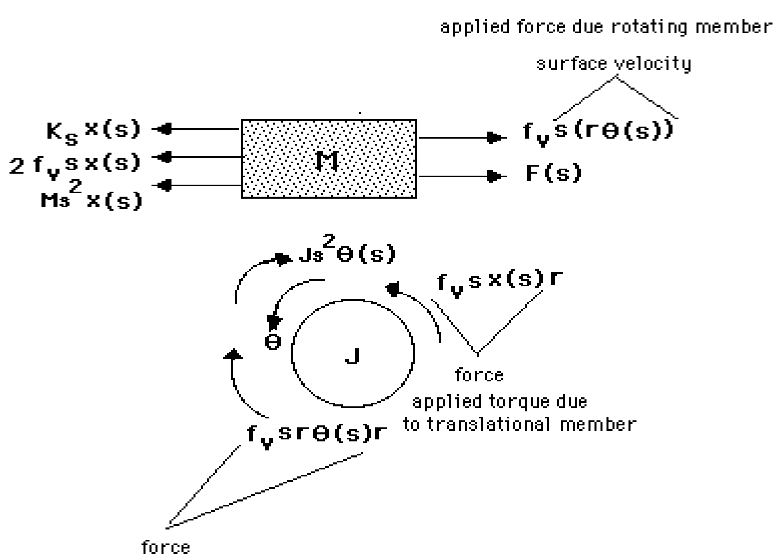
$$\frac{\theta_6(s)}{\theta_1(s)} = \frac{N_1N_3}{N_2N_4}\left(\frac{J_{eq}s^2+(D_{eq}+D)s+ (K_2+K_{eq})}{Ds+K_2}\right),$$

where $J_{eq} = \left[J_1\left(\frac{N_4N_2}{N_3N_1}\right)^2 + (J_2+J_3)\left(\frac{N_4}{N_3}\right)^2 + (J_4+J_5)\right]$, $K_{eq} = K_1\left(\frac{N_4}{N_3}\right)^2$, and

$D_{eq} = D\left[\left(\frac{N_4N_2}{N_3N_1}\right)^2 + \left(\frac{N_4}{N_3}\right)^2 + 1\right]$.

39.

Draw the freebody diagrams,



Write the equations of motion from the translational and rotational freebody diagrams,

$$(Ms^2+2f_v s+K_2)X(s) \qquad -f_v rs\theta(s) = F(s)$$

$$-f_v rsX(s) +(Js^2+f_v r^2 s)\theta(s) = 0$$

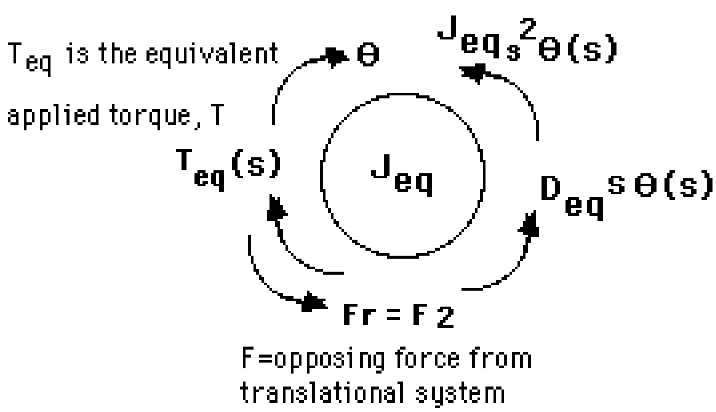
Solve for $\theta(s)$,

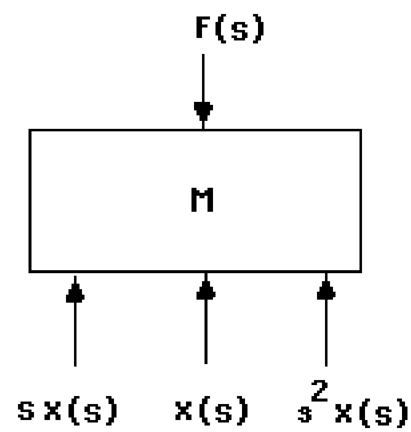
$$\theta(s) = \frac{\begin{vmatrix} Ms^2+2f_v s+K_2 & F(s) \\ -f_v rs & 0 \end{vmatrix}}{\begin{vmatrix} Ms^2+2f_v s+K_2 & -f_v rs \\ -f_v rs & Js^2+f_v r^2 s \end{vmatrix}} = \frac{f_v rF(s)}{JMs^3+(2Jf_v+Mf_v r^2)s^2+(JK_2+f_v^2 r^2)s+K_2 f_v r^2}$$

From which, $\dfrac{\theta(s)}{F(s)} = \dfrac{f_v r}{JMs^3+(2Jf_v+Mf_v r^2)s^2+(JK_2+f_v^2 r^2)s+K_2 f_v r^2}$.

40.

Draw a freebody diagram of the translational system and the rotating member connected to the translational system.





From the freebody diagram of the mass, $F(s) = (s^2+s+1)X(s)$. Summing torques on the rotating member,

$(J_{eq}s^2 +D_{eq}s)\theta(s) + F(s)2 = T_{eq}(s)$. Substituting F(s) above, $(J_{eq}s^2 +D_{eq}s)\theta(s) + (2s^2+2s+2)X(s) = T_{eq}(s)$. However, $\theta(s) = \dfrac{X(s)}{2}$. Substituting and simplifying,

$$T_{eq} = \left[\left(\frac{J_{eq}}{2} +2\right)s^2 +\left(\frac{D_{eq}}{2} +2\right)s+2\right]X(s)$$

But, $J_{eq} = 1+1(4)^2 = 17$, $D_{eq} = 1(2)^2 +1 = 5$, and $T_{eq}(s) = 4T(s)$. Therefore, $\left[\frac{21}{2}s^2 + \frac{9}{2}s+2\right]X(s) = 4T(s)$. Finally, $\frac{X(s)}{T(s)} = \frac{\frac{8}{21}}{s^2+\frac{9}{21}s+\frac{4}{21}}$.

41.

Writing the equations of motion,

$$(J_1s^2+K_1)\theta_1(s) \quad - K_1\theta_2(s) \quad = T(s)$$
$$-K_1\theta_1(s) + (J_2s^2+D_3s+K_1)\theta_2(s) +F(s)r \quad -D_3s\theta_3(s) = 0$$
$$-D_3s\theta_2(s) + (J_2s^2+D_3s)\theta_3(s) = 0$$

where F(s) is the opposing force on J_2 due to the translational member and r is the radius of J_2. But, for the translational member,

$$F(s) = (Ms^2+f_vs+K_2)X(s) = (Ms^2+f_vs+K_2)r\theta(s)$$

Substituting F(s) back into the second equation of motion,

$$(J_1s^2+K_1)\theta_1(s) \quad - K_1\theta_2(s) \quad = T(s)$$

$$-K_1\theta_1(s) + [(J_2 + Mr^2)s^2+(D_3 + f_vr^2)s+(K_1 + K_2r^2)]\theta_2(s) \quad -D_3s\theta_3(s) = 0$$

$$-D_3s\theta_2(s) + (J_2s^2+D_3s)\theta_3(s) = 0$$

Notice that the translational components were reflected as equivalent rotational components by the square of the radius. Solving for $\theta_2(s)$, $\theta_2(s) = \frac{K_1(J_3s^2+D_3s)T(s)}{\Delta}$, where Δ is the determinant formed from the coefficients of the three equations of motion. Hence,

$$\frac{\theta_2(s)}{T(s)} = \frac{K_1(J_3s^2+D_3s)}{\Delta}$$

Since

$$X(s) = r\theta_2(s), \quad \frac{X(s)}{T(s)} = \frac{rK_1(J_3s^2+D_3s)}{\Delta}$$

42.

$$\frac{K_t}{R_a} = \frac{T_{stall}}{E_a} = \frac{100}{50} = 2\ ;\ K_b = \frac{E_a}{\omega_{no-load}} = \frac{50}{150} = \frac{1}{3}$$

Also,

$$J_m = 2+18\left(\frac{1}{3}\right)^2 = 4;\ D_m = 2+36\left(\frac{1}{3}\right)^2 = 6.$$

Thus,

$$\frac{\theta_m(s)}{E_a(s)} = \frac{2/4}{s(s+\frac{1}{4}(6+\frac{2}{3}))} = \frac{1/2}{s(s+\frac{5}{3})}$$

Since $\theta_L(s) = \frac{1}{3}\,\theta_m(s)$,

$$\frac{\theta_L(s)}{E_a(s)} = \frac{\frac{1}{6}}{s(s+\frac{5}{3})}.$$

43.

The parameters are:

$$\frac{K_t}{R_a} = \frac{T_s}{E_a} = \frac{5}{5} = 1;\ K_b = \frac{E_a}{\omega} = \frac{5}{\frac{600}{\pi}2\pi\frac{1}{60}} = \frac{1}{4};\ J_m = 16\left(\frac{1}{4}\right)^2 + 4\left(\frac{1}{2}\right)^2 + 1 = 3;\ D_m = 32\left(\frac{1}{4}\right)^2 = 2$$

Thus,

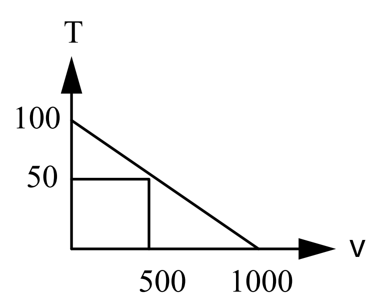
$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{1}{3}}{s(s+\frac{1}{3}(2+(1)(\frac{1}{4})))} = \frac{\frac{1}{3}}{s(s+0.75)}$$

Since $\theta_2(s) = \frac{1}{4}\,\theta_m(s)$,

$$\frac{\theta_2(s)}{E_a(s)} = \frac{\frac{1}{12}}{s(s+0.75)}.$$

44.

The following torque-speed curve can be drawn from the data given:



Therefore, $\frac{K_t}{R_a} = \frac{T_{stall}}{E_a} = \frac{100}{10} = 10$; $K_b = \frac{E_a}{\omega_{no-load}} = \frac{10}{1000} = \frac{1}{100}$. Also, $J_m = 5+100\left(\frac{1}{5}\right)^2 = 9$;

$D_m = 1$. Thus,

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{10}{9}}{s(s+\frac{1}{9}(1+0.1))} = \frac{\frac{10}{9}}{s(s+0.122)}$$. Since $\theta_L(s) = \frac{1}{5}\theta_m(s)$, $\frac{\theta_L(s)}{E_a(s)} = \frac{\frac{10}{45}}{s(s+0.122)} = \frac{0.222}{s(s+0.122)}$.

45.

From Eqs. (2.45) and (2.46),

$$R_a I_a(s) + K_b s\theta(s) = E_a(s) \tag{1}$$

Also,

$T_m(s) = K_t I_a(s) = (J_m s^2 + D_m s)\theta(s)$. Solving for $\theta(s)$ and substituting into Eq. (1), and simplifying yields

$$\frac{I_a(s)}{E_a(s)} = \frac{1}{R_a}\frac{(s+\frac{D_m}{J_m})}{s+\frac{R_a D_m + K_b K_t}{R_a J_m}} \tag{2}$$

Using $T_m(s) = K_t I_a(s)$ in Eq. (2),

$$\frac{T_m(s)}{E_a(s)} = \frac{K_t}{R_a}\frac{(s+\frac{D_m}{J_m})}{s+\frac{R_a D_m + K_b K_t}{R_a J_m}}$$

46.

For the rotating load, assuming all inertia and damping has been reflected to the load, $(J_{eqL}s^2 + D_{eqL}s)\theta_L(s) + F(s)r = T_{eq}(s)$, where F(s) is the force from the translational system, r=2 is the radius of the rotational member, J_{eqL} is the equivalent inertia at the load of the rotational load and the armature, and D_{eqL} is the equivalent damping at the load of the rotational load and the armature. Since $J_{eqL} = 1(2)^2 + 1 = 5$, and $D_{eqL} = 1(2)^2 + 1 = 5$, the equation of motion becomes, $(5s^2+5s)\theta_L(s) + F(s)r = T_{eq}(s)$. For the translational system, $(s^2+s)X(s) = F(s)$. Since $X(s) = 2\theta_L(s)$, $F(s) = (s^2+s)2\theta_L(s)$. Substituting F(s) into the rotational equation, $(9s^2+9s)\,\theta_L(s) = T_{eq}(s)$. Thus, the equivalent inertia at the load is 9, and the equivalent damping at the load is 9. Reflecting these back to the armature, yields an equivalent inertia of $\frac{9}{4}$ and an equivalent damping of $\frac{9}{4}$. Finally, $\frac{K_t}{R_a} = 1$;

$K_b = 1$. Hence, $\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{4}{9}}{s(s+\frac{4}{9}(\frac{9}{4}+1))} = \frac{\frac{4}{9}}{s(s+\frac{13}{9})}$. Since $\theta_L(s) = \frac{1}{2}\ \theta_m(s)$, $\frac{\theta_L(s)}{E_a(s)} = \frac{\frac{2}{9}}{s(s+\frac{13}{9})}$. But

$X(s) = r\theta_L(s) = 2\theta_L(s)$. therefore, $\frac{X(s)}{E_a(s)} = \frac{\frac{4}{9}}{s(s+\frac{13}{9})}$.

47.

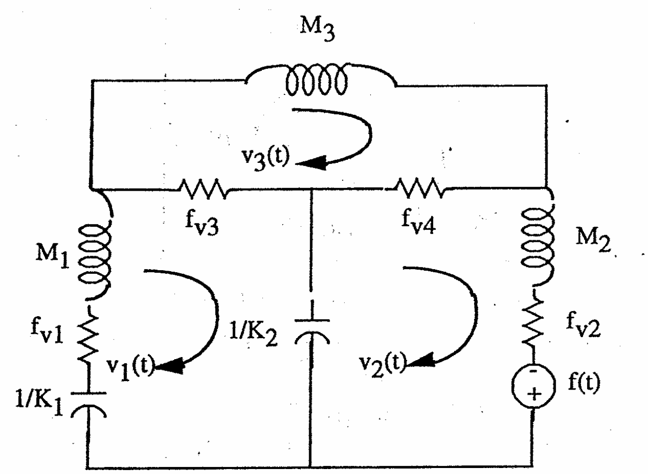
The equations of motion in terms of velocity are:

$$[M_1s + (f_{v1} + f_{v3}) + \frac{K_1}{s} + \frac{K_2}{s}]V_1(s) - \frac{K_2}{s}V_2(s) - f_{v3}V_3(s) = 0$$

$$-\frac{K_2}{s}V_1(s) + [M_2s + (f_{v2} + f_{v4}) + \frac{K_2}{s}]V_2(s) - f_{v4}V_3(s) = F(s)$$

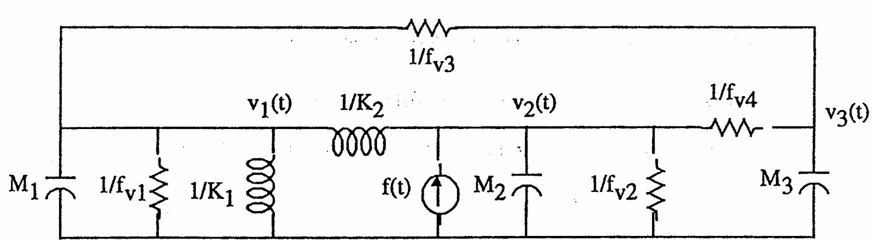
$$-f_{v3}V_1(s) - f_{v4}V_2(s) + [M_3s + f_{v3} + f_{v4}]V_3(S) = 0$$

For the series analogy, treating the equations of motion as mesh equations yields



In the circuit, resistors are in ohms, capacitors are in farads, and inductors are in henries.

For the parallel analogy, treating the equations of motion as nodal equations yields



In the circuit, resistors are in ohms, capacitors are in farads, and inductors are in henries.

48.

Writing the equations of motion in terms of angular velocity, $\Omega(s)$ yields

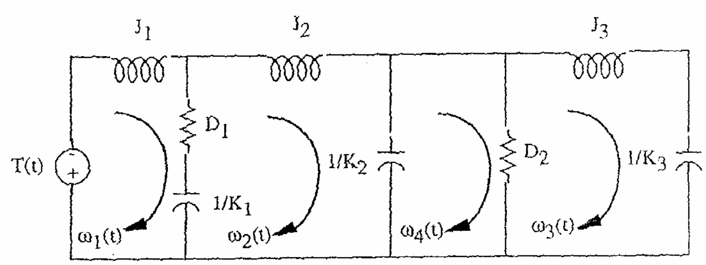
$$(J_1 s + D_1 + \frac{K_1}{s})\Omega_1(s) - (D_1 + \frac{K_1}{s})\Omega_2(s) = T(s)$$

$$-(D_1 + \frac{K_1}{s})\Omega_1(s) + (J_2 s + D_1 + \frac{(K_1 + K_2)}{s})\Omega_2(s) = 0$$

$$-\frac{K_2}{s}\Omega_2(s) - D_2\Omega_3(s) + (D_2 + \frac{K_2}{s})\Omega_4(s) = 0$$

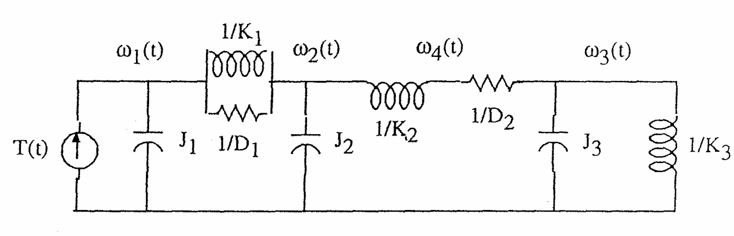
$$(J_3 s + D_2 + \frac{K_3}{s})\Omega_3(s) - D_2\Omega_4(s) = 0$$

For the series analogy, treating the equations of motion as mesh equations yields



In the circuit, resistors are in ohms, capacitors are in farads, and inductors are in henries.

For the parallel analogy, treating the equations of motion as nodal equations yields



In the circuit, resistors are in ohms, capacitors are in farads, and inductors are in henries.

49.

An input r_1 yields $c_1 = 5r_1+7$. An input r_2 yields $c_2 = 5r_2 +7$. An input $r_1 +r_2$ yields, $5(r_1+r_2)+7 = 5r_1+7+5r_2 = c_1+c_2-7$. Therefore, not additive. What about homogeneity? An input of Kr_1 yields $c = 5Kr_1+7 \neq Kc_1$. Therefore, not homogeneous. The system is not linear.

50.

a. Let $x = \delta x+0$. Therefore,

$\ddot{\delta x}+3\dot{\delta x}+2\delta x = \sin(0+\delta x)$

But, $\sin(0+\delta x) = \sin 0 + \left.\frac{d \sin x}{dx}\right|_{x=0} \delta x = 0+\cos x \Big|_{x=0} \delta x = \delta x$

Therefore, $\ddot{\delta x}+3\dot{\delta x}+2\delta x = \delta x$. Collecting terms, $\ddot{\delta x}+3\dot{\delta x}+\delta x = 0$.

b. Let $x = \delta x+\pi$. Therefore,
$\ddot{\delta x}+3\dot{\delta x}+2\delta x = \sin(\pi+\delta x)$

But, $\sin(\pi+\delta x) = \sin \pi + \left.\frac{d \sin x}{dx}\right|_{x=\pi} \delta x = 0+\cos x \Big|_{x=\pi} \delta x = -\delta x$

Therefore, $\ddot{\delta x}+3\dot{\delta x}+2\delta x = -\delta x$. Collecting terms, $\ddot{\delta x}+3\dot{\delta x}+3\delta x = 0$.

51.

If $x = 0 + \delta x$,
$\dddot{\delta x} + 10\ddot{\delta x} + 31\dot{\delta x} + 30\delta x = e^{-(\delta x)}$

But $e^{-(\delta x)} = e^{-0} + \left.\frac{de^{-x}}{dx}\right|_{x=0} \delta x = 1 - e^{-x}\Big|_{x=0} \delta x = 1 - \delta x$

Therefore, $\dddot{\delta x} + 10\ddot{\delta x} + 31\dot{\delta x} + 30\delta x = 1 - \delta x$, or, $\dddot{\delta x} + 10\ddot{\delta x} + 31\dot{\delta x} + 31\delta x = 1$.

52.

The given curve can be described as follows:

$f(x) = -4$; $-\infty<x<-2$;

$f(x) = 2x$; $-2<x<2$;

$f(x) = 4$; $2<x<+\infty$

Thus,

a. $\ddot{x} + 15\dot{x} + 50x = -4$

b. $\ddot{x} + 15\dot{x} + 50x = 2x$, or $\ddot{x} + 15\dot{x} + 48x = 0$

c. $\ddot{x} + 15\dot{x} + 50x = 4$

53.

The relationship between the nonlinear spring's displacement, $x_s(t)$ and its force, $f_s(t)$ is

$x_s(t) = 1 - e^{-f_s(t)}$

Solving for the force,

$$f_s(t) = -\ln(1 - x_s(t)) \quad (1)$$

Writing the differential equation for the system by summing forces,

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} - \ln(1 - x(t)) = f(t) \quad (2)$$

Letting $x(t) = x_0 + \delta x$ and $f(t) = 1 + \delta f$, linearize $\ln(1 - x(t))$.

$$\ln(1-x) - \ln(1-x_0) = \left.\frac{d\ln(1-x)}{dx}\right|_{x=x_0} \delta x$$

Solving for $\ln(1 - x)$,

$$\ln(1-x) = \ln(1-x_0) - \left.\frac{1}{1-x}\right|_{x=x_0} \delta x = \ln(1-x_0) - \frac{1}{1-x_0}\delta x \qquad (3)$$

When f = 1, δx = 0. Thus from Eq. (1), $1 = -\ln(1 - x_0)$. Solving for x_0,

$1 - x_0 = e^{-1}$, or $x_0 = 0.6321$.

Substituting $x_0 = 0.6321$ into Eq. (3),

$$\ln(1-x) = \ln(1 - 0.6321) - \frac{1}{1-0.6321}\delta x = -1 - 2.718\delta x$$

Placing this value into Eq. (2) along with $x(t) = x_0 + \delta x$ and $f(t) = 1 + \delta f$, yields the linearized differential equation,

$$\frac{d^2\delta x}{dt^2} + \frac{d\delta x}{dt} + 1 + 2.718\delta x = 1 + \delta f$$

or

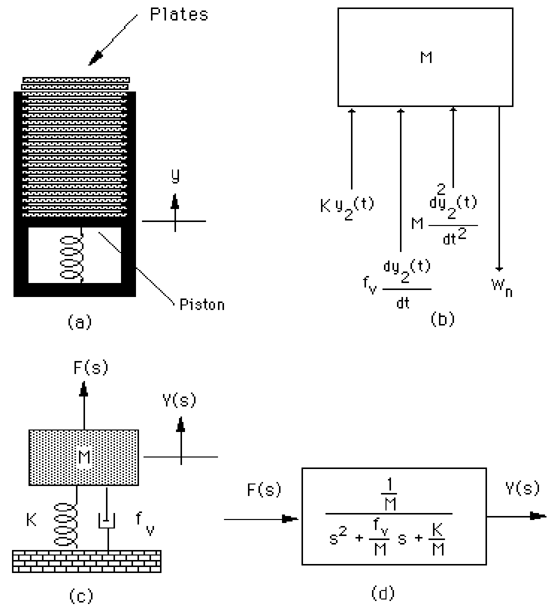
$$\frac{d^2\delta x}{dt^2} + \frac{d\delta x}{dt} + 2.718\delta x = \delta f$$

Taking the Laplace transform and rearranging yields the transfer function,

$$\frac{\delta x(s)}{\delta f(s)} = \frac{1}{s^2 + s + 2.718}$$

54.

First assume there are n plates without the top plate positioned at a displacement of $y_2(t)$ where $y_2(t) = 0$ is the position of the unstretched spring. Assume the system consists of mass M, where M is the mass of the dispensing system and the n plates, viscous friction, f_v, where the viscous friction originates where the piston meets the sides of the cylinder, and of course the spring with spring constant, K. Now, draw the freebody diagram shown in Figure (b) where W_n is the total weight of the n dishes and the piston. If we now consider the current position, $y_2(0)$,



Restaurant Plate Dispenser

the equilibrium point and define a new displacement, $y_1(t)$, which is measured from equilibrium, we can write the force in the spring as $Ky_2(t) = Ky_2(0) + Ky_1(t)$. Changing variables from $y_2(t)$ to $y_1(t)$, we sum forces and get,

$$M\frac{d^2y_1}{dt^2} + f_v\frac{dy_1}{dt} + Ky_1 + Ky_2(0) + W_n = 0 \qquad (1)$$

where $\frac{d^2y_2}{dt^2} = \frac{d^2y_1}{dt^2}$ and $\frac{dy_2}{dt} = \frac{dy_1}{dt}$. But, $Ky_2(0) = -W_n$, since it is the component of the spring force that balances the weight at equilibrium when $y_1 = 0$. Thus, the differential equation becomes,

$$M\frac{d^2y_1}{dt^2} + f_v\frac{dy_1}{dt} + Ky_1 = 0 \qquad (2)$$

When the top plate is added, the spring is further compressed by an amount, $\frac{W_D}{K}$, where W_D is the weight of the single dish, and K is the spring constant. We can think of this displacement as an initial condition. Thus, $y_1(0-) = -\frac{W_D}{K}$ and $\frac{dy_1}{dt}(0-) = 0$, and $y_1(t) = 0$ is the equilibrium position of the spring with n plates rather than the unstretched position. Taking the Laplace transform of equation (2), using the initial conditions,

$$M(s^2Y_1(s) + s\frac{W_D}{K}) + f_v(sY_1(s) + \frac{W_D}{K}) + KY_1(s) = 0 \quad (3)$$

or

$$(Ms^2 + f_vs + K)Y_1(s) = -\frac{W_D}{K}(Ms + f_v) \quad (4)$$

Now define a new position reference, Y(s), which is zero when the spring is compressed with the initial condition,

$$Y(s) = Y_1(s) + \frac{W_D}{Ks} \quad (5)$$

or

$$Y_1(s) = Y(s) - \frac{W_D}{Ks} \quad (6)$$

Substituting $Y_1(s)$ in Equation (4), we obtain,

$$(Ms^2 + f_vs + K)Y(s) = \frac{W_D}{s} = F(s) \quad (7)$$

a differential equation that has an input and zero initial conditions. The schematic is shown in Figure (c). Forming the transfer function, $\frac{Y(s)}{F(s)}$, we show the final result in Figure (d), where for the removal of the top plate, F(s) is always a step, $F(s) = \frac{W_D}{s}$.

55.

Writing the equations of motion,

$$(17.2s^2 + 160s + 7000)Y_f(s) \quad - (130s + 7000)Y_h(s) \quad - 0Y_{cat}(s) = F_{up}(s)$$

$$-(130s+7000)Y_f(s) \quad + (9.1s^s + 130s + 89300)Y_h(s) \quad - 82300Y_{cat}(s) = 0$$

$$-0Y_f(s) \quad - 82300Y_h(s) \quad + 1.6173 \times 10^6\, Y_{cat}(s) = 0$$

These equations are in the form **AY=F**, where det(**A**) **=** $2.5314 \times 10^8\ (s^2 + 15.47 s + 9283)(s^2 + 8.119 s + 376.3)$

Using Cramer's rule:

$$\frac{Y_{cat}(s)}{F_{up}(s)} = \frac{0.04227(s+53.85)}{(s^2+15.47s+9283)(s^2+8.119s+376.3)}$$

$$\frac{Y_h(s)}{F_{up}(s)} = \frac{0.8306(s+53.85)}{(s^2+15.47s+9283)(s^2+8.119s+376.3)}$$

$$\frac{Y_h(s)-Y_{cat}(s)}{F_{up}(s)} = \frac{0.7883(s+53.85)}{(s^2+15.47s+9283)(s^2+8.119s+376.3)}$$

T H R E E

Modeling in the Time Domain

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: State-Space Representation

For the power amplifier, $\frac{E_a(s)}{V_p(s)} = \frac{150}{s+150}$. Taking the inverse Laplace transform, $\dot{e}_a + 150e_a = 150v_p$. Thus, the state equation is

$$\dot{e}_a = -150e_a + 150v_p$$

For the motor and load, define the state variables as $x_1 = \theta_m$ and $x_2 = \dot{\theta}_m$. Therefore,

$$\dot{x}_1 = x_2 \qquad (1)$$

Using the transfer function of the motor, cross multiplying, taking the inverse Laplace transform, and using the definitions for the state variables,

$$\dot{x}_2 = -\frac{1}{J_m}\left(D_m + \frac{K_tK_a}{R_a}\right)x_2 + \frac{K_t}{R_aJ_m}e_a \qquad (2)$$

Using the gear ratio, the output equation is

$$y = 0.2x_1 \qquad (3)$$

Also, $J_m = J_a + 5\left(\frac{1}{5}\right)^2 = 0.05+0.2 = 0.25$, $D_m = D_a + 3\left(\frac{1}{5}\right)^2 = 0.01+0.12 = 0.13$, $\frac{K_t}{R_aJ_m} = \frac{1}{(5)(0.25)} = 0.8$, and $\frac{1}{J_m}\left(D_m + \frac{K_tK_a}{R_a}\right) = 1.32$. Using Eqs. (1), (2), and (3) along with the previous values, the state and output equations are,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -1.32 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}e_a \; ; \; y = \begin{bmatrix} 0.2 & 0 \end{bmatrix}\mathbf{x}$$

Aquifer: State-Space Representation

$$C_1\frac{dh_1}{dt} = q_{i1}-q_{o1}+q_2-q_1+q_{21} = q_{i1}-0+G_2(h_2-h_1)-G_1h_1+G_{21}(H_1-h_1) =$$

$$-(G_2+G_1+G_{21})h_1+G_2h_2+q_{i1}+G_{21}H_1$$

$$C_2\frac{dh_2}{dt} = q_{i2}-q_{02}+q_3-q_2+q_{32} = q_{i2}-q_{o2}+G_3(h_3-h_2)-G_2(h_2-h_1)+0 = G_2h_1-[G_2+G_3]h_2+G_3h_3+q_{i2}-q_{o2}$$

$$C_3\frac{dh_3}{dt} = q_{i3}-q_{o3}+q_4-q_3+q_{43} = q_{i3}-q_{o3}+0-G_3(h_3-h_2)+0 = G_3h_2-G_3h_3+q_{i3}-q_{o3}$$

Dividing each equation by C_i and defining the state vector as $\mathbf{x} = [h_1\ \ h_2\ \ h_3]^T$

$$\dot{\mathbf{x}} = \begin{bmatrix} \frac{-(G_1+G_2+G_3)}{C_1} & \frac{G_2}{C_1} & 0 \\ \frac{G_2}{C_2} & \frac{-(G_2+G_3)}{C_2} & \frac{G_3}{C_2} \\ 0 & \frac{G_3}{C_3} & \frac{-G_3}{C_3} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{q_{i1}+G_{21}H_1}{C_1} \\ \frac{q_{i2}-q_{o2}}{C_2} \\ \frac{q_{i3}-q_{o3}}{C_3} \end{bmatrix}u(t)$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}$$

where u(t) = unit step function.

ANSWERS TO REVIEW QUESTIONS

1. (1) Can model systems other than linear, constant coefficients; (2) Used for digital simulation

2. Yields qualitative insight

3. That smallest set of variables that completely describe the system

4. The value of the state variables

5. The vector whose components are the state variables

6. The n-dimensional space whose bases are the state variables

7. State equations, an output equation, and an initial state vector (initial conditions)

8. Eight

9. Forms linear combinations of the state variables and the input to form the desired output

10. No variable in the set can be written as a linear sum of the other variables in the set.

11. (1) They must be linearly independent; (2) The number of state variables must agree with the order of the differential equation describing the system; (3) The degree of difficulty in obtaining the state equations for a given set of state variables.

12. The variables that are being differentiated in each of the linearly independent energy storage elements

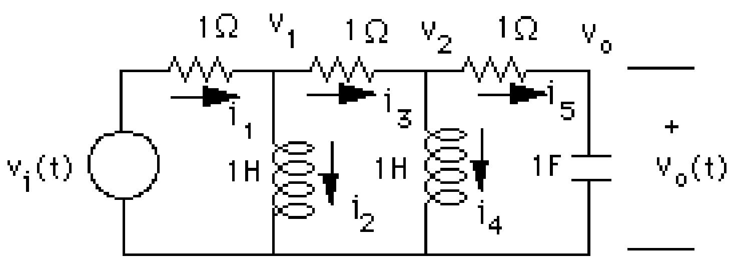
13. Yes, depending upon the choice of circuit variables and technique used to write the system equations. For example, a three -loop problem with three energy storage elements could yield three simultaneous second-order differential equations which would then be described by six, first-order differential equations. This exact situation arose when we wrote the differential equations for mechanical systems and then proceeded to find the state equations.

14. The state variables are successive derivatives.

SOLUTIONS TO PROBLEMS

1.

Add the branch currents and node voltages to the network.



Write the differential equation for each energy storage element.

$$\frac{di_2}{dt} = v_1$$
$$\frac{di_4}{dt} = v_2$$
$$\frac{dv_o}{dt} = i_5$$

Therefore, the state vector is $\mathbf{x} = \begin{bmatrix} i_2 \\ i_4 \\ v_o \end{bmatrix}$

Now obtain v_1, v_2, and i_5 in terms of the state variables. First find i_1 in terms of the state variables.

$$-v_i + i_1 + i_3 + i_5 + v_o = 0$$

But $i_3 = i_1 - i_2$ and $i_5 = i_3 - i_4$. Thus,

$$-v_i + i_1 + (i_1 - i_2) + (i_3 - i_4) + v_o = 0$$

Making the substitution for i_3 yields

$$-v_i + i_1 + (i_1 - i_2) + ((i_1 - i_2) - i_4) + v_o = 0$$

Solving for i_1

$$i_1 = \frac{2}{3}i_2 + \frac{1}{3}i_4 - \frac{1}{3}v_o + \frac{1}{3}v_i$$

Thus,

$$v_1 = v_i - i_1 = -\frac{2}{3}i_2 - \frac{1}{3}i_4 + \frac{1}{3}v_o + \frac{2}{3}v_i$$

Also,

$$i_3 = i_1 - i_2 = -\frac{1}{3}i_2 + \frac{1}{3}i_4 - \frac{1}{3}v_o + \frac{1}{3}v_i$$

and

$$i_5 = i_3 - i_4 = -\frac{1}{3}i_2 - \frac{2}{3}i_4 - \frac{1}{3}v_o + \frac{1}{3}v_i$$

Finally,

$$v_2 = i_5 + v_o = -\frac{1}{3}i_2 - \frac{2}{3}i_4 + \frac{2}{3}v_o + \frac{1}{3}v_i$$

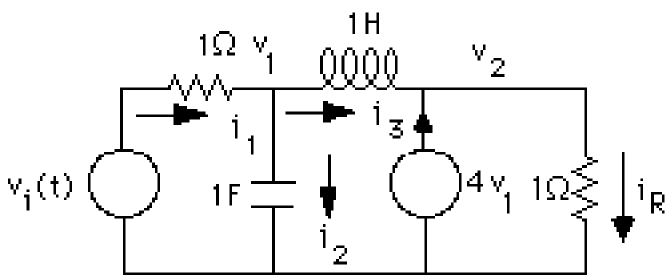
Using v_1, v_2, and i_5, the state equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}$$

2.

Add branch currents and node voltages to the schematic and obtain,



Write the differential equation for each energy storage element.

$$\frac{dv_1}{dt} = i_2$$
$$\frac{di_3}{dt} = v_L$$

Therefore the state vector is $\mathbf{x} = \begin{bmatrix} v_1 \\ i_3 \end{bmatrix}$

Now obtain v_L and i_2, in terms of the state variables,

$$v_L = v_1 - v_2 = v_1 - i_R = v_1 - (i_3 + 4v_1) = -3v_1 - i_3$$

$$i_2 = i_1 - i_3 = (v_i - v_1) - i_3 = -v_1 - i_3 + v_i$$

Also, the output is

$$y = i_R = 4v_1 + i_3$$

Hence,

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -1 \\ -3 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 4 & 1 \end{bmatrix}\mathbf{x}$$

3.

Let C_1 be the grounded capacitor and C_2 be the other. Now, writing the equations for the energy storage components yields,

$$\frac{di_L}{dt} = v_i - v_{C_1}$$
$$\frac{dv_{C_1}}{dt} = i_1 - i_2 \qquad (1)$$
$$\frac{dv_{C_2}}{dt} = i_2 - i_3$$

Thus the state vector is $\mathbf{x} = \begin{bmatrix} i_L \\ v_{C_1} \\ v_{C_2} \end{bmatrix}$. Now, find the three loop currents in terms of the state variables and the input.

Writing KVL around Loop 2 yields $v_{C_1} = v_{C_2} + i_2$ .Or,

$$i_2 = v_{C_1} - v_{C_2}$$

Writing KVL around the outer loop yields $i_3 + i_2 = v_i$ Or,

$$i_3 = v_i - i_2 = v_i - v_{C_1} + v_{C_2}$$

Also, $i_1 - i_3 = i_L$. Hence,

$$i_1 = i_L + i_3 = i_L + v_i - v_{C_1} + v_{C_2}$$

Substituting the loop currents in equations (1) yields the results in vector-matrix form,

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_{C_1}}{dt} \\ \frac{dv_{C_2}}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} i_L \\ v_{C_1} \\ v_{C_2} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} v_i$$

Since $v_o = i_2 = v_{C_1} - v_{C_2}$, the output equation is

$$y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} i_L \\ v_{C_1} \\ v_{C_2} \end{bmatrix}$$

4.

Equations of motion in Laplace:

$$(s^2 + 3s + 1)x_1(s) - (s+1)x_2(s) - sx_3(s) = 0$$
$$-(s+1)x_1(s) + (s^2 + 2s + 1)x_2(s) - sx_3(s) = F(s)$$
$$-sx_1(s) - sx_2(s) + (s^2 + 3s)x_3(s) = 0$$

Equations of motion in the time domain:

$$\frac{d^2x_1}{dt^2} + 3\frac{dx_1}{dt} + x_1 - \frac{dx_2}{dt} - x_2 - \frac{dx_3}{dt} = 0$$
$$\frac{d^2x_2}{dt^2} + 2\frac{dx_2}{dt} + x_2 - \frac{dx_1}{dt} - x_1 - \frac{dx_3}{dt} = f(t)$$
$$\frac{d^2x_3}{dt^2} + 3\frac{dx_3}{dt} + x_1 - \frac{dx_2}{dt} - \frac{dx_1}{dt} = 0$$

Define state variables:

$$z_1 = x_1 \quad \text{or} \quad x_1 = z_1 \tag{1}$$
$$z_2 = \frac{dx_1}{dt} \quad \text{or} \quad \frac{dx_1}{dt} = z_2 \tag{2}$$
$$z_3 = x_2 \quad \text{or} \quad x_2 = z_3 \tag{3}$$
$$z_4 = \frac{dx_2}{dt} \quad \text{or} \quad \frac{dx_2}{dt} = z_4 \tag{4}$$
$$z_5 = x_3 \quad \text{or} \quad x_3 = z_5 \tag{5}$$
$$z_6 = \frac{dx_3}{dt} \quad \text{or} \quad \frac{dx_3}{dt} = z_5 \tag{6}$$

Substituting Eq. (1) in (2), (3) in (4), and (5) in (6), we obtain, respectively:

$$\frac{dz_1}{dt} = z_2 \tag{7}$$

$$\frac{dz_3}{dt} = z_4 \tag{8}$$

$$\frac{dz_5}{dt} = z_6 \tag{9}$$

Substituting Eqs. (1) through (6) into the equations of motion in the time domain and solving for the derivatives of the state variables and using Eqs. (7) through (9) yields the state equations:

$$\frac{dz_1}{dt} = z_2$$

$$\frac{dz_2}{dt} = -z_1 - 3z_2 + z_3 + z_4 + z_6$$

$$\frac{dz_3}{dt} = z_4$$

$$\frac{dz_4}{dt} = z_1 + z_2 - z_3 - 2z_4 + z_6 + f(t)$$

$$\frac{dz_5}{dt} = z_6$$

$$\frac{dz_6}{dt} = z_2 + z_4 - 3z_6$$

The output is $x_3 = z_5$.

In vector-matrix form:

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -3 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} f(t)$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{z}$$

5.

Writing the equations of motion,

$$(s^2+2s+1)X_1(s)-sX_2(s)-(s+1)X_3(s)=0$$
$$-sX_1(s)+(s^2+2s+1)X_2(s)-(s+1)X_3(s)=0$$
$$-(s+1)X_1(s)-(s+1)X_2(s)+(s^2+2s+2)X_3(s)=F(s)$$

Taking the inverse Laplace transform,

$$\ddot{x}_1+2\dot{x}_1+x_1-\dot{x}_2-\dot{x}_3-x_3=0$$
$$-\dot{x}_1+\ddot{x}_2+2\dot{x}_2+x_2-\dot{x}_3-x_3=0$$
$$-\dot{x}_1-x_1-\dot{x}_2-x_2+\ddot{x}_3+2\dot{x}_3+2x_3=f(t)$$

Simplifying,

$$\ddot{x}_1=-2\dot{x}_1-x_1+\dot{x}_2+\dot{x}_3+x_3$$
$$\ddot{x}_2=\dot{x}_1-2\dot{x}_2-x_2+\dot{x}_3+x_3$$
$$\ddot{x}_3=\dot{x}_1+x_1+\dot{x}_2+x_2-2\dot{x}_3-2x_3+f(t)$$

Defining the state variables,

$$z_1=x_1;\ z_2=\dot{x}_1;\ z_3=x_2;\ z_4=\dot{x}_2;\ z_5=x_3;\ z_6=\dot{x}_3$$

Writing the state equations using the simplified equations above yields,

$$\dot{z}_1=\dot{x}_1=z_2$$
$$\dot{z}_2=\ddot{x}_1=-2z_2-z_1+z_4+z_6+z_5$$
$$\dot{z}_3=\dot{x}_2=z_4$$
$$\dot{z}_4=\ddot{x}_2=z_2-2z_4-z_3+z_6+z_5$$
$$\dot{z}_5=\dot{x}_3=z_6$$
$$\dot{z}_6=\ddot{x}_3=z_2+z_1+z_4+z_3-2z_6-2z_5+f(t)$$

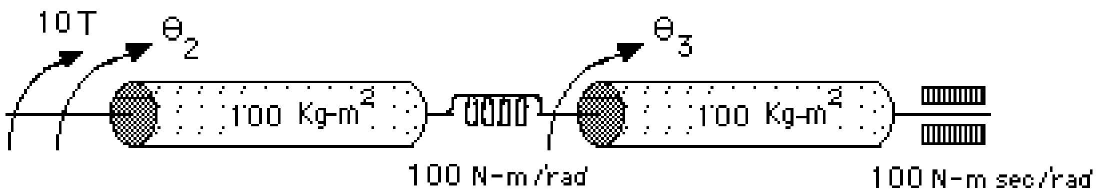
Converting to vector-matrix form,

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & -2 & -2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} f(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{z}$$

6.

Drawing the equivalent network,



Writing the equations of motion,

$$(100s^2 + 100)\theta_2 - 100\theta_3 = 10T$$
$$-100\theta_2 + (100s^2 + 100s + 100)\theta_3 = 0$$

Taking the inverse Laplace transform and simplifying,

$$\ddot{\theta}_2 + \theta_2 - \theta_3 = \frac{1}{10}T$$
$$-\theta_2 + \ddot{\theta}_3 + \dot{\theta}_3 + \theta_3 = 0$$

Defining the state variables as

$$x_1 = \theta_2,\ x_2 = \dot{\theta}_2,\ x_3 = \theta_3,\ x_4 = \dot{\theta}_3$$

Writing the state equations using the equations of motion and the definitions of the state variables

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \ddot{\theta}_2 = -\theta_2 + \theta_3 + \frac{1}{10}T = -x_1 + x_3 + \frac{1}{10}T \\
\dot{x}_3 &= x_4 \qquad , \\
\dot{x}_4 &= \ddot{\theta}_3 = \theta_2 - \theta_3 - \dot{\theta}_3 = x_1 - x_3 - x_4 \\
y &= 10\theta_2 = 10x_1
\end{aligned}$$

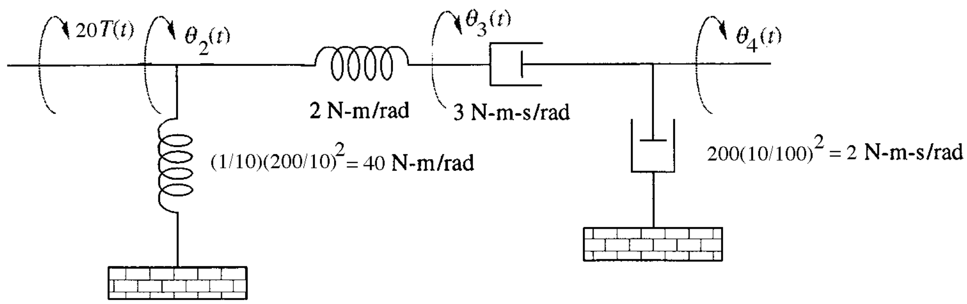
In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{10} \\ 0 \\ 0 \end{bmatrix}T$$

$$y = \begin{bmatrix} 10 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

7.

Drawing the equivalent circuit,



Writing the equations of motion,

$$\begin{aligned}
&42\theta_2(s) - 2\theta_3(s) = 20T(s) \\
&-2\theta_2(s) + (3s+2)\theta_3(s) - 3s\theta_4(s) = 0 \\
&-3s\theta_3(s) + 5s\theta_4(s) = 0
\end{aligned}$$

Taking the inverse Laplace transform,

$$42\,\theta_2(t) - 2\theta_3(t) = 20T(t) \qquad (1)$$

$$-2\theta_2(t) + 3\,\dot{\theta}_3(t) + 2\theta_3 - 3\,\dot{\theta}_4(t) = 0 \qquad (2)$$

$$-3\,\dot{\theta}_3(t) + 5\,\dot{\theta}_4(t) = 0 \qquad (3)$$

From (3),

$$\dot{\theta}_3(t) = \frac{5}{3}\dot{\theta}_4(t) \text{ and } \theta_3(t) = \frac{5}{3}\theta_4(t) \quad (4)$$

assuming zero initial conditions.

From (1)

$$\theta_2(t) = \frac{1}{21}\theta_3(t) + \frac{10}{21}T(t) = \frac{5}{63}\theta_4(t) + \frac{10}{21}T(t) \quad (5)$$

Substituting (4) and (5) into (2) yields the state equation (notice there is only one equation),

$$\dot{\theta}_4(t) = -\frac{100}{63}\theta_4(t) + \frac{10}{21}T(t)$$

The output equation is given by,

$$\theta_L(t) = \frac{1}{10}\theta_4(t)$$

8.

Solving Eqs. (3.44) and (3.45) in the text for the transfer functions $\frac{X_1(s)}{F(s)}$ and $\frac{X_2(s)}{F(s)}$:

$$X_1(s) = \frac{\left|\begin{pmatrix} 0 & -K \\ F & M_2 s^2 + K \end{pmatrix}\right|}{\left|\begin{pmatrix} M_1 s^2 + D s + K & -K \\ -K & M_2 s^2 + K \end{pmatrix}\right|} \text{ and } X_2(s) = \frac{\left|\begin{pmatrix} M_1 s^2 + D s + K & 0 \\ -K & F \end{pmatrix}\right|}{\left|\begin{pmatrix} M_1 s^2 + D s + K & -K \\ -K & M_2 s^2 + K \end{pmatrix}\right|}$$

Thus,

$$\frac{X_1(s)}{F(s)} = \frac{K}{M_2 M_1 s^4 + D M_2 s^3 + K M_2 s^2 + K M_1 s^2 + D K s}$$

and

$$\frac{X_2(s)}{F(s)} = \frac{M_1 s^2 + D s + K}{M_2 M_1 s^4 + D M_2 s^3 + K M_2 s^2 + K M_1 s^2 + D K s}$$

Multiplying each of the above transfer functions by *s* to find velocity yields pole/zero cancellation at the origin and a resulting transfer function that is third order.

9.

a. . Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & -7 & -10 & -20 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 100 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

b. Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -30 & -1 & -6 & -9 & -8 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 30 & 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

10.

Program:

```
'a'
num=100;
den=[1 20 10 7 100];
G=tf(num,den)
[Acc,Bcc,Ccc,Dcc]=tf2ss(num,den);
Af=flipud(Acc);
A=fliplr(Af)
B=flipud(Bcc)
C=fliplr(Ccc)
'b'
num=30;
den=[1 8 9 6 1 30];
G=tf(num,den)
[Acc,Bcc,Ccc,Dcc]=tf2ss(num,den);
Af=flipud(Acc);
A=fliplr(Af)
B=flipud(Bcc)
C=fliplr(Ccc)
```

Computer response:

```
ans =

a

Transfer function:
               100
---------------------------------
s^4 + 20 s^3 + 10 s^2 + 7 s + 100




A =

     0     1     0     0
     0     0     1     0
     0     0     0     1
  -100    -7   -10   -20

B =

     0
     0
     0
     1

C =

   100     0     0     0

ans =

b
```

```
Transfer function:
                 30
------------------------------------
s^5 + 8 s^4 + 9 s^3 + 6 s^2 + s + 30

A =

     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
     0     0     0     0     1
   -30    -1    -6    -9    -8

B =

     0
     0
     0
     0
     1

C =

    30     0     0     0     0
```

11.

a. Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -5 & -1 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 10 & 5 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

b. Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -8 & -10 & -9 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 3 & 7 & 12 & 2 & 1 \end{bmatrix}\mathbf{x}$$

12.

Program:

```
'a'
num=[5 10];
den=[1 2 1 5 10]
G=tf(num,den)
[Acc,Bcc,Ccc,Dcc]=tf2ss(num,den);
Af=flipud(Acc);
A=fliplr(Af)
B=flipud(Bcc)
```

```
C=fliplr(Ccc)
'b'
num=[1 2 12 7 3];
den=[1 9 10 8 0 0]
G=tf(num,den)
[Acc,Bcc,Ccc,Dcc]=tf2ss(num,den);
Af=flipud(Acc);
A=fliplr(Af)
B=flipud(Bcc)
C=fliplr(Ccc)
```

Computer response:

```
ans =

a

den =

     1     2     1     5    10


Transfer function:
          5 s + 10
----------------------------
s^4 + 2 s^3 + s^2 + 5 s + 10


A =

     0     1     0     0
     0     0     1     0
     0     0     0     1
   -10    -5    -1    -2

B =

     0
     0
     0
     1

C =

    10     5     0     0

ans =

b

den =

     1     9    10     8     0     0


Transfer function:
s^4 + 2 s^3 + 12 s^2 + 7 s + 3
------------------------------
 s^5 + 9 s^4 + 10 s^3 + 8 s^2


A =

     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
     0     0     0     0     1
     0     0    -8   -10    -9
```

```
B =

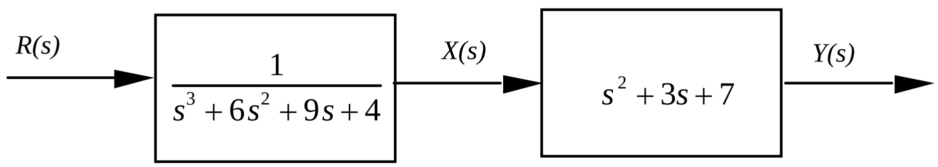
     0
     0
     0
     0
     1

C =

     3     7    12     2     1
```

13.

The transfer function can be represented as a block diagram as follows:



Writing the differential equation for the first box:

$$\dddot{x} + 6\ddot{x} + 9\dot{x} + 4x = r(t)$$

Defining the state variables:

$$x_1 = x$$
$$x_2 = \dot{x}$$
$$x_3 = \ddot{x}$$

Thus,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -4x - 9\dot{x} - 6\ddot{x} + r(t) = -4x_1 - 9x_2 - 6x_3 + r(t)$$

From the second box,

$$y = \ddot{x} + 3\dot{x} + 7x = 7x_1 + 3x_2 + x_3$$

In vector-matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$y = \begin{bmatrix} 7 & 3 & 1 \end{bmatrix}\mathbf{x}$$

14.

a. $G(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -3 & -2 & -5\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}0\\ 0\\ 10\end{bmatrix};\ \mathbf{C}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^3+5s^2+2s+3}\begin{bmatrix}s^2+5s+2 & s+5 & 1\\ -3 & s(s+5) & s\\ -3s & -2s-3 & s^2\end{bmatrix}$$

Therefore, $G(s)=\dfrac{10}{s^3+5s^2+2s+3}$.

b. $G(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{A}=\begin{pmatrix}2 & 3 & -8\\ 0 & 5 & 3\\ -3 & -5 & -4\end{pmatrix};\ \mathbf{B}=\begin{pmatrix}1\\ 4\\ 6\end{pmatrix};\ \mathbf{C}=(1,3,6)$$

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^3-3s^2-27s+157}\begin{pmatrix}s^2-s-5 & 3s+52 & -8s+49\\ -9 & s^2+2s-32 & 3s-6\\ -3s+15 & -5s+1 & s^2-7s+10\end{pmatrix}$$

Therefore, $G(s)=\dfrac{49s^2-349s+452}{s^3-3s^2-27s+157}$.

c. $G(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{A}=\begin{bmatrix}3 & -5 & 2\\ 1 & -8 & 7\\ -3 & -6 & 2\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}5\\ -3\\ 2\end{bmatrix};\ \mathbf{C}=\begin{bmatrix}1 & -4 & 3\end{bmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^3+3s^2+19s-133}\begin{bmatrix}(s^2+6s+26) & -(5s+2) & (2s-19)\\ (s-23) & (s^2-5s+12) & (7s-19)\\ -(3s+30) & -(6s-33) & (s^2+5s-19)\end{bmatrix}$$

Therefore, $G(s)=\dfrac{23s^2-48s-7}{s^3+3s^2+19s-133}$.

15.

Program:

```
'a'
A=[0 1 3 0;0 0 1 0;0 0 0 1;-7 -9 -2 -3];
B=[0;5;8;2];
C=[1 3 4 6];
D=0;
statespace=ss(A,B,C,D)
```

```
[num,den]=ss2tf(A,B,C,D);
G=tf(num,den)
'b'
A=[3 1 0 4 -2;-3 5 -5 2 -1;0 1 -1 2 8;-7 6 -3 -4 0;-6 0 4 -3 1];
B=[2;7;6;5;4];
C=[1 -2 -9 7 6];
D=0;
statespace=ss(A,B,C,D)
[num,den]=ss2tf(A,B,C,D);
G=tf(num,den)
```

Computer response:

```
ans =

a


a =
       x1  x2  x3  x4
   x1   0   1   3   0
   x2   0   0   1   0
   x3   0   0   0   1
   x4  -7  -9  -2  -3


b =
       u1
   x1   0
   x2   5
   x3   8
   x4   2


c =
       x1  x2  x3  x4
   y1   1   3   4   6


d =
       u1
   y1   0

Continuous-time model.

Transfer function:
59 s^3 - 164 s^2 - 1621 s - 260
-------------------------------
s^4 + 3 s^3 + 2 s^2 + 30 s + 7


ans =

b


a =
       x1  x2  x3  x4  x5
   x1   3   1   0   4  -2
   x2  -3   5  -5   2  -1
   x3   0   1  -1   2   8
   x4  -7   6  -3  -4   0
   x5  -6   0   4  -3   1


b =
       u1
   x1   2
```

```
   x2   7
   x3   6
   x4   5
   x5   4


c =
       x1  x2  x3  x4  x5
   y1   1  -2  -9   7   6


d =
       u1
   y1   0

Continuous-time model.

Transfer function:
-7 s^4 - 408 s^3 + 1708 s^2 + 1.458e004 s + 2.766e004
-----------------------------------------------------
   s^5 - 4 s^4 - 32 s^3 + 148 s^2 - 1153 s - 4480
```

16.

Program:

```
syms s
'a'
A=[0 1 3 0
0 0 1 0
0 0 0 1
-7 -9 -2 -3];
B=[0;5;8;2];
C=[1 3 4 6];
D=0;
I=[1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1];
'T(s)'
T=C*((s*I-A)^-1)*B+D;
T=simple(T);
pretty(T)
'b'
A=[3 1 0 4 -2
-3 5 -5 2 -1
0 1 -1 2 8
-7 6 -3 -4 0
-6 0 4 -3 1];
B=[2;7;6;5;4];
C=[1 -2 -9 7 6];
D=0;
I=[1 0 0 0 0
0 1 0 0 0
0 0 1 0 0
0 0 0 1 0
0 0 0 0 1];
'T(s)'
T=C*((s*I-A)^-1)*B+D;
T=simple(T);
pretty(T)
```

Computer response:

```
ans =

a
```

```
ans =

T(s)

                          2                       3
                -164 s  - 1621 s - 260 + 59 s
                ------------------------------
                  4       3       2
                 s  + 3 s  + 2 s  + 30 s + 7

ans =

b

ans =
T(s)

                               2         3       4
          14582 s + 1708 s  - 408 s  - 7 s  + 27665
          -----------------------------------------
            5       4        3         2
           s  - 4 s  - 32 s  + 148 s  - 1153 s - 4480
```

17.

Let the input be $\frac{d\theta_z}{dt} = \omega_z$, $x_1 = \theta_x$, $x_2 = \dot{\theta}_x$. Therefore,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{K_x}{J_x} x_1 - \frac{D_x}{J_x} x_2 + J\omega\omega_z$$

The output is θ_x.

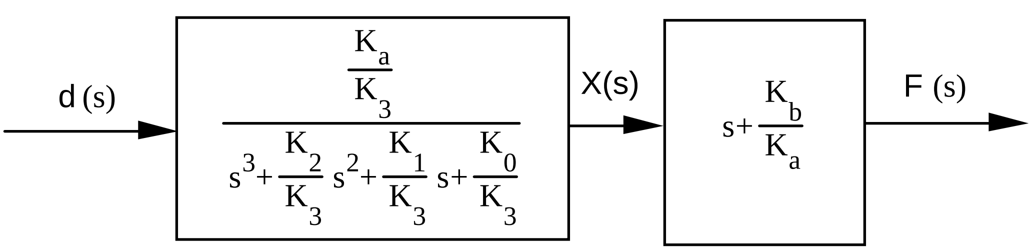
In vector-matrix form, $\theta_x = x_1$. Therefore, $y = x_1$.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{K_x}{J_x} & -\frac{D_x}{J_x} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ J\omega \end{bmatrix} \omega_z$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

18.

The equivalent cascade transfer function is as shown below.



For the first box, $\dddot{x}+\frac{K_2}{K_3}\ddot{x}+\frac{K_1}{K_3}\dot{x}+\frac{K_0}{K_3}x=\frac{K_a}{K_3}\delta(t)$.

Selecting the phase variables as the state variables: $x_1=x$, $x_2=\dot{x}$, $x_3=\ddot{x}$.

Writing the state and output equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -\frac{K_0}{K_3}x_1-\frac{K_1}{K_3}x_2-\frac{K_2}{K_3}x_3+\frac{K_a}{K_3}\delta(t)$$

$$y = \phi(t) = \dot{x}+\frac{K_b}{K_a}x = \frac{K_b}{K_a}x_1+x_2$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{K_0}{K_3} & -\frac{K_1}{K_3} & -\frac{K_2}{K_3} \end{bmatrix}\mathbf{x}+\begin{bmatrix} 0 \\ 0 \\ \frac{K_a}{K_3} \end{bmatrix}\delta(t)\ ;\ y=\begin{bmatrix} \frac{K_b}{K_a} & 1 & 0 \end{bmatrix}\mathbf{x}$$

19.

Since $T_m = J_{eq}\frac{d\omega_m}{dt}+D_{eq}\omega_m$, and $T_m = K_t i_a$,

$$J_{eq}\frac{d\omega_m}{dt}+D_{eq}\omega_m = K_t i_a \qquad (1)$$

Or,

$$\frac{d\omega_m}{dt} = -\frac{D_{eq}}{J_{eq}}\omega_m+\frac{K_t}{J_{eq}}i_a$$

But, $\omega_m = \frac{N_2}{N_1}\omega_L$.

Substituting in (1) and simplifying yields the first state equation,

$$\frac{d\omega_L}{dt} = -\frac{D_{eq}}{J_{eq}}\omega_L+\frac{K_t}{J_{eq}}\frac{N_1}{N_2}i_a$$

The second state equation is:

$$\frac{d\theta_L}{dt} = \omega_L$$

Since

$$e_a = R_a i_a+L_a\frac{di_a}{dt}+K_b\omega_m = R_a i_a+L_a\frac{di_a}{dt}+K_b\frac{N_2}{N_1}\omega_L,$$

the third state equation is found by solving for $\frac{di_a}{dt}$. Hence,

$$\frac{di_a}{dt} = -\frac{K_b}{L_a}\frac{N_2}{N_1}\omega_L-\frac{R_a}{L_a}i_a+\frac{1}{L_a}e_a$$

Thus the state variables are: $x_1 = \omega_L$, $x_2 = \theta_L$, and $x_3 = i_a$.

Finally, the output is $y = \theta_m = \frac{N_2}{N_1}\theta_L$.

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{D_{eq}}{J_{eq}} & 0 & \frac{K_t}{J_{eq}}\frac{N_1}{N_2} \\ 1 & 0 & 0 \\ -\frac{K_b}{L_a}\frac{N_2}{N_1} & 0 & -\frac{R_a}{L_a} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_a} \end{bmatrix} e_a \; ; \; y = \begin{bmatrix} 0 & \frac{N_2}{N_1} & 0 \end{bmatrix}\mathbf{x}$$

where,

$$\mathbf{x} = \begin{bmatrix} \omega_L \\ \theta_L \\ i_a \end{bmatrix}$$

20.

Writing the differential equations,

$$\frac{d^2x_1}{dt^2} + \frac{dx_1}{dt} + 2x_1^2 - \frac{dx_2}{dt} = 0$$

$$\frac{d^2x_2}{dt^2} + \frac{dx_2}{dt} - \frac{dx_1}{dt} = f(t)$$

Defining the state variables to be x_1, v_1, x_2, v_2, where v's are velocity,

$$\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_1 = -v_1 - 2x_1^2 + v_2$$

$$\dot{v}_2 = v_1 - v_2 + f(t)$$

Around $x_1 = 1$, $x_1 = 1+\delta x_1$, and $\dot{x}_1 = \delta\dot{x}_1$. Also,

$$x_1^2 = x_1^2\Big|_{x=1} + \frac{dx_1}{dt}\Big|_{x=1}\delta x_1 = 1+2x_1\Big|_{x=1}\delta x_1 = 1+2\delta x_1$$

Therefore, the state and output equations are,

$$\delta\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_1 = -v_1 - 2(1+2\delta x_1) + v_2$$

$$\dot{v}_2 = v_1 - v_2 + f(t)$$

$$y = x_2$$

In vector-matrix form,

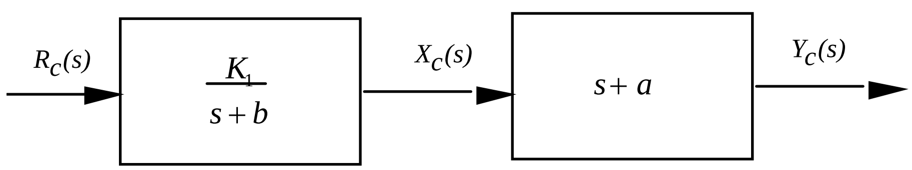
$$\begin{bmatrix} \delta\dot{x}_1 \\ \dot{x}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ f(t) \end{bmatrix}; \; y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}$$

where $f(t) = 2 + \delta f(t)$, since force in nonlinear spring is 2 N and must be balanced by 2 N force on damper.

21.

Controller:

The transfer function can be represented as a block diagram as follows:



Writing the differential equation for the first box,

$$\frac{K_1}{s+b}$$

and solving for $\dot{x}_c$,

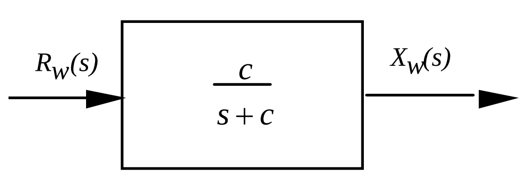
$$\dot{x}_c = -bx_c + K_1 r_c(t)$$

From the second box,

$$y_c = \dot{x}_c + ax_c = -bx_c + K_1 r_c(t) + ax_c$$
$$= (a-b)x_c + K_1 r_c(t)$$

Wheels:

The transfer function can be represented as a block diagram as follows:



Writing the differential equation for the block of the form,

$$\frac{c}{s+c}$$

and solving for $\dot{x}_w$,

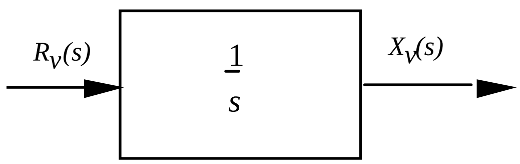
$$\dot{x}_w = -cx_w + cr_w(t)$$

The output equation is,

$$y_w = x_w$$

Vehicle:

The transfer function can be represented as a block diagram as follows:



Writing the differential equation for the block,

$$\frac{1}{s}$$

and solving for $\dot{x}_v$,

$$\dot{x}_v = r_v(t)$$

The output equation is

$$y_v = x_v$$

22.

$$\mathbf{A} = \begin{pmatrix} -1.702 & 50.72 & 263.38 \\ 0.22 & -1.418 & -31.99 \\ 0 & 0 & -14 \end{pmatrix} ; \mathbf{B} = \begin{pmatrix} -272.06 \\ 0 \\ 14 \end{pmatrix}$$

For $G_1(s)$, $\mathbf{C_1} = (1, 0, 0)$, and

$$G_1(s) = \mathbf{C_1}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$

Thus,

$$G_1(s) = \mathbf{C}_1 \left(\frac{1}{s^3 + 17.12s^2 + 34.935s - 122.43} \begin{bmatrix} s^2 + 15.418s + 19.852 & 50.72s + 710.08 & 263.38s - 1249.1 \\ 0.22s + 3.08 & s^2 + 15.702s + 23.828 & -31.99s + 3.4966 \\ 0 & 0 & s^2 + 3.12s - 8.745 \end{bmatrix} \right) \mathbf{B}$$

Or

$$G_1(s) = \frac{-272.06s^2 - 507.3s - 22888}{s^3 + 17.12s^2 + 34.935s - 122.43} = \frac{-272.06(s^2 + 1.8647s + 84.128)}{(s+14)(s-1.7834)(s+4.9034)}$$

For $G_2(s)$, $\mathbf{C_2} = (0,1,0)$, and

$$G_2(s) = \mathbf{C_2}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$

Thus,

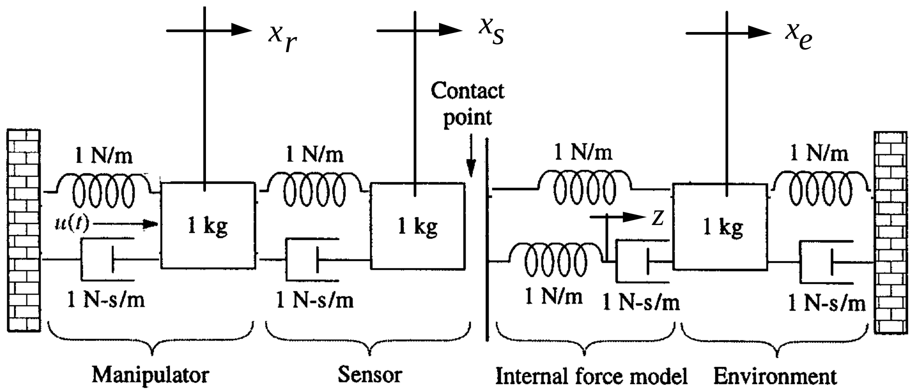
$$G_2(s) = \mathbf{C}_2\left(\frac{1}{s^3+17.12s^2+34.935s-122.43}\begin{bmatrix} s^2+15.418s+19.852 & 50.72s+710.08 & 263.38s-1249.1 \\ 0.22s+3.08 & s^2+15.702s+23.828 & -31.99s+3.4966 \\ 0 & 0 & s^2+3.12s-8.745 \end{bmatrix}\right)\mathbf{B}$$

Or

$$G_2(s) = \frac{-507.71s-788.99}{s^3+17.12s^2+34.935s-122.43} = \frac{-507.71(s+1.554)}{(s+14)(s-1.7834)(s+4.9034)}$$

23.

Adding displacements to the figure,



Writing the differential equations for noncontact,

$$\frac{d^2x_r}{dt^2} + 2\frac{dx_r}{dt} + 2x_r - x_s - \frac{dx_s}{dt} = u(t)$$

$$-\frac{dx_r}{dt} - x_r + \frac{d^2x_s}{dt^2} + \frac{dx_s}{dt} + x_s = 0$$

Define the state variables as,

$$x_1 = x_r;\ x_2 = \dot{x}_r;\ x_3 = x_s;\ x_4 = \dot{x}_s$$

Writing the state equations, using the differential equations and the definition of the state variables,

we get,

$$\dot{x}_1 = \dot{x}_r = x_2$$

$$\dot{x}_2 = \ddot{x}_r = -2x_1 - 2x_2 + x_3 + x_4 + u(t)$$

$$\dot{x}_3 = \dot{x}_s = x_4$$

$$\dot{x}_4 = \ddot{x}_s = x_1 + x_2 - x_3 - x_4$$

Assuming the output to be x_s, the output equation is,

$$y = x_3$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}$$

Writing the differential equations for contact,

$$\frac{d^2 x_r}{dt^2} + 2\frac{dx_r}{dt} + 2x_r - x_s - \frac{dx_s}{dt} = u(t)$$

$$-\frac{dx_r}{dt} - x_r + \frac{d^2 x_s}{dt^2} + \frac{dx_s}{dt} + x_s - z - x_e = 0$$

$$-x_s + \frac{dz}{dt} + z - \frac{dx_e}{dt} = 0$$

$$-x_s - \frac{dz}{dt} + \frac{d^2 x_e}{dt^2} + 2\frac{dx_e}{dt} + 2x_e = 0$$

Defining the state variables,

$$x_1 = x_r;\ x_2 = \dot{x}_r;\ x_3 = x_s;\ x_4 = \dot{x}_s;\ x_5 = z;\ x_6 = \dot{z};\ x_7 = x_e;\ x_8 = \dot{x}_e$$

Using the differential equations and the definitions of the state variables, we write the state equations.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - 2x_2 + x_3 + x_4 + u(t)$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = x_1 + x_2 - x_3 - x_4 + x_5 + x_7$$

$$\dot{x}_5 = x_6$$

Differentiating the third differential equation and solving for d^2z/dt^2 we obtain,

$$\dot{x}_6 = \frac{d^2 z}{dt^2} = \frac{dx_s}{dt} - \frac{dz}{dt} + \frac{d^2 x_e}{dt^2}$$

But, from the fourth differential equation,

$$\frac{d^2 x_e}{dt^2} = x_s + \frac{dz}{dt} - 2\frac{dx_e}{dt} - 2x_e = x_3 + x_6 - 2x_8 - 2x_7$$

Substituting this expression back into $\dot{x}_6$ along with the other definitions and then simplifying yields,

$$\dot{x}_6 = x_4 + x_3 - 2x_8 - 2x_7$$

Continuing,

$$\dot{x}_7 = x_8$$

$$\dot{x}_8 = x_3 + x_6 - 2x_7 - 2x_8$$

Assuming the output is x_s,

$$y = x_s$$

Hence, the solution in vector-matrix form is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

24.

Writing the equations of motion,

$$M_f \frac{d^2 y_f}{dt^2} + (f_{vf} + f_{vh}) \frac{dy_f}{dt} + K_h y_f - f_{vh} \frac{dy_h}{dt} - K_h y_h = f_{up}(t)$$

$$-f_{vh} \frac{dy_f}{dt} - K_h y_f + M_h \frac{d^2 y_h}{dt^2} + f_{vh} \frac{dy_h}{dt} + (K_h + K_s) y_h - K_s y_{cat} = 0$$

$$-K_s y_h + (K_s + K_{ave}) y_{cat} = 0$$

The last equation says that

$$y_{cat} = \frac{K_s}{(K_s + K_{ave})} y_h$$

Defining state variables for the first two equations of motion,

$$x_1 = y_h;\; x_2 = \dot{y}_h;\; x_3 = y_f;\; x_4 = \dot{y}_f$$

Solving for the highest derivative terms in the first two equations of motion yields,

$$\frac{d^2y_f}{dt^2} = -\frac{(f_{vf} + f_{vh})}{M_f}\frac{dy_f}{dt} - \frac{K_h}{M_f}y_f + \frac{f_{vh}}{M_f}\frac{dy_h}{dt} + \frac{K_h}{M_f}y_h + \frac{1}{M_f}f_{up}(t)$$

$$\frac{d^2y_h}{dt^2} = \frac{f_{vh}}{M_h}\frac{dy_f}{dt} + \frac{K_h}{M_h}y_f - \frac{f_{vh}}{M_h}\frac{dy_h}{dt} - \frac{(K_h + K_s)}{M_h}y_h + \frac{K_s}{M_h}y_{cat}$$

Writing the state equations,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{f_{vh}}{M_h}x_4 + \frac{K_h}{M_h}x_3 - \frac{f_{vh}}{M_h}x_2 - \frac{(K_h + K_s)}{M_h}x_1 + \frac{K_s}{M_h}\frac{K_s}{(K_s + K_{ave})}x_1$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{(f_{vf} + f_{vh})}{M_f}x_4 - \frac{K_h}{M_f}x_3 + \frac{f_{vh}}{M_f}x_2 + \frac{K_h}{M_f}x_1 + \frac{1}{M_f}f_{up}(t)$$

The output is y_h - y_{cat}. Therefore,

$$y = y_h - y_{cat} = y_h - \frac{K_s}{(K_s + K_{ave})}y_h = \frac{K_{ave}}{(K_s + K_{ave})}x_1$$

Simplifying, rearranging, and putting the state equations in vector-matrix form yields,

$$\dot{\mathrm{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{M_h}\left(\frac{K_s^2}{(K_s + K_{ave})} - (K_h + K_s)\right) & -\frac{f_{vh}}{M_h} & \frac{K_h}{M_h} & \frac{f_{vh}}{M_h} \\ 0 & 0 & 0 & 1 \\ \frac{K_h}{M_f} & \frac{f_{vh}}{M_f} & -\frac{K_h}{M_f} & -\frac{(f_{vf} + f_{vh})}{M_f} \end{bmatrix}\mathrm{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_f} \end{bmatrix} f_{up}(t)$$

$$y = \begin{bmatrix} \frac{K_{ave}}{(K_s + K_{ave})} & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

Substituting numerical values,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9353 & -14.29 & 769.2 & 14.29 \\ 0 & 0 & 0 & 1 \\ 406 & 7.558 & -406 & -9.302 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0581 \end{bmatrix} f_{up}(t)$$

$$y = \begin{bmatrix} 0.9491 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

F O U R

Time Response

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Open-Loop Response

The forward transfer function for angular velocity is,

$$G(s) = \frac{\omega_0(s)}{V_P(s)} = \frac{24}{(s+150)(s+1.32)}$$

a. $\omega_0(t) = A + Be^{-150t} + Ce^{-1.32t}$

b. $G(s) = \frac{24}{s^2+151.32s+198}$. Therefore, $2\zeta\omega_n = 151.32$, $\omega_n = 14.07$, and $\zeta = 5.38$.

c. $\omega_0(s) = \frac{24}{s(s^2+151.32s+198)} =$

$$\frac{24}{s(s+150)(s+1.32)} = 0.12121\frac{1}{s} + 0.0010761\frac{1}{s+150} - 0.12229\frac{1}{s+1.32}$$

Therefore, $\omega_0(t) = 0.12121 + .0010761\ e^{-150t} - 0.12229e^{-1.32t}$.

d. Using G(s),

$$\ddot{\omega}_0 + 151.32\,\dot{\omega}_0 + 198\omega_0 = 24v_p(t)$$

Defining,

$$x_1 = \omega_0$$

$$x_2 = \dot{\omega}_0$$

Thus, the state equations are,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -198x_1 - 151.32x_2 + 24v_p(t)$$

$$y = x_1$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -198 & -151.32 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 24 \end{bmatrix} v_p(t);\ y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}$$

e.

Program:

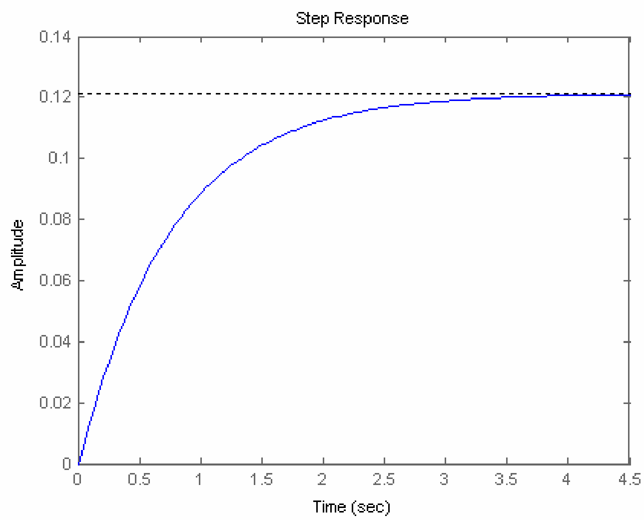
```
'Case Study 1 Challenge (e)'
num=24;
den=poly([-150  -1.32]);
G=tf(num,den)
step(G)
```

Computer response:

```
ans =

Case Study 1 Challenge (e)


Transfer function:
        24
-------------------
s^2 + 151.3 s + 198
```



Ship at Sea: Open-Loop Response

a. Assuming a second-order approximation: $\omega_n^2 = 2.25$, $2\zeta\omega_n = 0.5$. Therefore $\zeta = 0.167$, $\omega_n = 1.5$.

$$T_s = \frac{4}{\zeta\omega_n} = 16;\ T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 2.12\ ;$$

$$\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 58.8\%;\ \omega_n T_r = 1.169 \text{ therefore, } T_r = 0.77.$$

b. $$\theta(s) = \frac{2.25}{s\left(s^2+0.5s+2.25\right)} = \frac{1}{s} - \frac{s+0.5}{s^2+0.5s+2.25}$$

$$= \frac{1}{s} - \frac{(s+0.25)+\frac{0.25}{\sqrt{2.1875}}\sqrt{2.1875}}{(s+0.25)^2+2.1875}$$

$$= \frac{1}{s} - \frac{(s + 0.25) + 0.16903 \cdot 1.479}{(s + 0.25)^2 + 2.1875}$$

Taking the inverse Laplace transform,

$\theta(t) = 1 - e^{-0.25t} (\cos 1.479t + 0.16903 \sin 1.479t)$

c.

Program:

```
'Case Study 2 Challenge (C)'
'(a)'
numg=2.25;
deng=[1 0.5 2.25];
G=tf(numg,deng)
omegan=sqrt(deng(3))
zeta=deng(2)/(2*omegan)
Ts=4/(zeta*omegan)
Tp=pi/(omegan*sqrt(1-zeta^2))
pos=exp(-zeta*pi/sqrt(1-zeta^2))*100
t=0:.1:2;
[y,t]=step(G,t);
Tlow=interp1(y,t,.1);
Thi=interp1(y,t,.9);
Tr=Thi-Tlow
'(b)'
numc=2.25*[1 2];
denc=conv(poly([0 -3.57]),[1 2 2.25]);
[K,p,k]=residue(numc,denc)
'(c)'
[y,t]=step(G);
plot(t,y)
title('Roll Angle Response')
xlabel('Time(seconds)')
ylabel('Roll Angle(radians)')
```

Computer response:

```
ans =

Case Study 2 Challenge (C)


ans =

(a)


Transfer function:
       2.25
------------------
s^2 + 0.5 s + 2.25


omegan =

    1.5000


zeta =

    0.1667


Ts =

    16
```

```
Tp =

    2.1241


pos =

   58.8001


Tr =

    0.7801


ans =

(b)


K =

   0.1260
  -0.3431 + 0.1058i
  -0.3431 - 0.1058i
   0.5602


p =

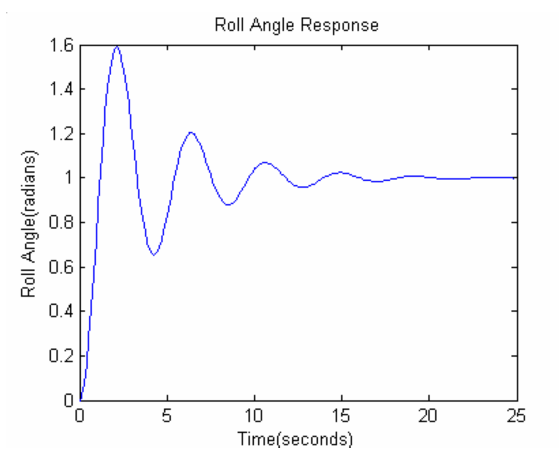
  -3.5700
  -1.0000 + 1.1180i
  -1.0000 - 1.1180i
        0


k =

     []


ans =

(c)
```



ANSWERS TO REVIEW QUESTIONS

1.Time constant

2. The time for the step response to reach 67% of its final value

3. The input pole

4. The system poles

5. The radian frequency of a sinusoidal response

6. The time constant of an exponential response

7. Natural frequency is the frequency of the system with all damping removed; the damped frequency of oscillation is the frequency of oscillation with damping in the system.

8. Their damped frequency of oscillation will be the same.

9. They will all exist under the same exponential decay envelop.

10. They will all have the same percent overshoot and the same shape although differently scaled in time.

11. ζ, ω_n, T_P, %OS, T_s

12. Only two since a second-order system is completely defined by two component parameters

13. (1) Complex, (2) Real, (3) Multiple real

14. Pole's real part is large compared to the dominant poles, (2) Pole is near a zero

15. If the residue at that pole is much smaller than the residues at other poles

16. No; one must then use the output equation

17. The Laplace transform of the state transition matrix is $(s\mathrm{I} - \mathbf{A})^{-1}$

18. Computer simulation

19. Pole-zero concepts give one an intuitive feel for the problem.

20. State equations, output equations, and initial value for the state-vector

21. Det(s**I**-**A**) = 0

SOLUTIONS TO PROBLEMS

1.

a. Overdamped Case:

$$C(s) = \frac{9}{s(s^2 + 9s + 9)}$$

Expanding into partial fractions,

$$C(s) = \frac{9}{s(s + 7.854)(s + 1.146)} = \frac{1}{s} + \frac{0.171}{(s + 7.854)} - \frac{1.171}{(s + 1.146)}$$

Taking the inverse Laplace transform,

$$c(t) = 1 + 0.171\, e^{-7.854t} - 1.171\, e^{-1.146t}$$

b. Underdamped Case:

$$C(s) = \frac{9}{s(s^2 + 3s + 9)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{(s^2 + 3s + 9)}$$

$$K_1 = \left. \frac{9}{(s^2 + 3s + 9)} \right|_{s \to 0} = 1$$

K_2 and K_3 can be found by clearing fractions with K_1 replaced by its value. Thus,

$$9 = (s^2 + 3s + 9) + (K_2 s + K_3)s$$

or

$$9 = s^2 + 3s + 9 + K_2 s^2 + K_3 s$$

Hence $K_2 = -1$ and $K_3 = -3$. Thus,

$$C(s) = \frac{1}{s} - \frac{s + 3}{(s^2 + 3s + 9)}$$

$$C(s) = \frac{1}{s} - \frac{(s + \frac{3}{2})}{(s + \frac{3}{2})^2 + \frac{27}{4}} - \frac{\frac{3}{2}}{(s + \frac{3}{2})^2 + \frac{27}{4}}$$

$$C(s) = \frac{1}{s} - \frac{(s + \frac{3}{2})}{(s + \frac{3}{2})^2 + \frac{27}{4}} - \frac{\frac{3}{\sqrt{27}}\sqrt{\frac{27}{4}}}{(s + \frac{3}{2})^2 + \frac{27}{4}}$$

$$c(t) = 1 - e^{\frac{-3}{2}t} \cos\sqrt{\frac{27}{4}}\,t - \frac{3}{\sqrt{27}} e^{\frac{-3}{2}t} \sin\sqrt{\frac{27}{4}}\,t$$

$$c(t) = 1 - \frac{2}{\sqrt{3}} e^{-3t/2} \cos\left(\sqrt{\frac{27}{4}}\,t - \phi\right)$$

$$= 1 - 1.155\, e^{-1.5t} \cos(2.598t - \phi)$$

where

$$\phi = \arctan\left(\frac{3}{\sqrt{27}}\right) = 30^o$$

c. Oscillatory Case:

$$C(s) = \frac{9}{s(s^2+9)}$$

$$C(s) = \frac{9}{s(s^2+9)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{(s^2+9)}$$

The evaluation of the constants in the numerator are found the same way as they were for the underdamped case. The results are $K_2 = -1$ and $K_3 = 0$. Hence,

$$C(s) = \frac{1}{s} - \frac{s}{(s^2+9)}$$

Therefore,

$$c(t) = 1 - \cos 3t$$

d. Critically Damped

$$C(s) = \frac{9}{s(s^2+6s+9)}$$

$$C(s) = \frac{9}{s(s^2+6s+9)} = \frac{K_1}{s} + \frac{K_2}{(s+3)^2} + \frac{K_3}{(s+3)}$$

The constants are then evaluated as

$$K_1 = \left.\frac{9}{(s^2+6s+9)}\right|_{s\to 0} = 1 \;;\; K_2 = \left.\frac{9}{s}\right|_{s\to -3} = -3 \;;\; K_3 = \left.\frac{d}{ds}\left(\frac{9}{s}\right)\right|_{s\to -3} = -1$$

Now, the transform of the response is

$$C(s) = \frac{9}{s(s^2+6s+9)} = \frac{1}{s} - \frac{3}{(s+3)^2} - \frac{1}{(s+3)}$$

$$c(t) = 1 - 3t\,e^{-3t} - e^{-3t}$$

2.

a. $C(s) = \frac{5}{s(s+5)} = \frac{1}{s} - \frac{1}{s+5}$. Therefore, $c(t) = 1 - e^{-5t}$.

Also, $T = \frac{1}{5}$, $T_r = \frac{2.2}{a} = \frac{2.2}{5} = 0.44$, $T_s = \frac{4}{a} = \frac{4}{5} = 0.8$.

b. $C(s) = \frac{20}{s(s+20)} = \frac{1}{s} - \frac{1}{s+20}$. Therefore, $c(t) = 1 - e^{-20t}$. Also, $T = \frac{1}{20}$,

$T_r = \frac{2.2}{a} = \frac{2.2}{20} = 0.11$, $T_s = \frac{4}{a} = \frac{4}{20} = 0.2$.

3.

Program:

```
'(a)'
num=5;
den=[1 5];
Ga=tf(num,den)
subplot(1,2,1)
step(Ga)
title('(a)')
'(b)'
num=20;
den=[1 20];
Gb=tf(num,den)
subplot(1,2,2)
step(Gb)
title('(b)')
```

Computer response:

```
ans =

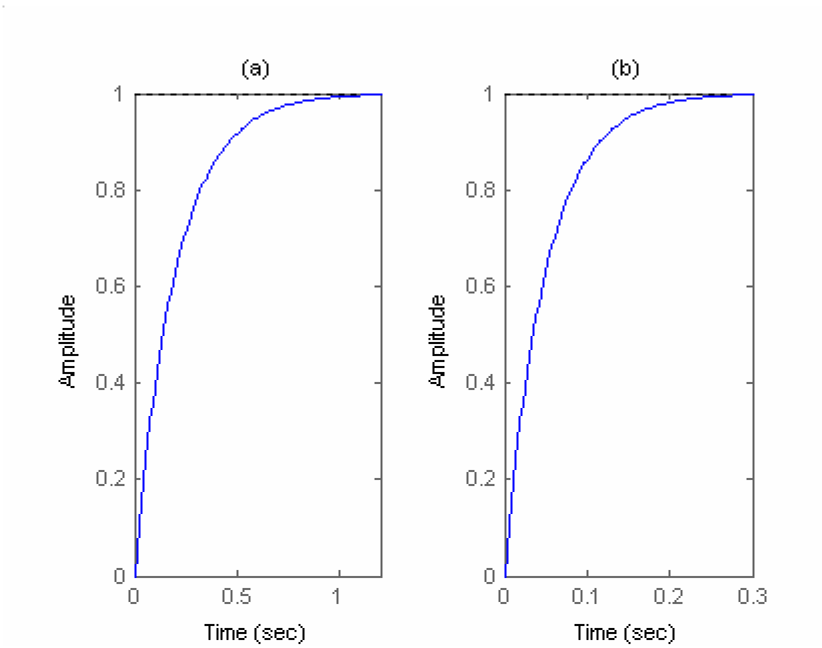
(a)

Transfer function:
  5
-----
s + 5

ans =

(b)

Transfer function:
  20
------
s + 20
```



4.

Using voltage division, $\frac{V_C(s)}{V_i(s)} = \frac{\frac{1}{Cs}}{(R+\frac{1}{Cs})} = \frac{2}{(s+2)}$. Since $V_i(s) = \frac{5}{s}$, $V_C(s) = \frac{10}{s(s+2)} = \frac{5}{s} - \frac{5}{s+2}$.

Therefore $v_C(t) = 5 - 5e^{-2t}$. Also, $T = \frac{1}{2}$, $T_r = \frac{2.2}{a} = \frac{2.2}{2} = 1.1$, $T_s = \frac{4}{a} = \frac{4}{2} = 2$.

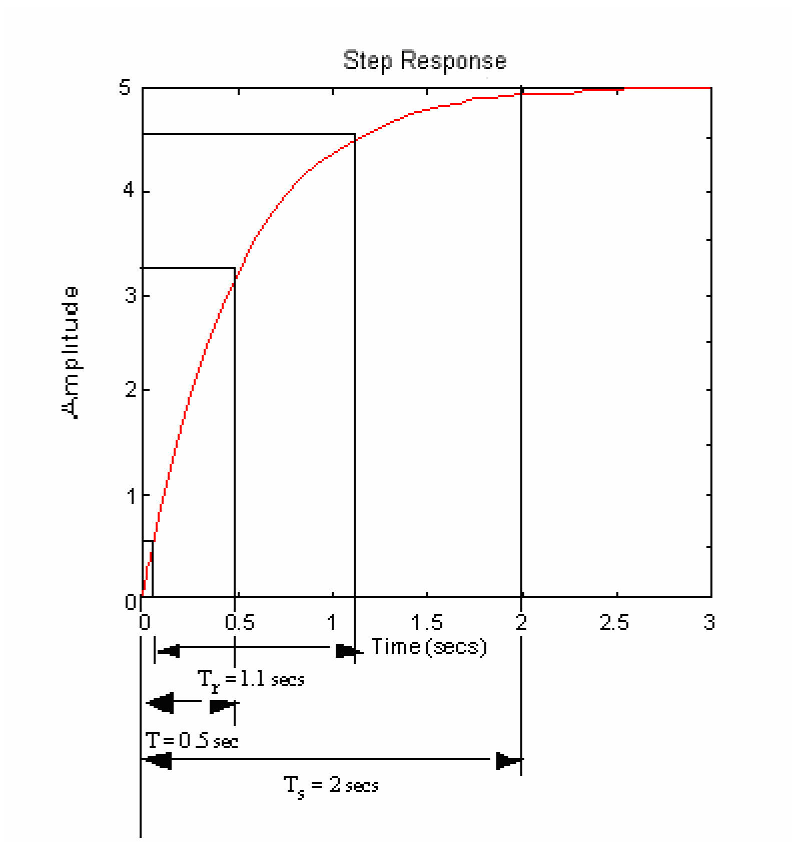
5.

Program:

```
clf
num=2;
den=[1 2];
G=tf(num,den)
step(5*G)
```

Computer response:

```
Transfer function:
  2
-----
s + 2
```



6.

Writing the equation of motion,

$$(Ms^2 + 8s)X(s) = F(s)$$

Thus, the transfer function is,

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + 8s}$$

Differentiating to yield the transfer function in terms of velocity,

$$\frac{sX(s)}{F(s)} = \frac{1}{Ms + 8} = \frac{1/M}{s + \frac{8}{M}}$$

Thus, the settling time, T_s, and the rise time, T_r, are given by

$$T_s = \frac{4}{8/M} = \frac{1}{2}M; \quad T_r = \frac{2.2}{8/M} = 0.275M$$

7.

Program:

```
Clf
M=1
num=1/M;
den=[1 8/M];
G=tf(num,den)
step(G)
pause
M=2
num=1/M;
den=[1 8/M];
G=tf(num,den)
step(G)
```

Computer response:

```
Transfer function:
 M =

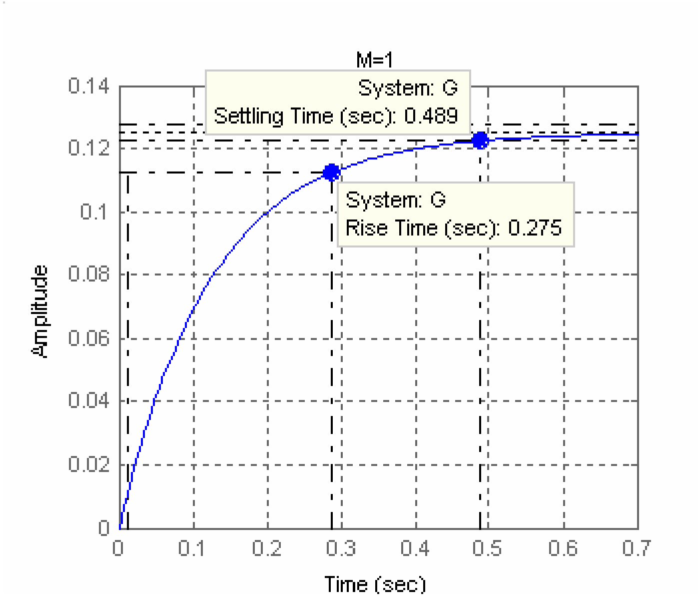
     1


Transfer function:
  1
-----
s + 8

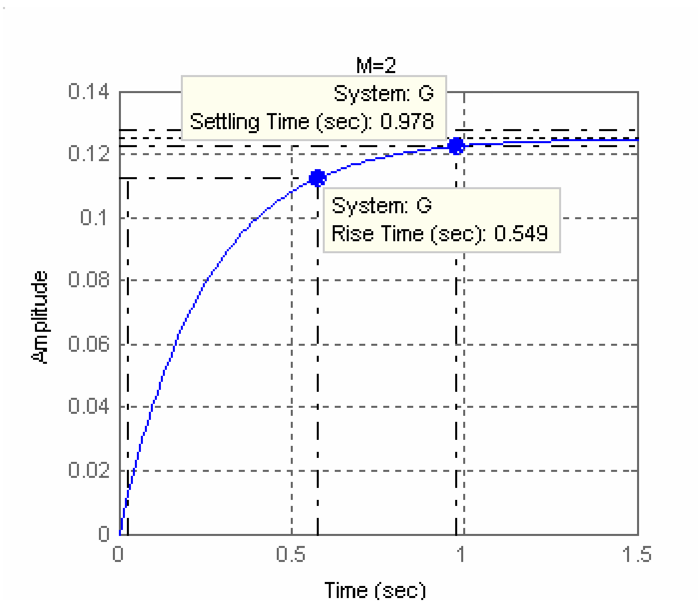

M =

     2


Transfer function:
 0.5
-----
s + 4
```



From plot, time constant = 0.125 s.



From plot, time constant = 0.25 s.

8.

a. Pole: -2; $c(t) = A + Be^{-2t}$; first-order response.

b. Poles: -3, -6; $c(t) = A + Be^{-3t} + Ce^{-6t}$; overdamped response.

c. Poles: -10, -20; Zero: -7; $c(t) = A + Be^{-10t} + Ce^{-20t}$; overdamped response.

d. Poles: $(-3+j3\sqrt{15})$, $(-3-j3\sqrt{15})$; $c(t) = A + Be^{-3t}\cos(3\sqrt{15}\ t + \phi)$; underdamped.

e. Poles: j3, -j3; Zero: -2; $c(t) = A + B\cos(3t + \phi)$; undamped.

f. Poles: -10, -10; Zero: -5; $c(t) = A + Be^{-10t} + Cte^{-10t}$; critically damped.

9.

Program:
```
p=roots([1 6 4 7 2])
```

Computer response:
```
p =

  -5.4917
  -0.0955 + 1.0671i
  -0.0955 - 1.0671i
  -0.3173
```

10.

$G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} 8 & -4 & 1 \\ -3 & 2 & 0 \\ 5 & 7 & -9 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix};\ \mathbf{C} = \begin{bmatrix} 2 & 8 & -3 \end{bmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s^3 - s^2 - 91s + 67}\begin{bmatrix} (s^2 + 7s - 18) & -(4s+29) & (s-2) \\ -(3s+27) & (s^2 + s - 77) & -3 \\ 5s - 31 & 7s - 76 & (s^2 - 10s + 4) \end{bmatrix}$$

Therefore, $G(s) = \dfrac{5s^2 + 136s - 1777}{s^3 - s^2 - 91s + 67}$.

Factoring the denominator, or using det($s\mathbf{I}-\mathbf{A}$), we find the poles to be 9.683, 0.7347, -9.4179.

11.

Program:
```
A=[8 -4 1;-3 2 0;5 7 -9]
B=[1;3;7]
C=[2 8 -3]
D=0
[numg,deng]=ss2tf(A,B,C,D,1);
G=tf(numg,deng)
poles=roots(deng)
```

Computer response:
```
A =
     8    -4     1
    -3     2     0
     5     7    -9
B =
     1
```

```
      3
      7
C =
      2      8     -3
D =
      0

Transfer function:
5 s^2 + 136 s - 1777
---------------------
s^3 - s^2 - 91 s + 67

poles =

   -9.4179
    9.6832
    0.7347
```

12.

Writing the node equation at the capacitor, $V_C(s)\,(\frac{1}{R_2} + \frac{1}{Ls} + Cs) + \frac{V_C(s) - V(s)}{R_1} = 0$.

Hence, $\frac{V_C(s)}{V(s)} = \frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{Ls} + Cs} = \frac{10s}{s^2+20s+500}$. The step response is $\frac{10}{s^2+20s+500}$. The poles

are at

$-10 \pm j20$. Therefore, $v_C(t) = Ae^{-10t}\cos(20t + \phi)$.

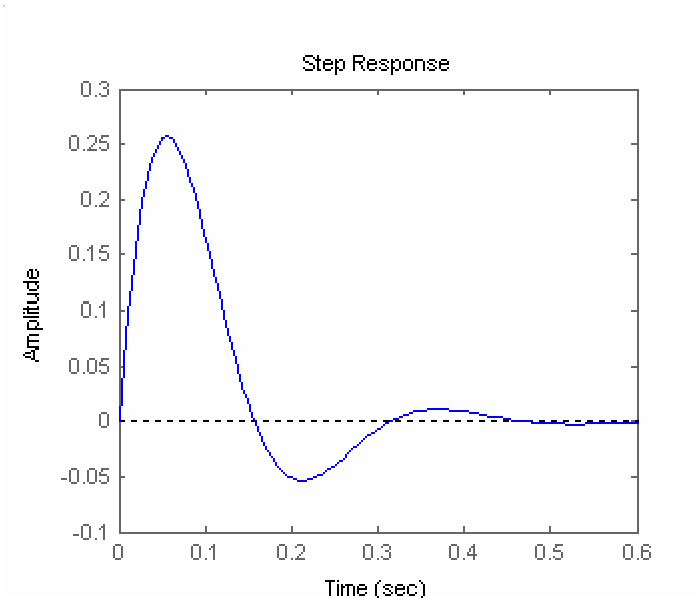
13.

Program:

```
num=[10 0];
den=[1 20 500];
G=tf(num,den)
step(G)
```

Computer response:

```
Transfer function:
      10 s
----------------
s^2 + 20 s + 500
```



14.

The equation of motion is: $(Ms^2+f_vs+K_s)X(s) = F(s)$. Hence, $\frac{X(s)}{F(s)} = \frac{1}{Ms^2+f_vs+K_s} = \frac{1}{s^2+s+5}$.

The step response is now evaluated: $X(s) = \frac{1}{s(s^2+s+5)} = \frac{1/5}{s} - \frac{\frac{1}{5}s+\frac{1}{5}}{(s+\frac{1}{2})^2+\frac{19}{4}} =$

$$\frac{\frac{1}{5}(s+\frac{1}{2}) + \frac{1}{5\sqrt{19}}\frac{\sqrt{19}}{2}}{(s+\frac{1}{2})^2+\frac{19}{4}} .$$

Taking the inverse Laplace transform, $x(t) = \frac{1}{5} - \frac{1}{5}\, e^{-0.5t} \left(\cos \frac{\sqrt{19}}{2}\, t + \frac{1}{\sqrt{19}} \sin \frac{\sqrt{19}}{2}\, t \right)$

$$= \frac{1}{5}\left[1 - 2\sqrt{\frac{5}{19}}\, e^{-0.5t} \cos\left(\frac{\sqrt{19}}{2}\, t - 12.92^o\right)\right].$$

15.

$$C(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s + \zeta\omega_n) + \zeta\omega_n}{(s+\zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2} = \frac{1}{s} - \frac{(s+\zeta\omega_n) + \frac{\zeta\omega_n}{\omega_n\sqrt{1 - \zeta^2}}\,\omega_n\sqrt{1 - \zeta^2}}{(s+\zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2}$$

Hence, $c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n\sqrt{1 - \zeta^2}\, t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_n\sqrt{1 - \zeta^2}\, t\right)$

$$= 1 - e^{-\zeta\omega_n t}\sqrt{1+\frac{\zeta^2}{1-\zeta^2}}\cos(\omega_n\sqrt{1-\zeta^2}\,t-\phi) = 1 - e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}}\cos(\omega_n\sqrt{1-\zeta^2}\,t-\phi),$$

where $\phi = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$

16.

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$. Dividing by 100 and taking the natural log of both sides, $\ln\left(\frac{\%OS}{100}\right) = -\frac{\zeta\pi}{\sqrt{1-\zeta^2}}$. Squaring both sides and solving for ζ^2, $\zeta^2 = \frac{\ln^2\left(\frac{\%OS}{100}\right)}{\pi^2+\ln^2\left(\frac{\%OS}{100}\right)}$. Taking the negative square root, $\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2+\ln^2\left(\frac{\%OS}{100}\right)}}$.

17.

a.

$$C(s) = \frac{2}{s(s+2)}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+2}$$

$$c(t) = 1 - e^{-2t}$$

b.

$$C(s) = \frac{5}{s(s+3)(s+6)}$$

$$C(s) = \frac{5}{18}\frac{1}{s} - \frac{5}{9}\frac{1}{s+3} + \frac{5}{18}\frac{1}{s+6}$$

$$c(t) = \frac{5}{18} - \frac{5}{9}e^{-3t} + \frac{5}{18}e^{-6t}$$

c.

$$C(s) = \frac{10(s+7)}{s(s+10)(s+20)}$$

$$C(s) = \frac{7}{20}\frac{1}{s} + \frac{3}{10}\frac{1}{s+10} - \frac{13}{20}\frac{1}{s+20}$$

$$c(t) = \frac{7}{20} + \frac{3}{10}e^{-10t} - \frac{13}{20}e^{-20t}$$

d.

$$C(s) = \frac{20}{s(s^2+6s+144)}$$

$$C(s) = \frac{5}{36}\frac{1}{s} - \frac{5}{36}\frac{s+6}{s^2+6s+144}$$

$$C(s) = \frac{5}{36}\frac{1}{s} - \frac{5}{36}\frac{(s+3)+\frac{3}{\sqrt{135}}\sqrt{135}}{(s+3)^2+135}$$

$$c(t) = \frac{5}{36} - \frac{5}{36}e^{-3t}\left(\cos\left[\sqrt{135}\right]t + \frac{3}{\sqrt{135}}\sin\left[\sqrt{135}\right]t\right)$$

e.

$$C(s) = \frac{s+2}{s(s^2+9)}$$

$$C(s) = \frac{2}{9}\frac{1}{s} + \frac{1}{9}\frac{-2s+9}{s^2+9}$$

$$C(s) = \frac{2}{9}\frac{1}{s} + \frac{1}{9}\frac{-2s+3.3}{s^2+9}$$

$$c(t) = \frac{2}{9} - \left(\frac{2}{9}\cos 3t - \frac{1}{3}\sin 3t\right)$$

f.

$$C(s) = \frac{s+5}{s(s+10)^2}$$

$$C(s) = \frac{1}{20}\frac{1}{s} - \frac{1}{20}\frac{1}{s+10} + \frac{1}{2}\frac{1}{(s+10)^2}$$

$$c(t) = \frac{1}{20} - \frac{1}{20}e^{-10t} + \frac{1}{2}te^{-10t}$$

18.

a. N/A

b. $s^2+9s+18$, $\omega_n^2 = 18$, $2\zeta\omega_n = 9$, Therefore $\zeta = 1.06$, $\omega_n = 4.24$, overdamped.

c. $s^2+30s+200$, $\omega_n^2 = 200$, $2\zeta\omega_n = 30$, Therefore $\zeta = 1.06$, $\omega_n = 14.14$, overdamped.

d. $s^2+6s+144$, $\omega_n^2 = 144$, $2\zeta\omega_n = 6$, Therefore $\zeta = 0.25$, $\omega_n = 12$, underdamped.

e. s^2+9, $\omega_n^2 = 9$, $2\zeta\omega_n = 0$, Therefore $\zeta = 0$, $\omega_n = 3$, undamped.

f. $s^2+20s+100$, $\omega_n^2 = 100$, $2\zeta\omega_n = 20$, Therefore $\zeta = 1$, $\omega_n = 10$, critically damped.

19.

$$X(s) = \frac{100^2}{s(s^2+100s+100^2)} = \frac{1}{s} - \frac{s+100}{(s+50)^2+7500} = \frac{1}{s} - \frac{(s+50)+50}{(s+50)^2+7500} = \frac{1}{s} - \frac{(s+50) + \frac{50}{\sqrt{7500}}\sqrt{7500}}{(s+50)^2+7500}$$

Therefore, $x(t) = 1 - e^{-50t}\left(\cos\sqrt{7500}\ t + \frac{50}{\sqrt{7500}}\sin\sqrt{7500}\ t\right)$

$= 1 - \frac{2}{\sqrt{3}}e^{-50t}\cos\left(50\sqrt{3}\ t - \tan^{-1}\frac{1}{\sqrt{3}}\right)$

20.

a. $\omega_n^2 = 16$ r/s, $2\zeta\omega_n = 3$. Therefore $\zeta = 0.375$, $\omega_n = 4$. $T_s = \frac{4}{\zeta\omega_n} = 2.667$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} =$ 0.8472 s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 28.06\ \%$; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1) = 1.4238$; therefore, $T_r = 0.356$ s.

b. $\omega_n^2 = 0.04$ r/s, $2\zeta\omega_n = 0.02$. Therefore $\zeta = 0.05$, $\omega_n = 0.2$. $T_s = \frac{4}{\zeta\omega_n} = 400$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 15.73$ s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 85.45$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore, $T_r = 5.26$ s.

c. $\omega_n^2 = 1.05 \times 10^7$ r/s, $2\zeta\omega_n = 1.6 \times 10^3$. Therefore $\zeta = 0.247$, $\omega_n = 3240$. $T_s = \frac{4}{\zeta\omega_n} = 0.005$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.001$ s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 44.92$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore, $T_r = 3.88\times10^{-4}$ s.

21.

Program:

```
'(a)'
clf
numa=16;
dena=[1 3 16];
Ta=tf(numa,dena)
omegana=sqrt(dena(3))
zetaa=dena(2)/(2*omegana)
Tsa=4/(zetaa*omegana)
Tpa=pi/(omegana*sqrt(1-zetaa^2))
Tra=(1.76*zetaa^3 - 0.417*zetaa^2 + 1.039*zetaa + 1)/omegana
percenta=exp(-zetaa*pi/sqrt(1-zetaa^2))*100
subplot(221)
step(Ta)
title('(a)')
'(b)'
numb=0.04;
denb=[1 0.02 0.04];
Tb=tf(numb,denb)
omeganb=sqrt(denb(3))
zetab=denb(2)/(2*omeganb)
Tsb=4/(zetab*omeganb)
Tpb=pi/(omeganb*sqrt(1-zetab^2))
Trb=(1.76*zetab^3 - 0.417*zetab^2 + 1.039*zetab + 1)/omeganb
percentb=exp(-zetab*pi/sqrt(1-zetab^2))*100
subplot(222)
step(Tb)
title('(b)')
'(c)'
numc=1.05E7;
denc=[1 1.6E3 1.05E7];
Tc=tf(numc,denc)
omeganc=sqrt(denc(3))
zetac=denc(2)/(2*omeganc)
Tsc=4/(zetac*omeganc)
Tpc=pi/(omeganc*sqrt(1-zetac^2))
Trc=(1.76*zetac^3 - 0.417*zetac^2 + 1.039*zetac + 1)/omeganc
percentc=exp(-zetac*pi/sqrt(1-zetac^2))*100
subplot(223)
step(Tc)
title('(c)')
```

Computer response:

```
ans =

(a)
```

```
Transfer function:
      16
--------------
s^2 + 3 s + 16


omegana =

     4


zetaa =

    0.3750


Tsa =

    2.6667


Tpa =

    0.8472


Tra =

    0.3559


percenta =

   28.0597


ans =

(b)


Transfer function:
       0.04
-------------------
s^2 + 0.02 s + 0.04


omeganb =

    0.2000


zetab =

    0.0500


Tsb =

   400


Tpb =
```

```
   15.7276


Trb =

    5.2556


percentb =

   85.4468


ans =

(c)


Transfer function:
       1.05e007
-----------------------
s^2 + 1600 s + 1.05e007


omeganc =

  3.2404e+003


zetac =

    0.2469


Tsc =

    0.0050


Tpc =

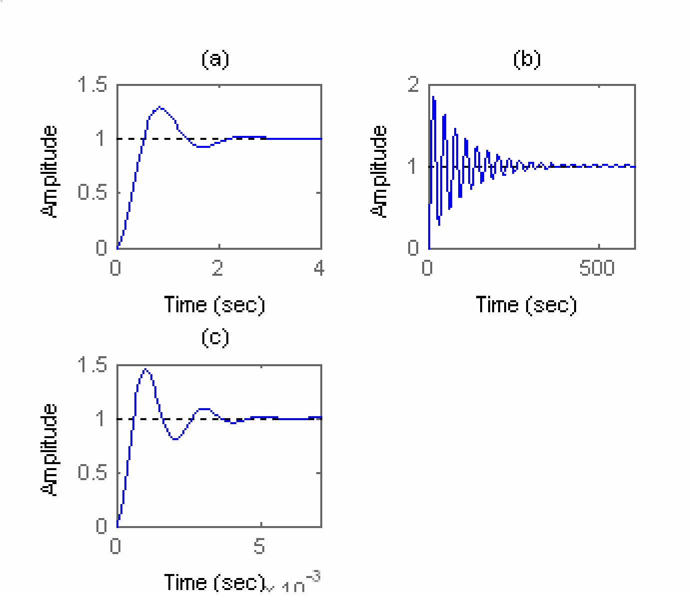
    0.0010


Trc =

  3.8810e-004


percentc =

   44.9154
```



22.

Program:

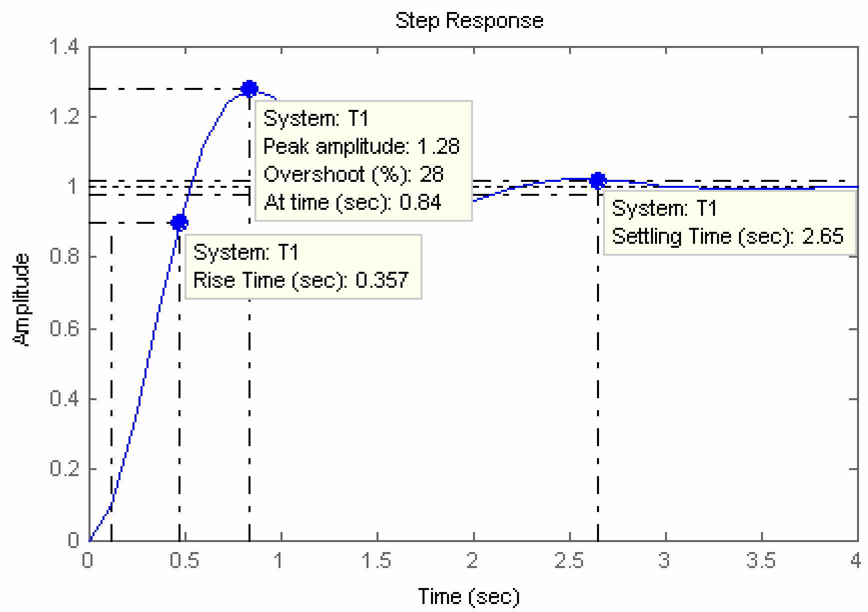
```
T1=tf(16,[1 3 16])
T2=tf(0.04,[1 0.02 0.04])
T3=tf(1.05e7,[1 1.6e3 1.05e7])
ltiview
```

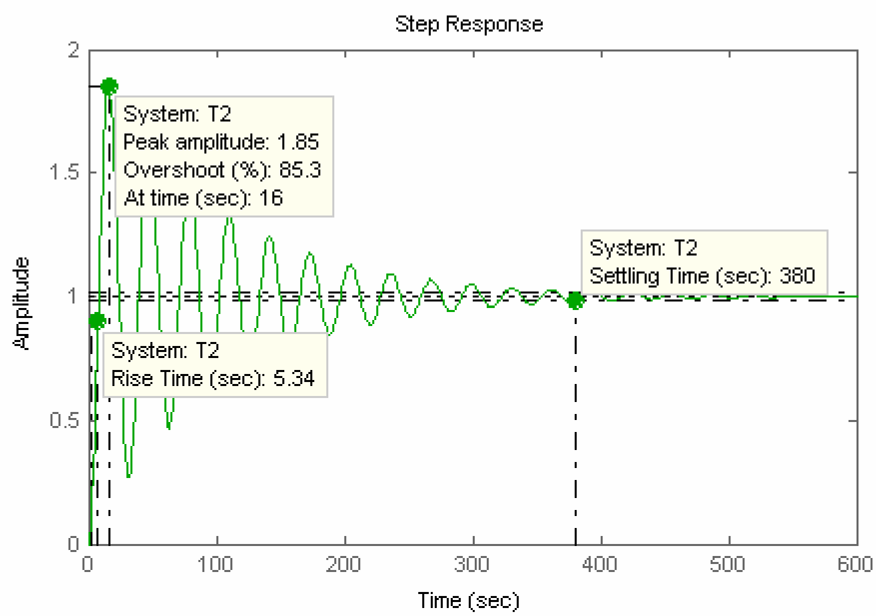
Computer response:

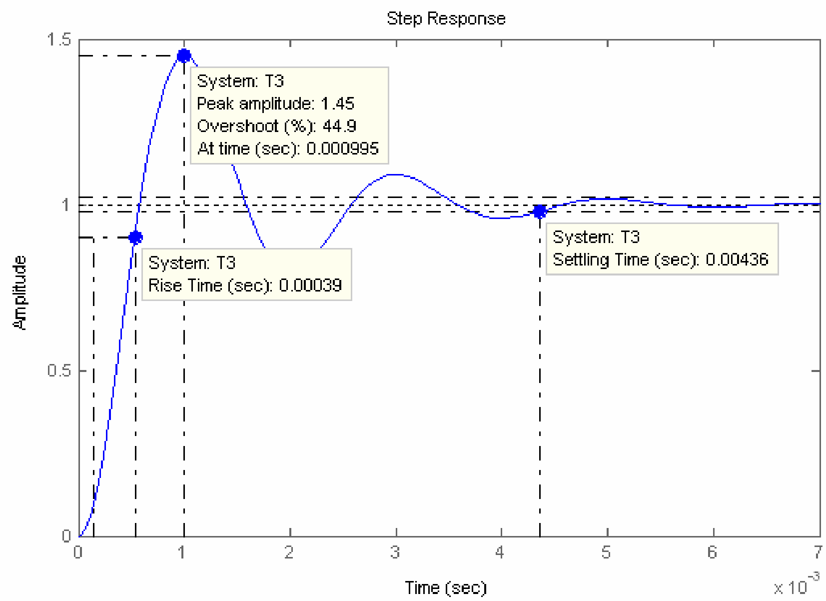
```
Transfer function:
      16
--------------
s^2 + 3 s + 16

Transfer function:
       0.04
-------------------
s^2 + 0.02 s + 0.04


Transfer function:
       1.05e007
-----------------------
s^2 + 1600 s + 1.05e007
```


Step Response
System: T1
Peak amplitude: 1.28
Overshoot (%): 28
At time (sec): 0.84
System: T1
Settling Time (sec): 2.65
System: T1
Rise Time (sec): 0.357
Amplitude
Time (sec)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0.5
1.5
2
2.5
3
3.5
4


Step Response
System: T2
Peak amplitude: 1.85
Overshoot (%): 85.3
At time (sec): 16
System: T2
Settling Time (sec): 380
System: T2
Rise Time (sec): 5.34
Amplitude
Time (sec)
0
0.5
1
1.5
2
100
200
300
400
500
600



23.

a. $\zeta = \dfrac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.56$, $\omega_n = \dfrac{4}{\zeta T_s} = 11.92$. Therefore, poles $= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$= -6.67 \pm j9.88$.

b. $\zeta = \dfrac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.591$, $\omega_n = \dfrac{\pi}{T_p\sqrt{1-\zeta^2}} = 0.779$.

Therefore, poles $= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -0.4605 \pm j0.6283$.

c. $\zeta\omega_n = \dfrac{4}{T_s} = 0.571$, $\omega_n\sqrt{1-\zeta^2} = \dfrac{\pi}{T_p} = 1.047$. Therefore, poles $= -0.571 \pm j1.047$.

24.

$$\text{Re} = \frac{4}{T_s} = 4;\quad \zeta = \frac{-\ln(12.3/100)}{\sqrt{\pi^2 + \ln^2(12.3/100)}} = 0.5549$$

$$\text{Re} = \zeta\omega_n = 0.5549\omega_n = 4;\quad \therefore \omega_n = 7.21$$

$$\text{Im} = \omega_n\sqrt{1-\zeta^2} = 6$$

$$\therefore G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{51.96}{s^2 + 8s + 51.96}$$

25.

a. Writing the equation of motion yields, $(3s^2 + 15s + 33)X(s) = F(s)$

Solving for the transfer function,

$$\frac{X(s)}{F(s)} = \frac{1/3}{s^2 + 5s + 11}$$

b. $\omega_n^2 = 11$ r/s, $2\zeta\omega_n = 5$. Therefore $\zeta = 0.754$, $\omega_n = 3.32$. $T_s = \frac{4}{\zeta\omega_n} = 1.6$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.44$ s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 2.7$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore, $T_r = 0.69$ s.

26.

Writing the loop equations,

$$(s^2 + s)\theta_1(s) - s\,\theta_2(s) = T(s)$$

$$-s\,\theta_1(s) + (s+1)\theta_2(s) = 0$$

Solving for $\theta_2(s)$,

$$\theta_2(s) = \frac{\begin{vmatrix} s^2+s & T(s) \\ -s & 0 \end{vmatrix}}{\begin{vmatrix} s^2+s & -s \\ -s & s+1 \end{vmatrix}} = \frac{T(s)}{s^2 + s + 1}$$

Forming the transfer function,

$$\frac{\theta_2(s)}{T(s)} = \frac{1}{s^2 + s + 1}$$

Thus $\omega_n = 1$, $2\zeta\omega_n = 1$. Thus, $\zeta = 0.5$. From Eq. (4.38), %OS = 16.3%. From Eq. (4.42), $T_s = 8$ seconds. From Eq. (4.34), $T_p = 3.63$ seconds.

27.

a. $$\frac{24.542}{s(s^2 + 4s + 24.542)} = \frac{1}{s} - \frac{s+4}{(s+2)^2 + 20.542} = \frac{1}{s} - \frac{(s+2) + \frac{2}{4.532}4.532}{(s+2)^2 + 20.542}.$$

Thus $c(t) = 1 - e^{-2t}(\cos 4.532t + 0.441 \sin 4.532t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8^0)$.

b.

$$\frac{245.42}{s(s+10)(s^2 + 4s + 24.542)} = \frac{1}{s} - 0.29029\frac{1}{s+10} - \frac{0.70971s + 5.7418}{s^2 + 4s + 24.542}$$

$$\frac{245.42}{s(s+10)(s^2 + 4s + 24.542)} = \frac{1}{s} - 0.29029\frac{1}{s+10} - \frac{0.70971s + 5.7418}{(s+2)^2 + 20.542}$$

$$\frac{245.42}{s(s+10)(s^2 + 4s + 24.542)} = \frac{1}{s} - 0.29029\frac{1}{s+10} - \frac{0.70971(s+2) + \frac{4.3223}{\sqrt{20.542}}\sqrt{20.542}}{(s+2)^2 + 20.542}$$

$$\frac{245.42}{s(s+10)(s^2+4s+24.542)} = \frac{1}{s} - 0.29029\frac{1}{s+10} - \frac{0.70971(s+2)+0.95367\sqrt{20.542}}{(s+2)^2+20.542}$$

Therefore, $c(t) = 1 - 0.29e^{-10t} - e^{-2t}(0.71\cos 4.532t + 0.954\sin 4.532t)$

$= 1 - 0.29e^{-10t} - 1.189\cos(4.532t - 53.34^o)$.

c.

$$\frac{73.626}{s(s+3)(s^2+4s+24.542)} = \frac{1}{s} - 1.1393\frac{1}{s+3} + \frac{0.13926s - 2.8607}{s^2+4s+24.542}$$

$$\frac{73.626}{s(s+3)(s^2+4s+24.542)} = \frac{1}{s} - 1.1393\frac{1}{s+3} + \frac{0.13926(s+2) - \frac{3.1393}{\sqrt{20.542}}\sqrt{20.542}}{(s+2)^2+20.542}$$

$$\frac{73.626}{s(s+3)(s^2+4s+24.542)} = \frac{1}{s} - 1.1393\frac{1}{s+3} + \frac{0.13926(s+2) - 0.69264\sqrt{20.542}}{(s+2)^2+20.542}$$

Therefore, $c(t) = 1 - 1.14e^{-3t} + e^{-2t}(0.14\cos 4.532t - 0.69\sin 4.532t)$

$= 1 - 1.14e^{-3t} + 0.704\cos(4.532t + 78.53^o)$.

28.

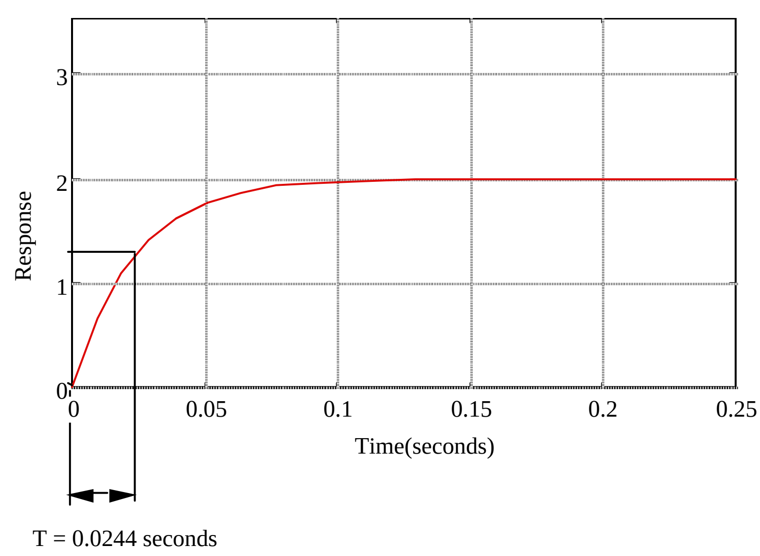
Since the third pole is more than five times the real part of the dominant pole, $s^2+1.204s+2.829$ determines the transient response. Since $2\zeta\omega_n = 1.204$, and $\omega_n = \sqrt{2.829} = \omega_n = 1.682$, $\zeta = 0.358$,

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 30\%$, $T_s = \frac{4}{\zeta\omega_n} = 6.64$ sec, $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 2$ sec; $\omega_n T_r = 1.4$,

therefore, $T_r = 0.832$.

29.

a. Measuring the time constant from the graph, T = 0.0244 seconds.

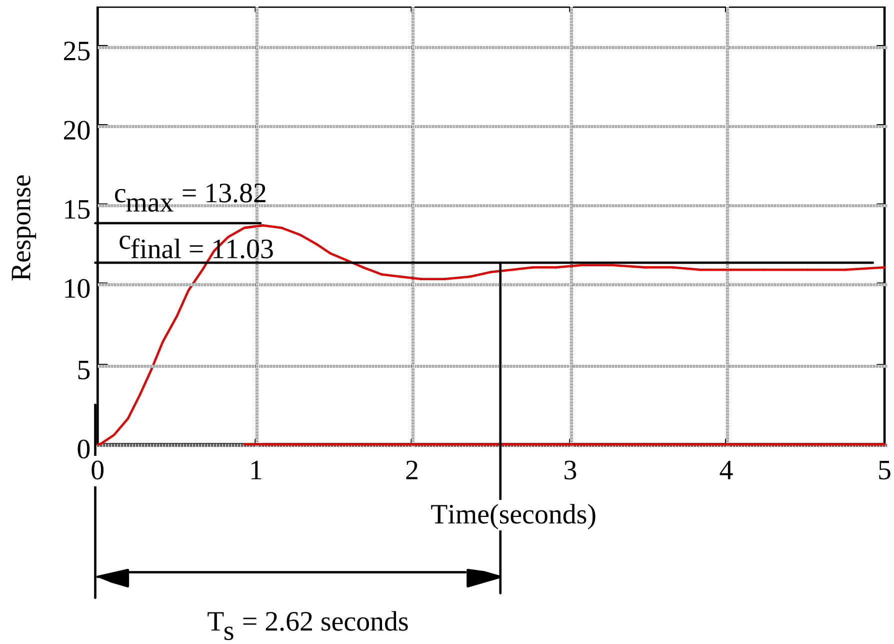


Estimating a first-order system, $G(s) = \frac{K}{s+a}$. But, a = 1/T = 40.984, and $\frac{K}{a}$ = 2. Hence, K = 81.967.

Thus,

$$G(s) = \frac{81.967}{s+40.984}$$

b. Measuring the percent overshoot and settling time from the graph: %OS = (13.82-11.03)/11.03 = 25.3%,



and T_s = 2.62 seconds. Estimating a second-order system, we use Eq. (4.39) to find $\zeta = 0.4$, and Eq. (4.42) to find $\omega_n = 3.82$. Thus, $G(s) = \frac{K}{s^2+2\zeta\omega_n s+\omega_n^2}$. Since $C_{final} = 11.03$, $\frac{K}{\omega_n^2} = 11.03$. Hence,

K = 160.95. Substituting all values,

$$G(s) = \frac{160.95}{s^2+3.056s+14.59}$$

c. From the graph, %OS = 40%. Using Eq. (4.39), $\zeta = 0.28$. Also from the graph,

$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 4$. Substituting $\zeta = 0.28$, we find $\omega_n = 0.818$.

Thus,

$$G(s) = \frac{K}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{0.669}{s^2+0.458s+0.669}.$$

30.

a.

$$\frac{s+3}{s(s+2)(s^2+3s+10)} = \frac{3}{20}\frac{1}{s} - \frac{1}{16}\frac{1}{s+2} - \frac{1}{80}\frac{7s+31}{\left(s+\frac{3}{2}\right)^2+\frac{31}{4}}$$

$$\frac{s+3}{s(s+2)(s^2+3s+10)} = \frac{3}{20}\frac{1}{s} - \frac{1}{16}\frac{1}{s+2} - \frac{1}{80}\frac{7\left(s+\frac{3}{2}\right)+7.3638\sqrt{\frac{31}{4}}}{\left(s+\frac{3}{2}\right)^2+\frac{31}{4}}$$

Since the amplitude of the sinusoids are of the same order of magnitude as the residue of the pole at -2, pole-zero cancellation cannot be assumed.

b.

$$\frac{s+2.5}{s(s+2)(s^2+4s+20)} = \frac{1}{16}\frac{1}{s} - \frac{1}{64}\frac{1}{s+2} - \frac{1}{64}\frac{3s+14}{s^2+4s+20}$$

$$\frac{s+2.5}{s(s+2)(s^2+4s+20)} = \frac{1}{16}\frac{1}{s} - \frac{1}{64}\frac{1}{s+2} - \frac{1}{64}\frac{3(s+2)+2\sqrt{16}}{(s+2)^2+16}$$

Since the amplitude of the sinusoids are of the same order of magnitude as the residue of the pole at -2, pole-zero cancellation cannot be assumed.

c.

$$\frac{s+2.1}{s(s+2)(s^2+s+5)} = 0.21\frac{1}{s} - 0.0071429\frac{1}{s+2} - \frac{0.20286s+0.21714}{s^2+1s+5}$$

$$\frac{s+2.1}{s(s+2)(s^2+s+5)} = 0.21\frac{1}{s} - 0.0071429\frac{1}{s+2} - \frac{0.20286\left(s+\frac{1}{2}\right)+0.053093\sqrt{\frac{19}{4}}}{\left(s+\frac{1}{2}\right)^2+\frac{19}{4}}$$

Since the amplitude of the sinusoids are of two orders of magnitude larger than the residue of the pole at -2, pole-zero cancellation can be assumed. Since $2\zeta\omega_n = 1$, and $\omega_n = \sqrt{5} = 2.236$, $\zeta = 0.224$, $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}}x100 = 48.64\%$, $T_s = \frac{4}{\zeta\omega_n} = 8$ sec, $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.44$ sec; $\omega_n T_r = 1.23$, therefore, $T_r = 0.55$.

d.

$$\frac{s+2.01}{s(s+2)(s^2+5s+20)} = 0.05025\frac{1}{s} - 0.00035714\frac{1}{s+2} - \frac{0.049893s+0.25018}{\left(s+\frac{5}{2}\right)^2+\frac{55}{4}}$$

$$\frac{s+2.01}{s(s+2)(s^2+5s+20)} = 0.05025\frac{1}{s} - 0.00035714\frac{1}{s+2} - \frac{0.049893\left(s+\frac{5}{2}\right)+0.03383\sqrt{\frac{55}{4}}}{\left(s+\frac{5}{2}\right)^2+\frac{55}{4}}$$

Since the amplitude of the sinusoids are of two orders of magnitude larger than the residue of the pole at -2, pole-zero cancellation can be assumed. Since $2\zeta\omega_n = 5$, and $\omega_n = \sqrt{20} = 4.472$, $\zeta = 0.559$,

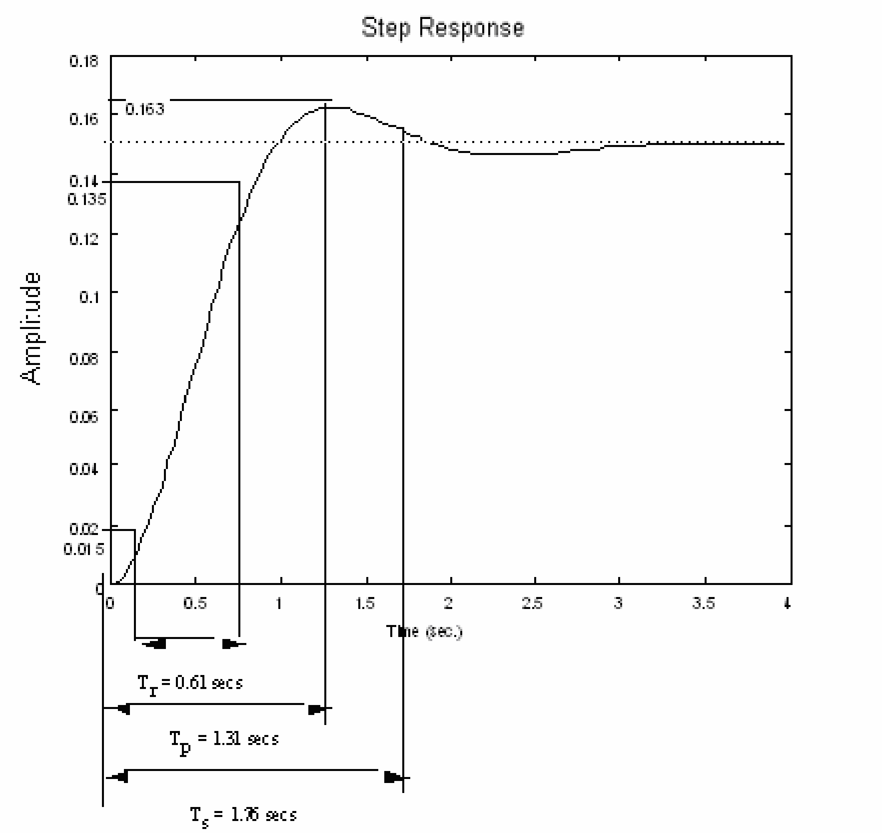
$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 12.03\%$, $T_s = \frac{4}{\zeta\omega_n}$ = 1.6 sec, $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ = 0.847 sec; $\omega_n T_r$ =

1.852, therefore, T_r = 0.414.

31.

Program:
```
%Form sC(s) to get transfer function
clf
num=[1 3];
den=conv([1 3 10],[1 2]);
T=tf(num,den)
step(T)
```

Computer response:
```
Transfer function:
          s + 3
-----------------------
s^3 + 5 s^2 + 16 s + 20
```



$\%OS = \frac{(0.163 - 0.15)}{0.15} = 8.67\%$

32.

Only part **c** can be approximated as a second-order system. From the exponentially decaying cosine the poles are located at $s_{1,2} = -2 \pm j9.796$. Thus,

$$T_s = \frac{4}{|\mathrm{Re}|} = \frac{4}{2} = 2\text{ s};\ T_p = \frac{\pi}{|\mathrm{Im}|} = \frac{\pi}{9.796} = 0.3207\text{ s}$$

Also, $\omega_n = \sqrt{2^2 + 9.796^2} = 10$ and $\zeta\omega_n = |\mathrm{Re}| = 2$. Hence, $\zeta = 0.2$, yielding 52.66 percent overshoot.

33.

a.

(1) $$C_{a1}(s) = \frac{1}{s^2+3s+36} = \frac{\frac{1}{\sqrt{33.75}}\sqrt{33.75}}{(s+1.5)^2+33.75} = \frac{0.17213\sqrt{33.75}}{(s+1.5)^2+33.75} = \frac{0.17213\cdot 5.8095}{(s+1.5)^2+33.75}$$

Taking the inverse Laplace transform

$$C_{a1}(t) = 0.17213\, e^{-1.5t} \sin 5.8095t$$

(2) $$C_{a2}(s)s = \frac{2}{s(s^2+3s+36)} = \frac{1}{18}\frac{1}{s} - \frac{\frac{1}{18}s + \frac{1}{6}}{s^2+3s+36} =$$

$$\frac{1}{18}\frac{1}{s} - \frac{\frac{1}{18}\left(s+\frac{3}{2}\right) + \frac{0.083333}{\sqrt{33.75}}\sqrt{33.75}}{\left(s+\frac{3}{2}\right)^2+33.75}$$

$$= 0.055556\frac{1}{s} - \frac{0.055556\left(s+\frac{3}{2}\right) + 0.014344\sqrt{33.75}}{\left(s+\frac{3}{2}\right)^2+33.75}$$

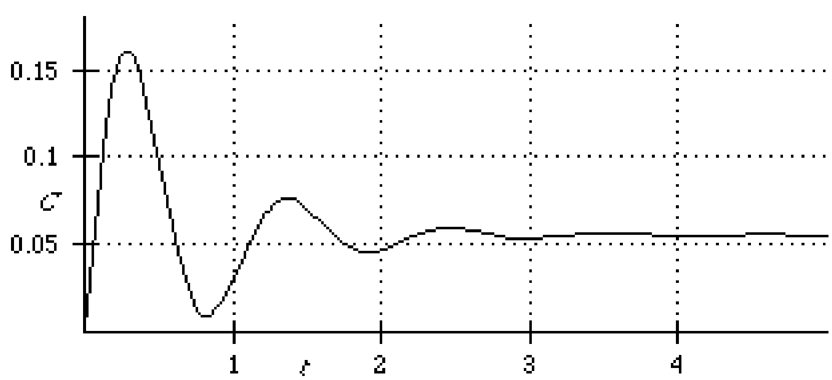
Taking the inverse Laplace transform

$$C_{a2}(t) = 0.055556 - e^{-1.5t}(0.055556 \cos 5.809t + 0.014344 \sin 5.809t)$$

The total response is found as follows:

$C_{at}(t) = C_{a1}(t) + C_{a2}(t) = 0.055556 - e^{-1.5t}(0.055556 \cos 5.809t - 0.157786 \sin 5.809t)$

Plotting the total response:



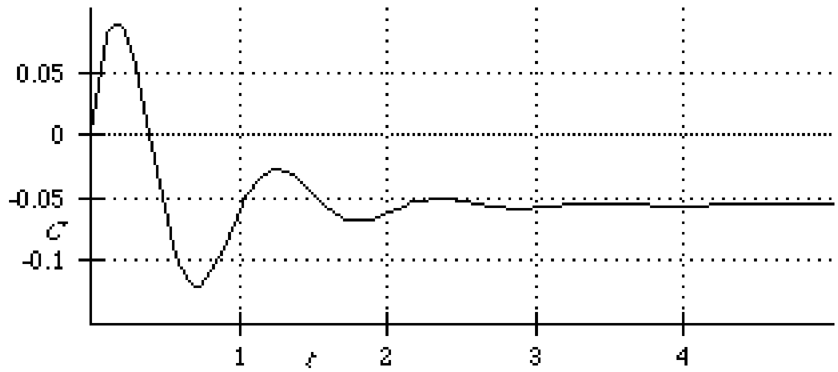
b.

(1) Same as (1) from part (a), or $C_{b1}(t) = C_{a1}(t)$

(2) Same as the negative of (2) of part (a), or $C_{b2}(t) = -C_{a2}(t)$

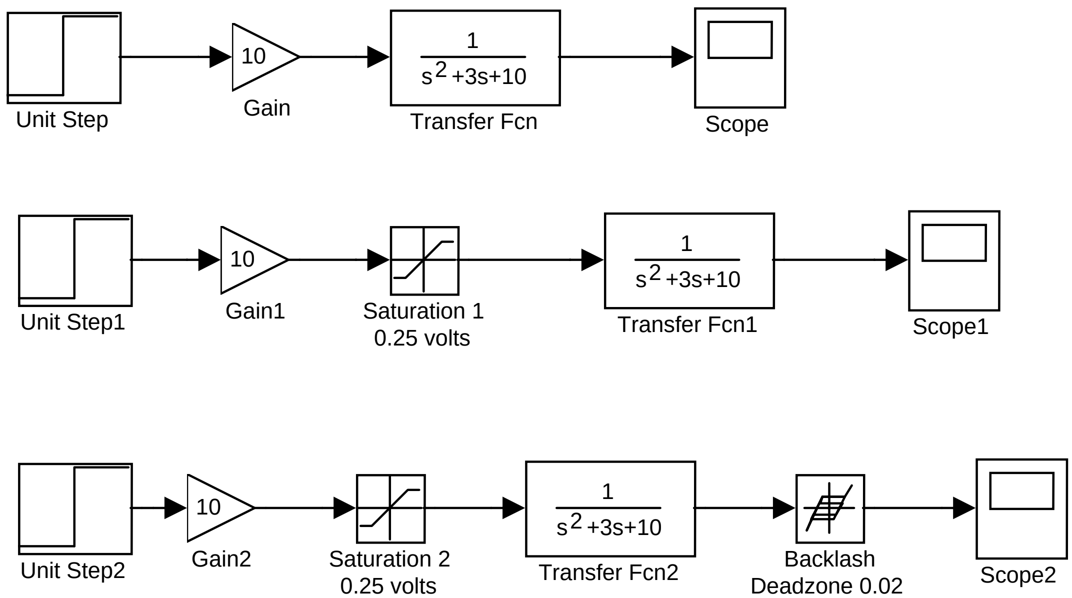
The total response is

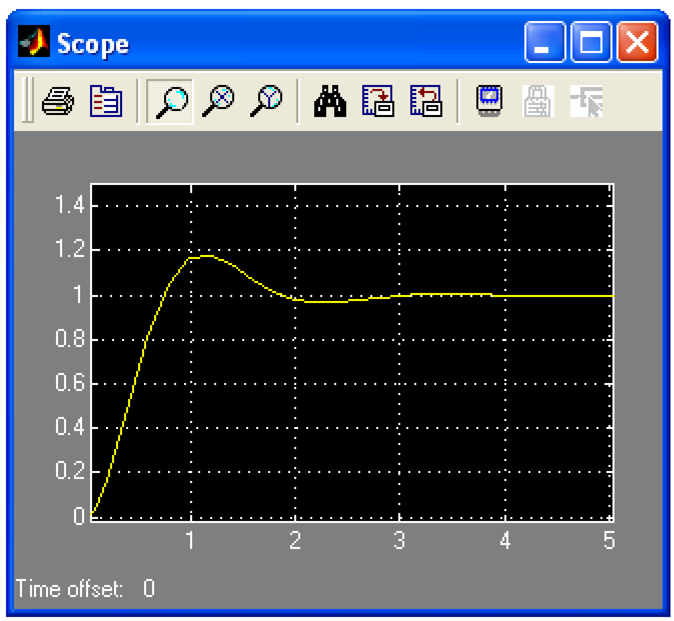
$C_{bt}(t) = C_{b1}(t) + C_{b2}(t) = C_{a1}(t) - C_{a2}(t) = -0.055556 + e^{-1.5t}$ (0.055556 cos 5.809t + 0.186474 sin 5.809t)

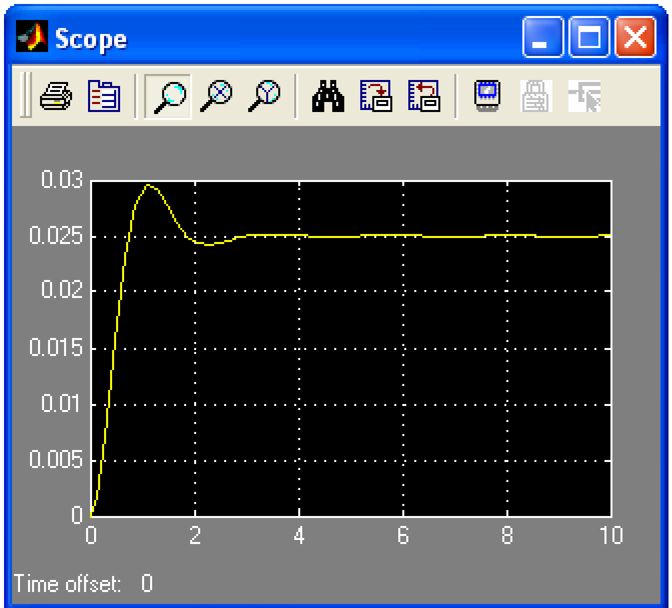


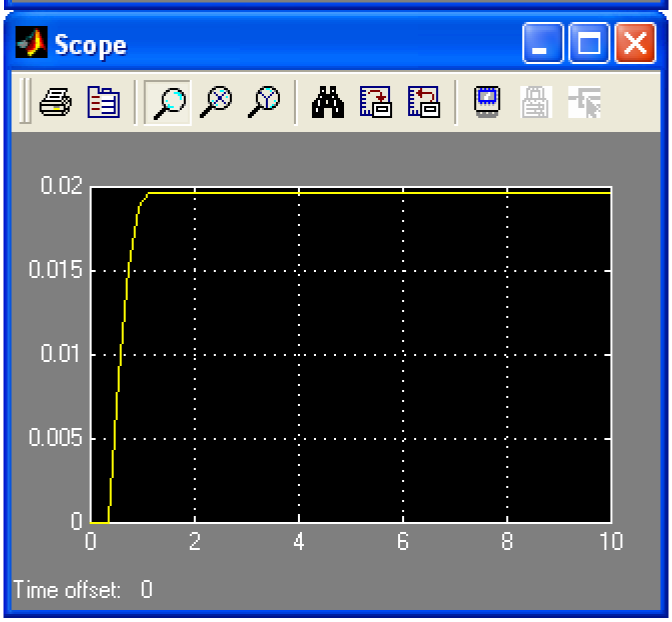
Notice the nonminimum phase behavior for $C_{bt}(t)$.

34.


10
1
s 2 +3s+10
Unit Step
Gain
Transfer Fcn
Scope
10
1
s 2 +3s+10
Unit Step1
Gain1
Saturation 1
0.25 volts
Transfer Fcn1
Scope1
10
1
s 2 +3s+10
Unit Step2
Gain2
Saturation 2
0.25 volts
Transfer Fcn2
Backlash
Deadzone 0.02
Scope2


Scope
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
Time offset: 0





35.

$$s\mathbf{I} - \mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ -3 & -5 \end{bmatrix} = \begin{bmatrix} (s+2) & 1 \\ 3 & (s+5) \end{bmatrix}$$

$$|s\mathbf{I} - \mathbf{A}| = s^2 + 7s + 7$$

Factoring yields poles at –5.7913 and –1.2087.

36.

a.

$$s\mathbf{I}-\mathbf{A}=s\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0&2&3\\0&6&5\\1&4&2\end{bmatrix}=\begin{bmatrix}s&-2&-3\\0&(s-6)&-5\\-1&-4&(s-2)\end{bmatrix}$$

$$|s\mathbf{I}-\mathbf{A}|=s^3-8s^2-11s+8$$

b. Factoring yields poles at 9.111, 0.5338, and –1.6448.

37.

$$\mathbf{x}=(s\mathbf{I}-\mathbf{A})^{-1}(\mathbf{x_0}+\mathbf{B}u)$$

$$\mathbf{x}=\left(s\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}1&2\\-3&-1\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}2\\1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\cdot 3\frac{1}{s^2+9}\right)$$

$$\mathbf{x}=\begin{pmatrix}\dfrac{2s^3+4s^2+21s+45}{[s^2+9][s^2+5]}\\[2ex]\dfrac{s^3-7s^2+12s-75}{[s^2+9][s^2+5]}\end{pmatrix}$$

$$y=(1,2)\mathbf{x}$$

$$y=\left(\frac{4s^3-10s^2+45s-105}{[s^2+9][s^2+5]}\right)$$

38.

$$\mathbf{x}=(s\mathbf{I}-\mathbf{A})^{-1}(\mathbf{x_0}+\mathbf{B}u)$$

$$\mathbf{x}=\left(s\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0&1&0\\-2&-4&1\\0&0&-6\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}\frac{1}{s+1}\right)$$

$$\mathbf{x}=\begin{pmatrix}\dfrac{1}{[s+6][s+1][s+0.58579][s+3.4142]}\\[2ex]\dfrac{s}{[s+6][s+1][s+0.58579][s+3.4142]}\\[2ex]\dfrac{s^2+4s+2}{[s+6][s+1][s+0.58579][s+3.4142]}\end{pmatrix}$$

$$y=(1,0,0)\mathbf{x}$$

$$y=\left(\frac{1}{[s+6][s+1][s+0.58579][s+3.4142]}\right)$$

39.

$$\mathbf{x}=(s\mathbf{I}-\mathbf{A})^{-1}(\mathbf{x_0}+\mathbf{B}u)$$

$$\mathbf{x}=\left(s\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}-2&0\\-1&-1\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{s}\right)$$

$$\mathbf{x}=\begin{pmatrix}\dfrac{s+1}{s[s+2]}\\[2ex]\dfrac{1}{s[s+1][s+2]}\end{pmatrix}$$

$$y(s) = (0, 1)\,x$$
$$y(s) = \left(\frac{1}{s\,[s+1]\,[s+2]}\right)$$
$$y(s) = \left(\frac{1}{2}\,\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\,\frac{1}{s+2}\right)$$
$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}\,e^{-2t}$$

40.

$$\mathbf{x} = (s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{x_0} + \mathbf{B}u)$$

$$\mathbf{x} = \left(s\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}-3 & 1 & 0\\ 0 & -6 & 1\\ 0 & 0 & -5\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}\frac{1}{s}\right)$$

$$\mathbf{x} = \begin{bmatrix}\dfrac{1}{s(s+3)(s+5)}\\ \dfrac{1}{s(s+5)}\\ \dfrac{1}{s(s+5)}\end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix}\dfrac{1}{15} - \dfrac{1}{6}e^{-3t} + \dfrac{1}{10}e^{-5t}\\ \dfrac{1}{5} - \dfrac{1}{5}e^{-5t}\\ \dfrac{1}{5} - \dfrac{1}{5}e^{-5t}\end{bmatrix}$$

$$y(t) = \begin{bmatrix}0 & 1 & 1\end{bmatrix}\mathbf{x} = \frac{2}{5} - \frac{2}{5}e^{-5t}$$

41.

Program:

```
A=[-3 1 0;0 -6 1;0 0 -5];
B=[0;1;1];
C=[0 1 1];
D=0;
S=ss(A,B,C,D)
step(S)
```

Computer response:

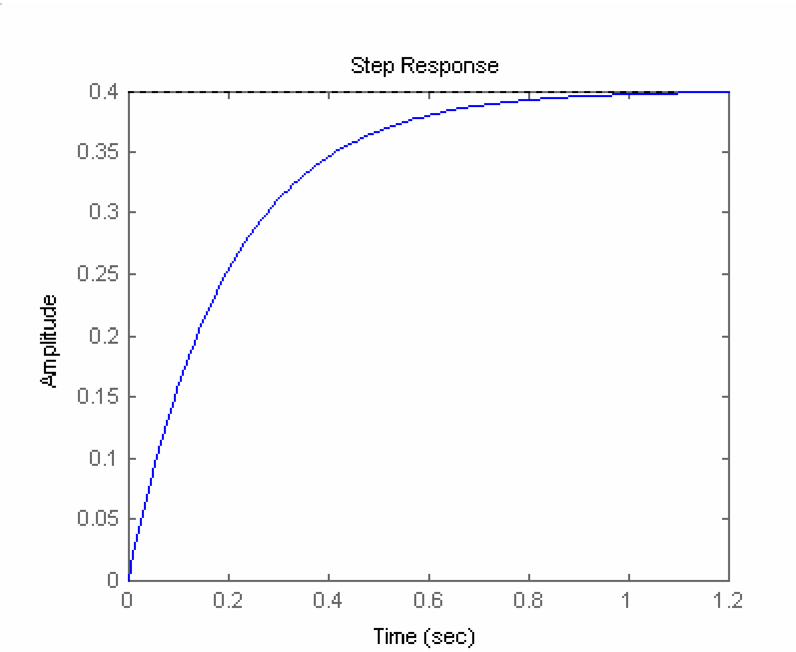
```
a =
        x1  x2  x3
   x1  -3   1   0
   x2   0  -6   1
   x3   0   0  -5
```

```
b =
        u1
   x1   0
   x2   1
   x3   1


c =
        x1  x2  x3
   y1   0   1   1


d =
        u1
   y1   0

Continuous-time model.
```



42.

Program:

```
syms s                      %Construct symbolic object for
                            %frequency variable 's'.
'a'                         %Display label
A=[-3 1 0;0 -6 1;0 0 -5]    %Create matrix A.
B=[0;1;1];                  %Create vector B.
C=[0 1 1];                  %Create C vector
X0=[1;1;0]                  %Create initial condition vector,X(0).
U=1/s;                      %Create U(s).
I=[1 0 0;0 1 0;0 0 1];      %Create identity matrix.
X=((s*I-A)^-1)*(X0+B*U);    %Find Laplace transform of state vector.
x1=ilaplace(X(1))           %Solve for X1(t).
x2=ilaplace(X(2))           %Solve for X2(t).
x3=ilaplace(X(3))           %Solve for X3(t).
y=C*[x1;x2;x3]              %Solve for output, y(t).
y=simplify(y)               %Simplify y(t).
'y(t)'                      %Display label.
pretty(y)                   %Pretty print y(t).
```

Computer response:

```
ans =

a
```

```
A =
    -3     1     0
     0    -6     1
     0     0    -5

X0 =
     1
     1
     0
x1 =

7/6*exp(-3*t)-1/3*exp(-6*t)+1/15+1/10*exp(-5*t)

x2 =

exp(-6*t)+1/5-1/5*exp(-5*t)

x3 =

1/5-1/5*exp(-5*t)

y =

2/5+exp(-6*t)-2/5*exp(-5*t)

y =

2/5+exp(-6*t)-2/5*exp(-5*t)

ans =

y(t)
                        2/5 + exp(-6 t) - 2/5 exp(-5 t)
```

43.

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^2 + 5\lambda + 1$$

$$|\lambda \mathbf{I} - \mathbf{A}| = (\lambda + 0.20871)(\lambda + 4.7913)$$

$$P = -0.20871$$

$$Q = -4.7913$$

$$\Phi = \begin{pmatrix} A_1 e^{-0.20871t} + A_2 e^{-4.7913t} & A_5 e^{-0.20871t} + A_6 e^{-4.7913t} \\ A_3 e^{-0.20871t} + A_4 e^{-4.7913t} & A_7 e^{-0.20871t} + A_8 e^{-4.7913t} \end{pmatrix}$$

$$\Phi_0 = \begin{pmatrix} A_2 + A_1 & A_6 + A_5 \\ A_4 + A_3 & A_8 + A_7 \end{pmatrix}$$

$$\frac{\partial}{\partial t}\Phi = \begin{pmatrix} -0.20871 A_1 e^{-0.20871t} - 4.7913 A_2 e^{-4.7913t} & -0.20871 A_5 e^{-0.20871t} - 4.7913 A_6 e^{-4.7913t} \\ -0.20871 A_3 e^{-0.20871t} - 4.7913 A_4 e^{-4.7913t} & -0.20871 A_7 e^{-0.20871t} - 4.7913 A_8 e^{-4.7913t} \end{pmatrix}$$

$$\frac{d}{\partial t}\Phi_0 = \begin{pmatrix} -4.7913 A_2 - 0.20871 A_1 & -4.7913 A_6 - 0.20871 A_5 \\ -4.7913 A_4 - 0.20871 A_3 & -4.7913 A_8 - 0.20871 A_7 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} A_2 + A_1 & A_6 + A_5 \\ A_4 + A_3 & A_8 + A_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -4.7913 A_2 - 0.20871 A_1 & -4.7913 A_6 - 0.20871 A_5 \\ -4.7913 A_4 - 0.20871 A_3 & -4.7913 A_8 - 0.20871 A_7 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -5 \end{pmatrix}$$

Solving for A_i's two at a time, and substituting into the state-transition matrix

$$\Phi = \begin{pmatrix} 1.0455\,e^{-0.20871\,t} - 0.045545\,e^{-4.7913\,t} & 0.21822\,e^{-0.20871\,t} - 0.21822\,e^{-4.7913\,t} \\ -0.21822\,e^{-0.20871\,t} + 0.21822\,e^{-4.7913\,t} & -0.045545\,e^{-0.20871\,t} + 1.0455\,e^{-4.7913\,t} \end{pmatrix}$$

To find **x**(t),

$$x = \Phi\, x_0$$

$$x = \begin{pmatrix} 1.0455\,e^{-0.20871\,t} - 0.045545\,e^{-4.7913\,t} & 0.21822\,e^{-0.20871\,t} - 0.21822\,e^{-4.7913\,t} \\ -0.21822\,e^{-0.20871\,t} + 0.21822\,e^{-4.7913\,t} & -0.045545\,e^{-0.20871\,t} + 1.0455\,e^{-4.7913\,t} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} 1.0455\,e^{-0.20871\,t} - 0.045545\,e^{-4.7913\,t} \\ -0.21822\,e^{-0.20871\,t} + 0.21822\,e^{-4.7913\,t} \end{pmatrix}$$

To find the output,

$$y = (1,2)\,x$$

$$y = (1,2)\begin{pmatrix} 1.0455\,e^{-0.20871\,t} - 0.045545\,e^{-4.7913\,t} \\ -0.21822\,e^{-0.20871\,t} + 0.21822\,e^{-4.7913\,t} \end{pmatrix}$$

$$y = \left(0.60911\,e^{-0.20871\,t} + 0.39089\,e^{-4.7913\,t}\right)$$

44.

$$\lambda I - A = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$$

$|\lambda\mathbf{I} - \mathbf{A}| = \lambda^2 + 1$

$$\Phi = \begin{pmatrix} A_1 \cos[t] + A_2 \sin[t] & A_5 \cos[t] + A_6 \sin[t] \\ A_3 \cos[t] + A_4 \sin[t] & A_7 \cos[t] + A_8 \sin[t] \end{pmatrix}$$

$$\frac{d}{\partial t}\Phi = \begin{pmatrix} A_2 \cos[t] - A_1 \sin[t] & A_6 \cos[t] - A_5 \sin[t] \\ A_4 \cos[t] - A_3 \sin[t] & A_8 \cos[t] - A_7 \sin[t] \end{pmatrix}$$

$$\Phi_0 = \begin{pmatrix} A_1 & A_5 \\ A_3 & A_7 \end{pmatrix}$$

$$\frac{d}{\partial t}\Phi_0 = \begin{pmatrix} A_2 & A_6 \\ A_4 & A_8 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & A_5 \\ A_3 & A_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A_2 & A_6 \\ A_4 & A_8 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Solving for the A_i's and substituting into the state-transition matrix,

$$\Phi = \begin{pmatrix} \cos[t] & \sin[t] \\ -\sin[t] & \cos[t] \end{pmatrix}$$

To find the state vector,

$$x = \int_0^t (\Phi[t-\tau]\, B\, u[\tau])\, d\tau$$

$$x = \int_0^t \left(\begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) d\tau$$

$$x = \int_0^t \begin{pmatrix} \sin[t-\tau] \\ \cos[t-\tau] \end{pmatrix} d\tau$$

$$t - \tau = \theta$$

$$x = \int_t^0 \begin{pmatrix} -\sin[\theta] \\ -\cos[\theta] \end{pmatrix} d\theta$$

$$x = \begin{pmatrix} 1-\cos[t] \\ \sin[t] \end{pmatrix}$$

$$y = (3, 2)\, x$$

$$\triangle\; y = (3, 2) \begin{pmatrix} 1-\cos[t] \\ \sin[t] \end{pmatrix}$$

$$y = (-3\cos[t] + 2\sin[t] + 3)$$

45.

$$|\lambda \mathbf{I} - \mathbf{A}| = (\lambda + 2)\,(\lambda + 0.5 - 2.3979i)\,(\lambda + 0.5 + 2.3979i)$$

Let the state-transition matrix be

$$\phi = \begin{bmatrix} A_1 e^{-.5t}\cos(2.3979t) + A_2 e^{-.5t}\sin(2.3979t) + A_3 e^{-2t} & A_{10} e^{-.5t}\cos(2.3979t) + A_{11} e^{-.5t}\sin(2.3979t) + A_{12} & \bullet \\ A_4 e^{-.5t}\cos(2.3979t) + A_5 e^{-.5t}\sin(2.3979t) + A_6 e^{-2t} & A_{13} e^{-.5t}\cos(2.3979t) + A_{14} e^{-.5t}\sin(2.3979t) + A_{15} & \bullet \\ A_7 e^{-.5t}\cos(2.3979t) + A_8 e^{-.5t}\sin(2.3979t) + A_9 e^{-2t} & A_{16} e^{-.5t}\cos(2.3979t) + A_{17} e^{-.5t}\sin(2.3979t) + A_{18} & \bullet \end{bmatrix}$$

Since $\phi(0) = \mathbf{I}$, $\dot{\Phi}(0) = \mathbf{A}$, and $\ddot{\phi}(0) = \mathbf{A}^2$, we can evaluate the coefficients, A_i's. Thus,

$$\begin{pmatrix} A_3 + A_1 & A_{12} + A_{10} & A_{21} + A_{19} \\ A_6 + A_4 & A_{15} + A_{13} & A_{24} + A_{22} \\ A_9 + A_7 & A_{18} + A_{16} & A_{27} + A_{25} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -2A_3 + 2.3979A_2 - 0.5A_1 & -2A_{12} + 2.3979A_{11} - 0.5A_{10} & -2A_{21} + 2.3979A_{20} - 0.5A_{19} \\ -2A_6 + 2.3979A_5 - 0.5A_4 & -2A_{15} + 2.3979A_{14} - 0.5A_{13} & -2A_{24} + 2.3979A_{23} - 0.5A_{22} \\ -2A_9 + 2.3979A_8 - 0.5A_7 & -2A_{18} + 2.3979A_{17} - 0.5A_{16} & -2A_{27} + 2.3979A_{26} - 0.5A_{25} \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 4A_3 - 2.3979A_2 - 5.4999A_1 & 4A_{12} - 2.3979A_{11} - 5.4999A_{10} & 4A_{21} - 2.3979A_{20} - 5.4999A_{19} \\ 4A_6 - 2.3979A_5 - 5.4999A_4 & 4A_{15} - 2.3979A_{14} - 5.4999A_{13} & 4A_{24} - 2.3979A_{23} - 5.4999A_{22} \\ 4A_9 - 2.3979A_8 - 5.4999A_7 & 4A_{18} - 2.3979A_{17} - 5.4999A_{16} & 4A_{27} - 2.3979A_{26} - 5.4999A_{25} \end{pmatrix} = \begin{pmatrix} 4 & -2 & 1 \\ 0 & -6 & -1 \\ 0 & 6 & -5 \end{pmatrix}$$

Solving for the A_i's taking three equations at a time,

$$\Phi = \begin{pmatrix} e^{-2t} & 0.125\,e^{-0.5t}\cos[2.3979\,t] + 0.33884\,e^{-0.5t}\sin[2.3979\,t] - 0.125\,e^{-2t} & -0.125\,e^{-0.5t}\cos[2.3979\,t] + 0.078194\,e^{-0.5t}\sin[2.3979\,t] + 0.125\,e^{-2t} \\ 0 & e^{-0.5t}\cos[2.3979\,t] + 0.20851\,e^{-0.5t}\sin[2.3979\,t] & 0.41703\,e^{-0.5t}\sin[2.3979\,t] \\ 0 & -2.502\,e^{-0.5t}\sin[2.3979\,t] & e^{-0.5t}\cos[2.3979\,t] - 0.20852\,e^{-0.5t}\sin[2.3979\,t] \end{pmatrix}$$

Using $\mathbf{x}(t) = \phi(t)\mathbf{x}(0) + \int_0^t \phi(t-\tau)\mathbf{B}\,\mathbf{u}(\tau)d\tau$, and $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}(t)$,

$$y = \int_0^t e^{-2(t-\tau)}\,d\tau$$

$$= \frac{1}{2} - \frac{1}{2}\,e^{-2t}$$

46.

Program:

```
syms s t tau                  %Construct symbolic object for
                              %frequency variable 's', 't', and 'tau.
'a'                           %Display label.
A=[-2 1 0;0 0 1;0 -6 -1]      %Create matrix A.
B=[1;0;0]                     %Create vector B.
C=[1 0 0]                     %Create vector C.
X0=[1;1;0]                    %Create initial condition vector,X(0).
I=[1 0 0;0 1 0;0 0 1];        %Create identity matrix.
'E=(s*I-A)^-1'                %Display label.
E=((s*I-A)^-1)                %Find Laplace transform of state
                              %transition matrix, (sI-A)^-1.
Fi11=ilaplace(E(1,1));        %Take inverse Laplace transform
Fi12=ilaplace(E(1,2));        %of each element
Fi13=ilaplace(E(1,3));
Fi21=ilaplace(E(2,1));
Fi22=ilaplace(E(2,2));
Fi23=ilaplace(E(2,3));
Fi31=ilaplace(E(3,1));
Fi32=ilaplace(E(3,2));        %to find state transition matrix.
Fi33=ilaplace(E(3,3));        %of (sI-A)^-1.
'Fi(t)'                       %Display label.
Fi=[Fi11 Fi12 Fi13            %Form Fi(t).
Fi21 Fi22 Fi23
Fi31 Fi32 Fi33];
pretty(Fi)                    %Pretty print state transition matrix, Fi.
Fitmtau=subs(Fi,t,t-tau);     %Form Fi(t-tau).
'Fi(t-tau)'                   %Display label.
pretty(Fitmtau)               %Pretty print Fi(t-tau).
x=Fi*X0+int(Fitmtau*B*1,tau,0,t);
                              %Solve for x(t).
x=simple(x);                  %Collect terms.
x=simplify(x);                %Simplify x(t).
x=vpa(x,3);
'x(t)'                        %Display label.
pretty(x)                     %Pretty print x(t).
y=C*x;                        %Find y(t)
y=simplify(y);
y=vpa(simple(y),3);
y=collect(y);
'y(t)'
pretty(y)                     %Pretty print y(t).
```

Computer response:

```
ans =

a



A =

    -2     1     0
     0     0     1
     0    -6    -1

B =

     1
     0
     0

C =

     1     0     0

X0 =

     1
     1
     0

ans =

E=(s*I-A)^-1

E =

[               1/(s+2), (s+1)/(s+2)/(s^2+s+6),     1/(s+2)/(s^2+s+6)]
[                     0,       (s+1)/(s^2+s+6),           1/(s^2+s+6)]
[                     0,          -6/(s^2+s+6),           s/(s^2+s+6)]

ans =

Fi(t)

       [                                       13
       [exp(-2 t) , - 1/8 exp(-2 t) + 1/8 %1 + --- %2 ,
       [                                       184

                                        ]
       1/8 exp(-2 t) - 1/8 %1 + 3/184 %2]
                                        ]

       [
       [0 , 1/23 %2 + %1 , - 1/23

            1/2                        1/2                            1/2
       (-23)    (exp((-1/2 + 1/2 (-23)   ) t) - exp((-1/2 - 1/2 (-23)   ) t))

       ]
       ]

       [
       [0 , 6/23

            1/2                        1/2                            1/2
       (-23)    (exp((-1/2 + 1/2 (-23)   ) t) - exp((-1/2 - 1/2 (-23)   ) t))
```

```
                    ]
        , - 1/23 %2 + %1]

                                    1/2
  %1 := exp(- 1/2 t) cos(1/2 23    t)

                            1/2           1/2
  %2 := exp(- 1/2 t) 23    sin(1/2 23    t)

ans =

Fi(t-tau)

        [
        [exp(-2 t + 2 tau) ,
        [

                                                       13       1/2
        - 1/8 exp(-2 t + 2 tau) + 1/8 %2 cos(%1) + --- %2 23    sin(%1) ,
                                                      184

                                                                1/2        ]
        1/8 exp(-2 t + 2 tau) - 1/8 %2 cos(%1) + 3/184 %2 23    sin(%1)]
                                                                           ]

        [              1/2                                   1/2
        [0 , 1/23 %2 23    sin(%1) + %2 cos(%1) , - 1/23 (-23)    (

                                  1/2
        exp((-1/2 + 1/2 (-23)    ) (t - tau))

                                     1/2            ]
         - exp((-1/2 - 1/2 (-23)    ) (t - tau)))]

        [              1/2                        1/2
        [0 , 6/23 (-23)    (exp((-1/2 + 1/2 (-23)    ) (t - tau))

                                     1/2
         - exp((-1/2 - 1/2 (-23)    ) (t - tau))) ,

                      1/2                         ]
        - 1/23 %2 23    sin(%1) + %2 cos(%1)]

                   1/2
  %1 := 1/2 23    (t - tau)

  %2 := exp(- 1/2 t + 1/2 tau)

ans =

x(t)

        [.375 exp(-2. t) + .125 exp(-.500 t) cos(2.40 t)

         + .339 exp(-.500 t) sin(2.40 t) + .500]

        [.209 exp(-.500 t) sin(2.40 t) + exp(-.500 t) cos(2.40 t)]

        [1.25 i (exp((-.500 + 2.40 i) t) - 1. exp((-.500 - 2.40 i) t))]

ans =

y(t)

  .375 exp(-2. t) + .125 exp(-.500 t) cos(2.40 t)
```

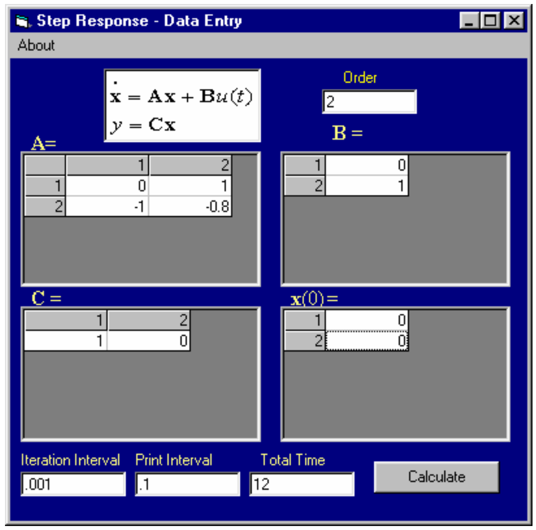
```
+ .339 exp(-.500 t) sin(2.40 t) + .500
```

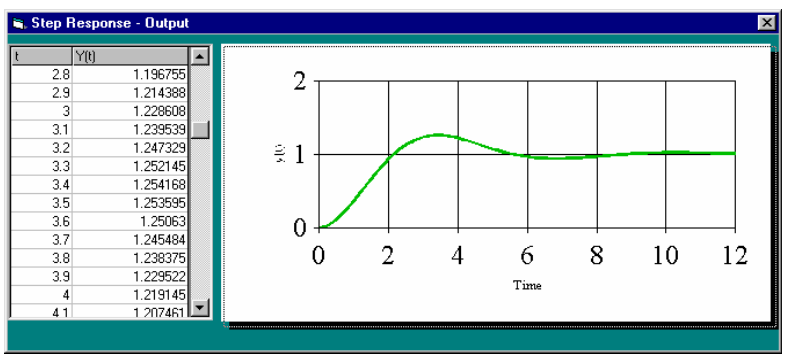
47.

The state-space representation used to obtain the plot is,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -0.8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t); \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

Using the Step Response software,



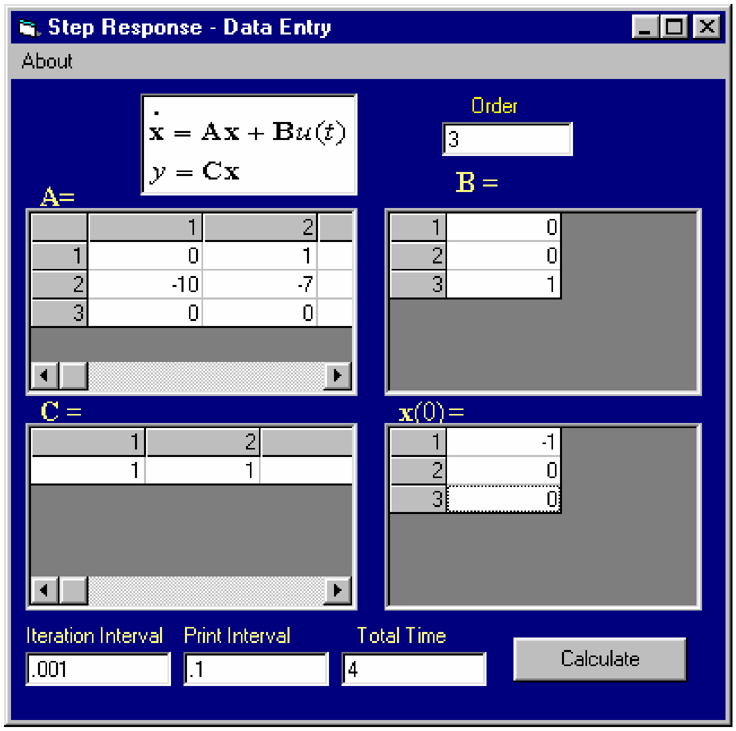


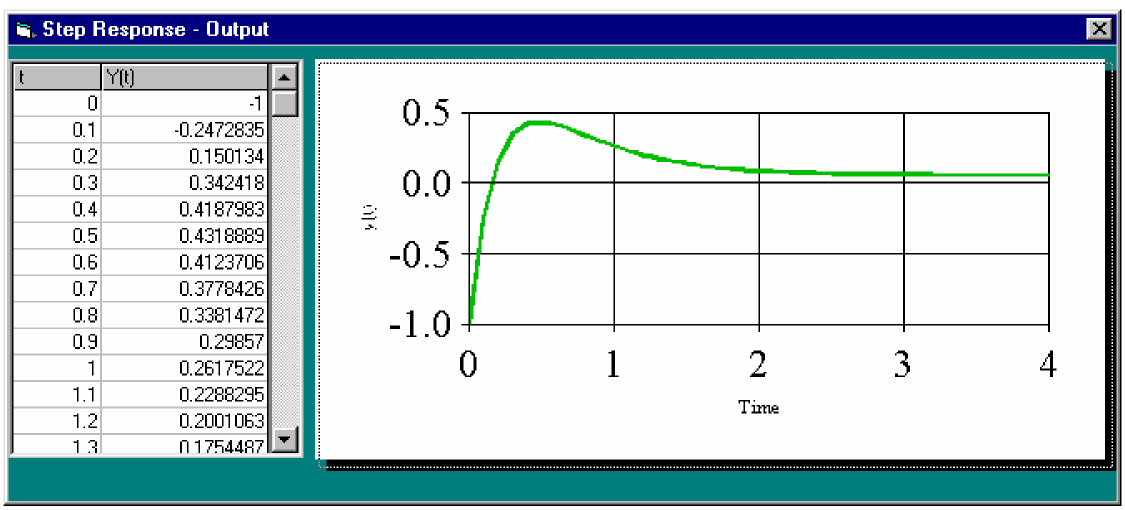
Calculating % overshoot, settling time, and peak time,

$2\zeta\omega_n = 0.8$, $\omega_n = 1$, $\zeta = 0.4$. Therefore, $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 25.38\%$, $T_s = \dfrac{4}{\zeta\omega_n} = 10$ sec,

$T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 3.43$ sec.

48.

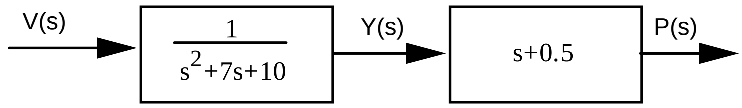




49.

a. $P(s) = \dfrac{s+0.5}{s(s+2)(s+5)} = \dfrac{1/20}{s} + \dfrac{1/4}{s+2} - \dfrac{3/10}{s+5}$. Therefore, $p(t) = \dfrac{1}{20} + \dfrac{1}{4}e^{-2t} - \dfrac{3}{10}e^{-5t}$.

b. To represent the system in state space, draw the following block diagram.



For the first block,

$$\ddot{y} + 7\dot{y} + 10y = v(t)$$

Let $x_1 = y$, and $x_2 = \dot{y}$. Therefore,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -10x_1 - 7x_2 + v(t)$$

Also,

$$p(t) = 0.5y + \dot{y} = 0.5x_1 + x_2$$

Thus,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1; \ p(t) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \mathbf{x}$$

c.

Program:

```
A=[0 1;-10 -7];
B=[0;1];
C=[.5 1];
D=0;
S=ss(A,B,C,D)
step(S)
```

Computer response:

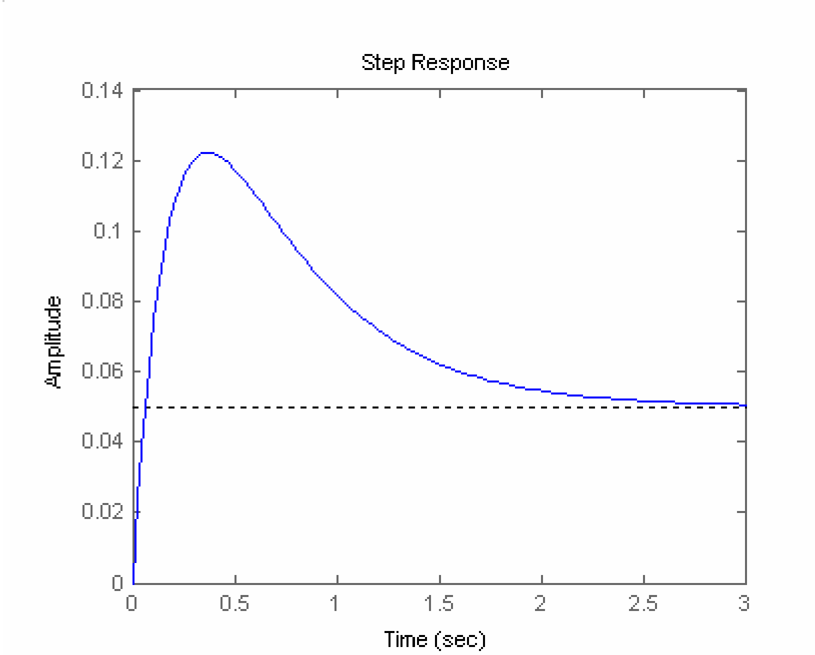
```
a =
          x1   x2
   x1      0    1
   x2    -10   -7


b =
        u1
   x1    0
   x2    1


c =
          x1   x2
   y1    0.5    1


d =
        u1
   y1    0

Continuous-time model.
```



50.

a. $\omega_n = \sqrt{10} = 3.16$; $2\zeta\omega_n = 4$. Therefore $\zeta = 0.632$. $\%OS = e^{-\xi\pi/\sqrt{1-\xi^2}} * 100 = 7.69\%$.

$T_s = \frac{4}{\xi\omega_n} = 2$ seconds. $T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 1.28$ seconds. From Figure 4.16, $T_r\omega_n = 1.93$.

Thus, $T_r = 0.611$ second. To justify second-order assumption, we see that the dominant poles are at $-2 \pm j2.449$. The third pole is at -10, or 5 times further. The second-order approximation is valid.

b. $G_e(s) = \frac{K}{(s+10)(s^2+4s+10)} = \frac{K}{s^3+14s^2+50s+100}$. Representing the system in phase-variable form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -100 & -50 & -14 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

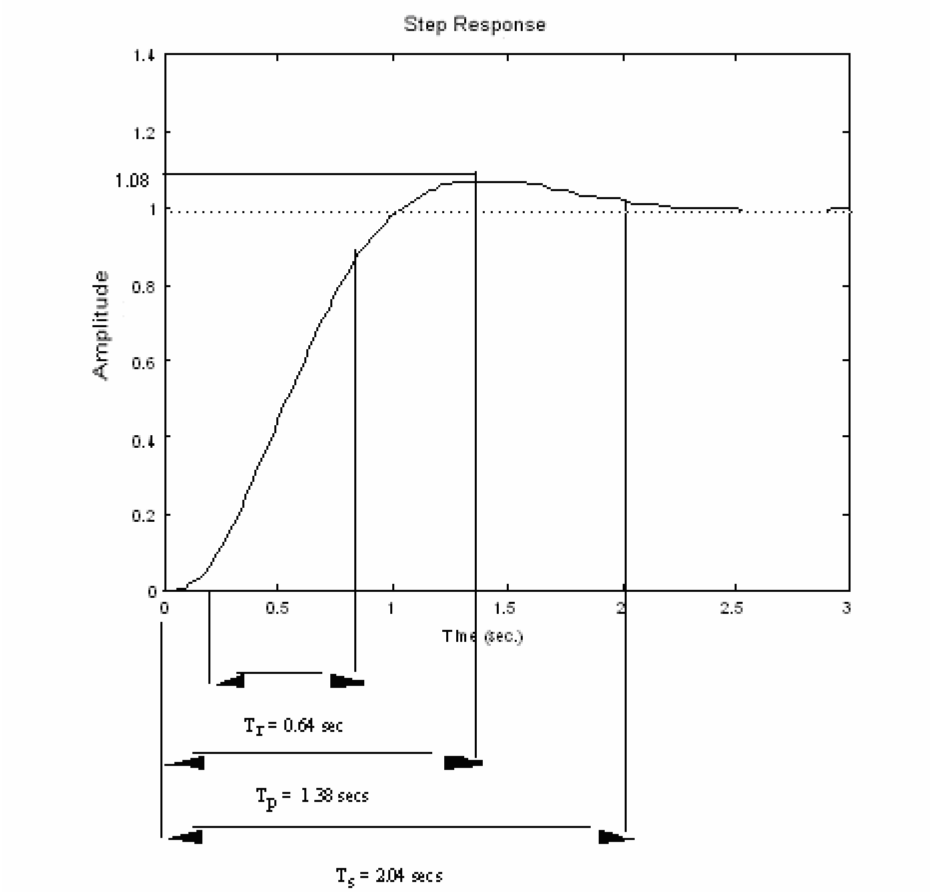
c.

Program:

```
numg=100;
deng=conv([1 10],[1 4 10]);
G=tf(numg,deng)
step(G)
```

Computer response:

```
Transfer function:
            100
-------------------------
s^3 + 14 s^2 + 50 s + 100
```



$$\%OS = \frac{(1.08-1)}{1} * 100 = 8\%$$

51.

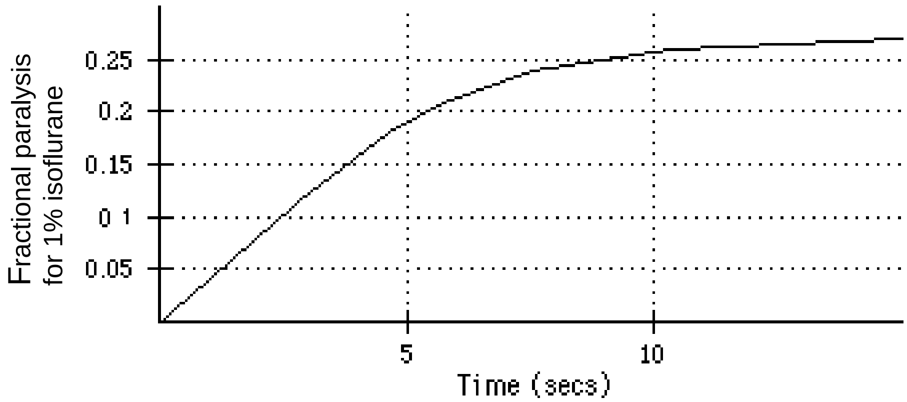
a. $\omega_n = \sqrt{0.28} = 0.529$; $2\zeta\omega_n = 1.15$. Therefore $\zeta = 1.087$.

b. $P(s) = U(s)\dfrac{7.63\text{x}10^{-2}}{s^2+1.15s+0.28}$, where $U(s) = \dfrac{2}{s}$. Expanding by partial fractions, $P(s) = \dfrac{0.545}{s}$ +

natural response terms. Thus percent paralysis = 54.5%.

c. $P(s) = \dfrac{7.63\text{x}10^{-2}}{s(s^2+1.15s+0.28)} = \dfrac{0.2725}{s} - \dfrac{0.48444}{s+0.35} + \dfrac{0.21194}{s+0.8}$.

Hence, $p(t) = 0.2725 - 0.48444e^{-0.35t} + 0.21194e^{-0.8t}$. Plotting,



d. $P(s) = \frac{K}{s} * \frac{7.63x10^{-2}}{s^2+1.15s+0.28} = \frac{1}{s}$ + natural response terms. Therefore, $\frac{7.63x10^{-2} K}{0.28} = 1$. Solving for K, K = 3.67%.

52.

a. Writing the differential equation,

$$\frac{dc(t)}{dt} = -k_{10}c(t) + \frac{i(t)}{V_d}$$

Taking the Laplace transform and rearranging,

$$(s+k_{10})C(s) = \frac{I(s)}{V_d}$$

from which the transfer function is found to be

$$\frac{C(s)}{I(s)} = \frac{\frac{1}{V_d}}{s+k_{10}}$$

For a step input, $I(s) = \frac{I_0}{s}$. Thus the response is

$$C(s) = \frac{\frac{I_0}{V_d}}{s(s+k_{10})} = \frac{I_0}{k_{10}V_d}\left(\frac{1}{s} - \frac{1}{s+k_{10}}\right)$$

Taking the inverse Laplace transform,

$$c(t) = \frac{I_0}{k_{10}V_d}(1-e^{-k_{10}t})$$

where the steady-state value, C_D, is

$$C_D = \frac{I_0}{k_{10}V_d}$$

Solving for $I_R = I_0$,

$$I_R = C_D k_{10} V_d$$

b. $T_r = \frac{2.2}{k_{10}}$; $T_s = \frac{4}{k_{10}}$

c. $I_R = C_D k_{10} V_d = 12 \frac{\mu g}{ml} \times 0.07\ hr^{-1} \times 0.6\ liters = 0.504 \frac{mg}{h}$

d. Using the equations of part b, where $k_{10} = 0.07$, $T_r = 31.43$ hrs, and $T_s = 57.14$ hrs.

SOLUTIONS TO DESIGN PROBLEMS

53.

Writing the equation of motion, $(f_v s+2)X(s)=F(s)$. Thus, the transfer function is $\frac{X(s)}{F(s)}=\frac{1/f_v}{s+\frac{2}{f_v}}$. Hence, $T_s=\frac{4}{a}=\frac{4}{\frac{2}{f_v}}=2f_v$, or $f_v=\frac{T_s}{2}$.

54.

The transfer function is, $F(s)=\frac{1/M}{s^2+\frac{1}{M}s+\frac{K}{M}}$. Now, $T_s=2=\frac{4}{|\text{Re}|}=\frac{4}{\frac{1}{2M}}=8M$. Thus, $M=\frac{1}{4}$. Substituting the value of M in the denominator of the transfer function yields, $s^2+4s+4K$. Identify the roots $s_{1,2}=-2\pm j2\sqrt{K-1}$. Using the imaginary part and substituting into the peak time equation yields $T_p=1=\frac{\pi}{|\text{Im}|}=\frac{\pi}{2\sqrt{K-1}}$, from which

$K=3.467$.

55.

Writing the equation of motion, $(Ms^2+f_v s+1)X(s)=F(s)$. Thus, the transfer function is $\frac{X(s)}{F(s)}=\frac{1/M}{s^2+\frac{f_v}{M}s+\frac{1}{M}}$. Since $T_s=10=\frac{4}{\zeta\omega_n}$, $\zeta\omega_n=0.4$. But, $\frac{f_v}{M}=2\zeta\omega_n=0.8$. Also, from Eq. (4.39) 30% overshoot implies ζ = 0.358. Hence, ω_n = 1.117. Now, 1/M = ω_n^2 = 1.248. Therefore, M = 0.801. Since $\frac{f_v}{M}=2\zeta\omega_n=0.8$, $f_v=0.641$.

56.

Writing the equation of motion: $(Js^2+s+K)\theta(s) = T(s)$. Therefore the transfer function is

$$\frac{\theta(s)}{T(s)}=\frac{\frac{1}{J}}{s^2+\frac{1}{J}s+\frac{K}{J}}.$$

$$\zeta=\frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2+\ln^2\left(\frac{\%OS}{100}\right)}}=0.358.$$

$$T_s=\frac{4}{\zeta\omega_n}=\frac{4}{\frac{1}{2J}}=8J=4.$$

Therefore $J=\frac{1}{2}$. Also, $T_s=4=\frac{4}{\zeta\omega_n}=\frac{4}{(0.358)\omega_n}$. Hence, ω_n = 2.793. Now, $\frac{K}{J}=\omega_n^2=7.803$. Finally, K = 3.901.

57.

Writing the equation of motion

$$[s^2+D(5)^2s+\frac{1}{4}(10)^2]\theta(s) = T(s)$$

The transfer function is

$$\frac{\theta(s)}{T(s)} = \frac{1}{s^2+25Ds+25}$$

Also,

$$\zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2+\ln^2(\frac{\%OS}{100})}} = 0.358$$

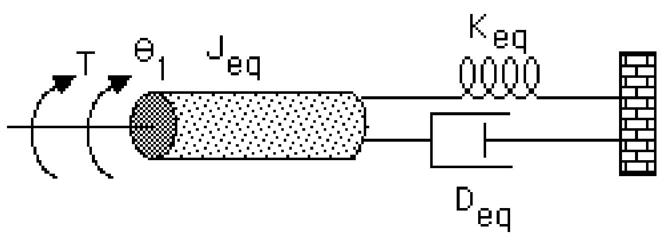
and

$$2\zeta\omega_n = 2(0.358)(5) = 25D$$

Therefore D = 0.14.

58.

The equivalent circuit is:



where $J_{eq} = 1+(\frac{N_1}{N_2})^2$; $D_{eq} = (\frac{N_1}{N_2})^2$; $K_{eq} = (\frac{N_1}{N_2})^2$. Thus,

$\frac{\theta_1(s)}{T(s)} = \frac{1}{J_{eq}s^2+D_{eq}s+K_{eq}}$. Letting $\frac{N_1}{N_2}$ = n and substituting the above values into the transfer function,

$\frac{\theta_1(s)}{T(s)} = \frac{\frac{1}{1+n^2}}{s^2+\frac{n^2}{1+n^2}s+\frac{n^2}{1+n^2}}$. Therefore, $\zeta\omega_n = \frac{n^2}{2(1+n^2)}$. Finally, $T_s = \frac{4}{\zeta\omega_n} = \frac{8(1+n^2)}{n^2} = 16$. Thus n = 1.

59.

Let the rotation of the shaft with gear N_2 be $\theta_L(s)$. Assuming that all rotating load has been reflected to the N_2 shaft, $(J_{eqL}s^2+D_{eqL}s+K)\theta_L(s)+F(s)r = T_{eq}(s)$, where F(s) is the force from the translational system, r = 2 is the radius of the rotational member, J_{eqL} is the equivalent inertia at the N_2 shaft, and D_{eqL} is the equivalent damping at the N_2 shaft. Since $J_{eqL} = 1(2)^2 + 1 = 5$ and D_{eqL} =

$1(2)^2 = 4$, the equation of motion becomes, $(5s^2+4s+K)\theta_L(s)+2F(s)=T_{eq}(s)$. For the translational system $(Ms^2+s)X(s)=F(s)$. Substituting *F(s)* into the rotational equation of motion, $(5s^2+4s+K)\theta_L(s)+(Ms^2+s)2X(s)=T_{eq}(s)$.

But, $\theta_L(s)=\dfrac{X(s)}{r}=\dfrac{X(s)}{2}$ and $T_{eq}(s)=2T(s)$. Substituting these quantities in the equation above yields $((5+4M)s^2+8s+K)\dfrac{X(s)}{4}=T(s)$. Thus, the transfer function is

$\dfrac{X(s)}{T(s)}=\dfrac{4/(5+4M)}{s^2+\dfrac{8}{(5+4M)}s+\dfrac{K}{(5+4M)}}$. Now, $T_s=10=\dfrac{4}{\text{Re}}=\dfrac{4}{\dfrac{8}{2(5+4M)}}=(5+4M)$.

Hence, *M* = 5/4. For 10% overshoot, $\zeta = 0.5912$ from Eq. (4.39). Hence,

$2\zeta\omega_n=\dfrac{8}{(5+4M)}=0.8$. Solving for ω_n yields $\omega_n = 0.6766$. But,

$\omega_n=\sqrt{\dfrac{K}{(5+4M)}}=\sqrt{\dfrac{K}{10}}=0.6766$. Thus, K = 4.578.

60.

The transfer function for the capacitor voltage is $\dfrac{V_C(s)}{V(s)}=\dfrac{\frac{1}{Cs}}{R+Ls+\frac{1}{Cs}}=\dfrac{10^6}{s^2+Rs+10^6}$.

For 20% overshoot, $\zeta=\dfrac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2+\ln^2(\frac{\%OS}{100})}}$ = 0.456. Therefore, $2\zeta\omega_n = R = 2(0.456)(10^3)$ = 912Ω.

61.

Solving for the capacitor voltage using voltage division, $V_C(s)=V_i(s)\dfrac{1/(CS)}{R+LS+\dfrac{1}{CS}}$. Thus, the transfer function is $\dfrac{V_C(s)}{V_i(s)}=\dfrac{1/(LC)}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}}$. Since $T_s=\dfrac{4}{|\text{Re}|}=10^{-3}$, $|\text{Re}|=\dfrac{R}{2L}=4000$. Thus $R=8\ \text{K}\Omega$. Also, since 20% overshoot implies a damping ratio of 0.46 and $2\zeta\omega_n=8000$, $\omega_n=8695.65=\dfrac{1}{\sqrt{LC}}$. Hence, $C=0.013\ \mu\text{F}$.

62.

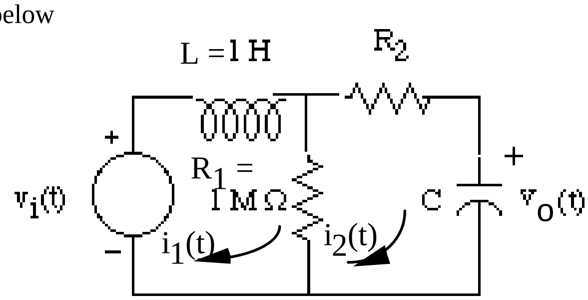
Using voltage division the transfer function is,

$$\frac{V_C(s)}{V_i(s)}=\frac{\frac{1}{Cs}}{R+Ls+\frac{1}{Cs}}=\frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Also, $T_s = 2x10^{-3} = \frac{4}{\mathrm{Re}} = \frac{4}{\frac{R}{2L}} = \frac{8L}{R}$. Thus, $\frac{R}{L} = 4000$. Using Eq. (4.39) with 15% overshoot, ζ = 0.5169. But, $2\zeta\omega_n = R/L$. Thus, $\omega_n = 3869 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{L(10^{-5})}}$. Therefore, L = 6.7 mH and R = 26.72 Ω.

63.

For the circuit shown below



write the loop equations as

$$(R_1 + Ls)I_1(s) - R_1 I_2(s) = V_i(s)$$
$$-R_1 I_1(s) + \left(R_1 + R_2 + \frac{1}{Cs}\right) I_2(s) = 0$$

Solving for $I_2(s)$

$$I_2(s) = \frac{\left|\begin{pmatrix} R_1 + Ls & V_i(s) \\ -R_1 & 0 \end{pmatrix}\right|}{\left|\begin{pmatrix} R_1 + Ls & -R_1 \\ -R_1 & R_1 + R_2 + \frac{1}{Cs} \end{pmatrix}\right|}$$

But, $V_o(s) = \frac{1}{Cs} I_2(s)$. Thus,

$$\frac{V_o(s)}{V_i(s)} = \frac{R_1}{(R_2 + R_1)CLs^2 + (CR_2R_1 + L)s + R_1}$$

Substituting component values,

$$\frac{V_o(s)}{V_i(s)} = 1000000 \frac{\frac{1}{(R_2 + 1000000)C}}{s^2 + \frac{(1000000CR_2 + 1)}{(R_2 + 1000000)C}s + 1000000\frac{1}{(R_2 + 1000000)C}}$$

For 15% overshoot, ζ = 0.517. For T_s = 0.001, $\zeta\omega_n = \frac{4}{0.001}$ = 4000. Hence, ω_n = 7736.9. Thus,

$$1000000 \frac{1}{(R_2 + 1000000)C} = 7736.9^2$$

or,

$$C = 0.016706 \frac{1}{R_2 + 1000000} \quad (1)$$

Also,

$$\frac{1000000\,C\,R_2 + 1}{(R_2 + 1000000)\,C} = 8000 \quad (2)$$

Solving (1) and (2) simultaneously, R_2 = 8003.7 Ω, and C = 1.6573 x 10^{-2} μF.

64.

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} (3.45 - 14000K_c) & -0.255x10^{-9} \\ 0.499x10^{11} & -3.68 \end{bmatrix}$$

$$= \begin{bmatrix} s - (3.45 - 14000K_c) & 0.255x10^{-9} \\ -0.499x10^{11} & s + 3.68 \end{bmatrix}$$

$$|s\mathbf{I} - \mathbf{A}| = s^2 + (0.23 + 0.14x10^5 K_c)s + (51520K_c + 0.0285)$$

$$(2\zeta\omega_n)^2 = [2 * 0.9]^2 * (51520K_c + 0.0285) = (0.23 + 0.14x10^5 K_c)^2$$

or

$$K_c^{\ 2} - 8.187x10^{-4} K_c - 2.0122x10^{-10} = 0$$

Solving for K_c,

$$K_c = 8.189x10^{-4}$$

65.

a. The transfer function from Chapter 2 is,

$$\frac{Y_h(s) - Y_{cat}(s)}{F_{up}(s)} = \frac{0.7883(s + 53.85)}{(s^2 + 15.47s + 9283)(s^2 + 8.119s + 376.3)}$$

The dominant poles come from $s^2 + 8.119s + 376.3$. Using this polynomial, $2\zeta\omega_n = 8.119$, and $\omega_n^2 = 376.3$. Thus, $\omega_n = 19.4$ and $\zeta = 0.209$. Using Eq. (4.38), %OS = 51.05%. Also, $T_s = \frac{4}{\zeta\omega_n} = 0.985$ s, and $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.166$ s. To find rise time, use Figure 4.16. Thus, $\omega_n T_r = 1.2136$ or $T_r = 0.0626$ s.

b. The other poles have a real part of 15.47/2 = 7.735. Dominant poles have a real part of 8.119/2 = 4.06. Thus, 7.735/4.06 = 1.91. This is not at least 5 times.

c.

Program:

```
syms s
numg=0.7883*(s+53.85);
deng=(s^2+15.47*s+9283)*(s^2+8.119*s+376.3);
'G(s) transfer function'
G=vpa(numg/deng,3);
pretty(G)
numg=sym2poly(numg);
deng=sym2poly(deng);
G=tf(numg,deng)
```

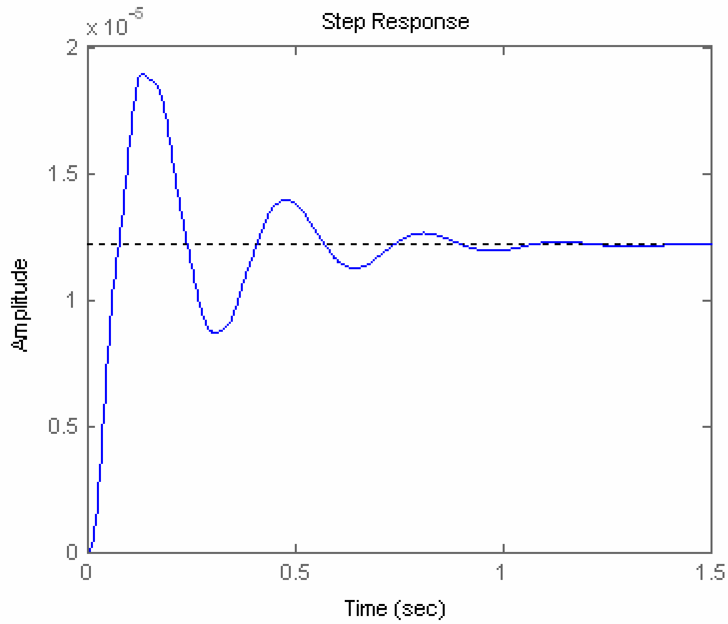
```
step(G)
```

Computer response:

```
ans =
G(s) transfer function

                    .788 s + 42.4
    -----------------------------------------
      2                      2
    (s   + 15.5 s + 9280.) (s   + 8.12 s + 376.)

Transfer function:
                 0.7883 s + 42.45
----------------------------------------------------
s^4 + 23.59 s^3 + 9785 s^2 + 8.119e004 s + 3.493e006
```



The time response shows 58 percent overshoot, T_s = 0.86 s, T_p = 0.13 s, T_r = 0.05 s.

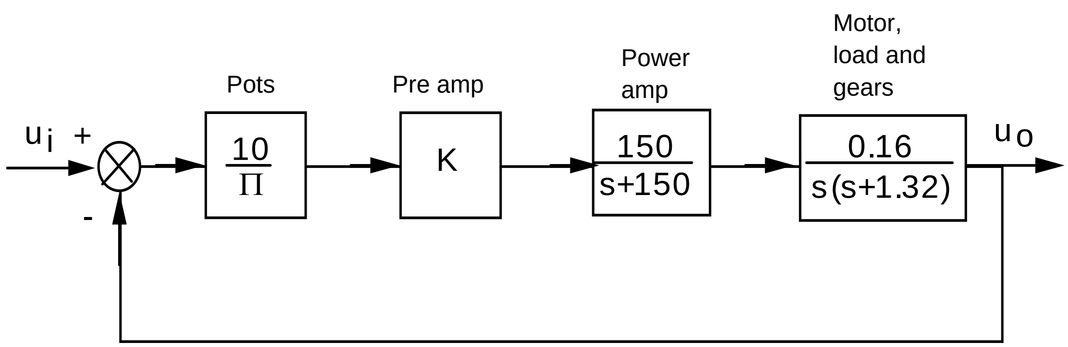
F I V E

Reduction of Multiple Subsystems

SOLUTIONS TO CASE STUDIES CHALLENGES

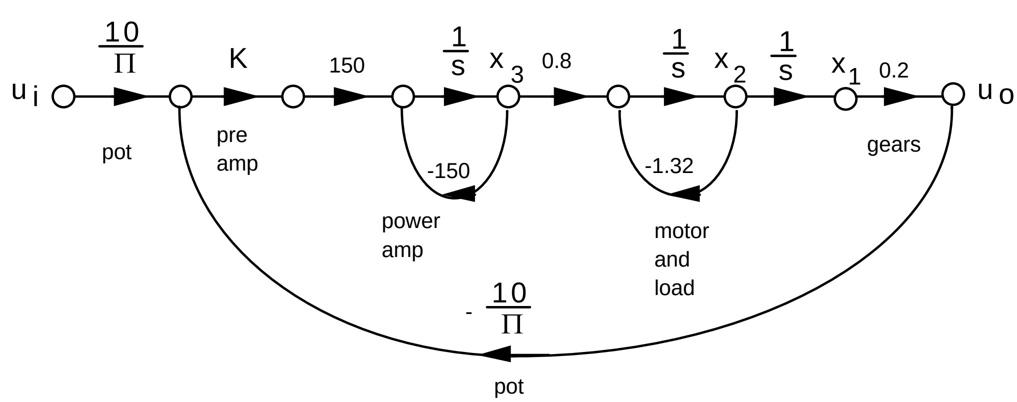
Antenna Control: Designing a Closed-Loop Response

a. Drawing the block diagram of the system:



Thus, $T(s) = \dfrac{76.39K}{s^3+151.32s^2+198s+76.39K}$

b. Drawing the signal flow-diagram for each subsystem and then interconnecting them yields:



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -1.32x_2 + 0.8x_3$$
$$\dot{x}_3 = -150x_3 + 150K\left(\frac{10}{\pi}(q_i - 0.2x_1)\right) = -95.49Kx_1 - 150x_3 + 477.46K\theta_i$$
$$\theta_o = 0.2x_1$$

In vector-matrix notation,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1.32 & 0.8 \\ -95.49K & 0 & -150 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 477.46K \end{bmatrix}\theta_i$$

$$\theta_o = \begin{bmatrix} 0.2 & 0 & 0 \end{bmatrix}\mathbf{x}$$

c. $T_1 = \left(\frac{10}{\pi}\right)(K)(150)\left(\frac{1}{s}\right)(0.8)\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(0.2) = \frac{76.39}{s^3}$

$$G_{L1} = \frac{-150}{s};\ G_{L2} = \frac{-1.32}{s};\ G_{L3} = (K)(150)\left(\frac{1}{s}\right)(0.8)\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(0.2)\left(\frac{-10}{\pi}\right) = \frac{-76.39K}{s^3}$$

Nontouching loops:

$$G_{L1}G_{L2} = \frac{198}{s^2}$$

$$\Delta = 1 - [G_{L1} + G_{L2} + G_{L3}] + [G_{L1}G_{L2}] = 1 + \frac{150}{s} + \frac{1.32}{s} + \frac{76.39K}{s^3} + \frac{198}{s^2}$$

$$\Delta_1 = 1$$

$$T(s) = \frac{T_1\Delta_1}{\Delta} = \frac{76.39K}{s^3+151.32s^2+198s+76.39K}$$

d. The equivalent forward path transfer function is $G(s) = \dfrac{\frac{10}{\pi}0.16K}{s(s+1.32)}$.

Therefore,

$$T(s) = \frac{2.55}{s^2+1.32s+2.55}$$

The poles are located at $-0.66 \pm j1.454$. $\omega_n = \sqrt{2.55} = 1.597$ rad/s; $2\zeta\omega_n = 1.32$, therefore, $\zeta = 0.413$.

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 24\%;\ T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.66} = 6.06 \text{ seconds};\ T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{1.454} =$$

2.16 seconds; Using Figure 4.16, the normalized rise time is 1.486. Dividing by the natural frequency, $T_r = \frac{1.486}{\sqrt{2.55}} = 0.93$ seconds.

e.

$$C(s) = \frac{2.55}{s(s^2 + 1.32s + 2.55)}$$

$$C(s) = \frac{1}{s} - \frac{1}{25}\,\frac{25s + 33}{s^2 + 1.32s + 2.55}$$

$$C(s) = \frac{1}{s} - \frac{1}{25}\,\frac{25(s + 0.66) + 11.347\sqrt{2.1144}}{(s + 0.66)^2 + \sqrt{2.1144}^2}$$

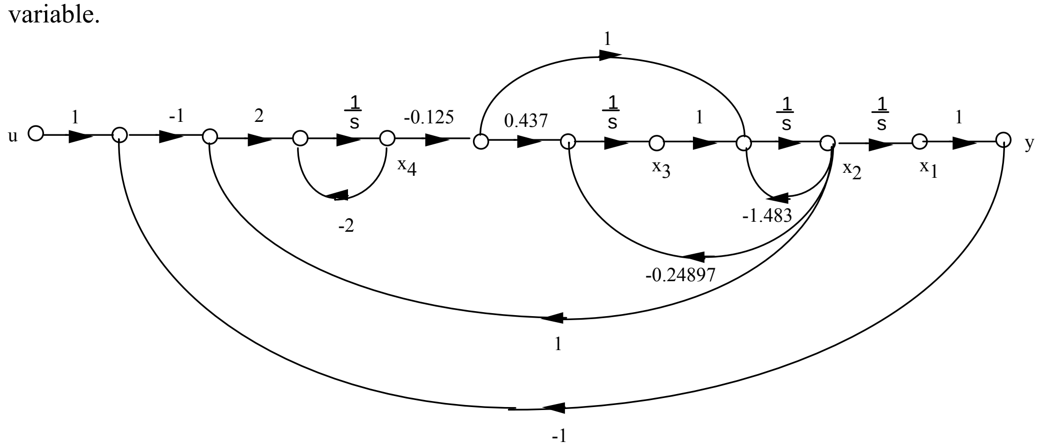
$$c(t) = 1 - e^{-0.66t}(\cos[1.454t] + 0.454\sin[1.454t])$$

f. Since $G(s) = \frac{0.51K}{s(s+1.32)}$, $T(s) = \frac{0.51K}{s^2+1.32s+0.51K}$. Also, $\zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} = 0.517$ for 15% overshoot; $\omega_n = \sqrt{0.51K}$; and $2\zeta\omega_n = 1.32$. Therefore, $\omega_n = \frac{1.32}{2\zeta} = \frac{1.32}{2(0.5147)} = 1.277 = \sqrt{0.51K}$.

Solving for K, K=3.2.

UFSS Vehicle: Pitch-Angle Control Representation

a. Use the observer canonical form for the vehicle dynamics so that the output yaw rate is a state variable.



b. Using the signal flow graph to write the state equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -1.483x_2 + x_3 - 0.125x_4$$
$$\dot{x}_3 = -0.24897x_2 - (0.125 * 0.437)x_4$$
$$\dot{x}_4 = 2x_1 + 2x_2 - 2x_4 - 2u$$

In vector-matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1.483 & 1 & -0.125 \\ 0 & -0.24897 & 0 & -0.054625 \\ 2 & 2 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}u$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

c.
Program:
```
numg1=-0.25*[1 0.437];
deng1=poly([-2 -1.29 -0.193 0]);
'G(s)'
G=tf(numg1,deng1)
numh1=[-1 0];
denh1=[0 1];
'H(s)'
H=tf(numh1,denh1)
'Ge(s)'
Ge=feedback(G,H)
'T(s)'
T=feedback(-1*Ge,1)
[numt,dent]=tfdata(T,'V');
[Acc,Bcc,Ccc,Dcc]=tf2ss(numt,dent)
```

Computer response:
```
ans =

G(s)

Transfer function:
           -0.25 s - 0.1093
-------------------------------------
s^4 + 3.483 s^3 + 3.215 s^2 + 0.4979 s

ans =

H(s)

Transfer function:
-s

ans =

Ge(s)

Transfer function:
           -0.25 s - 0.1093
-------------------------------------
s^4 + 3.483 s^3 + 3.465 s^2 + 0.6072 s
```

```
ans =

T(s)

Transfer function:
                0.25 s + 0.1093
-----------------------------------------------
s^4 + 3.483 s^3 + 3.465 s^2 + 0.8572 s + 0.1093

Acc =

   -3.4830   -3.4650   -0.8572   -0.1093
    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0

Bcc =

     1
     0
     0
     0

Ccc =

         0         0    0.2500    0.1093

Dcc =

     0
```

ANSWERS TO REVIEW QUESTIONS

1. Signals, systems, summing junctions, pickoff points

2. Cascade, parallel, feedback

3. Product of individual transfer functions, sum of individual transfer functions, forward gain divided by one plus the product of the forward gain times the feedback gain

4. Equivalent forms for moving blocks across summing junctions and pickoff points

5. As K is varied from 0 to ∞, the system goes from overdamped to critically damped to underdamped. When the system is underdamped, the settling time remains constant.

6. Since the real part remains constant and the imaginary part increases, the radial distance from the origin is increasing. Thus the angle θ is increasing. Since $\zeta = \cos\theta$ the damping ratio is decreasing.

7. Nodes (signals), branches (systems)

8. Signals flowing into a node are added together. Signals flowing out of a node are the sum of signals flowing into a node.

9. One

10. Phase-variable form, cascaded form, parallel form, Jordan canonical form, observer canonical form

11. The Jordan canonical form and the parallel form result from a partial fraction expansion.

12. Parallel form

13. The system poles, or eigenvalues

14. The system poles including all repetitions of the repeated roots

15. Solution of the state variables are achieved through decoupled equations. i.e. the equations are solvable individually and not simultaneously.

16. State variables can be identified with physical parameters; ease of solution of some representations

17. Systems with zeros

18. State-vector transformations are the transformation of the state vector from one basis system to another. i.e. the same vector represented in another basis.

19. A vector which under a matrix transformation is collinear with the original. In other words, the length of the vector has changed, but not its angle.

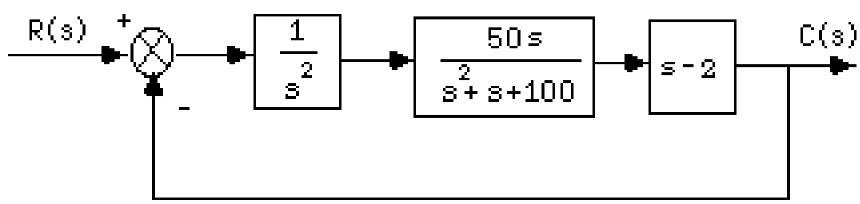
20. An eigenvalue is that multiple of the original vector that is the transformed vector.

21. Resulting system matrix is diagonal.

SOLUTIONS TO PROBLEMS

1.

a. Combine the inner feedback and the parallel pair.



Multiply the blocks in the forward path and apply the feedback formula to get,

$$T(s) = \frac{50(s-2)}{s^3+s^2+150s-100} .$$

b.

Program:

```
'G1(s)'
G1=tf(1,[1 0 0])
'G2(s)'
G2=tf(50,[1 1])
'G3(s)'
G3=tf(2,[1 0])
'G4(s)'
G4=tf([1 0],1)
'G5(s)'
G5=2
'Ge1(s)=G2(s)/(1+G2(s)G3(s))'
Ge1=G2/(1+G2*G3)
'Ge2(s)=G4(s)-G5(s)'
Ge2=G4-G5
'Ge3(s)=G1(s)Ge1(s)Ge2(s)'
Ge3=G1*Ge1*Ge2
```

```
'T(s)=Ge3(s)/(1+Ge3(s))'
T=feedback(Ge3,1);
T=minreal(T)
```

Computer response:

```
ans =

G1(s)

Transfer function:
 1
---
s^2

ans =

G2(s)

Transfer function:
 50
-----
s + 1

ans =

G3(s)

Transfer function:
2
-
s

ans =

G4(s)

Transfer function:
s

ans =

G5(s)

G5 =

     2

ans =

Ge1(s)=G2(s)/(1+G2(s)G3(s))


Transfer function:
      50 s^2 + 50 s
-------------------------
s^3 + 2 s^2 + 101 s + 100

ans =

Ge2(s)=G4(s)-G5(s)

Transfer function:
s - 2
```

```
ans =

Ge3(s)=G1(s)Ge1(s)Ge2(s)

Transfer function:
    50 s^3 - 50 s^2 - 100 s
-------------------------------
s^5 + 2 s^4 + 101 s^3 + 100 s^2

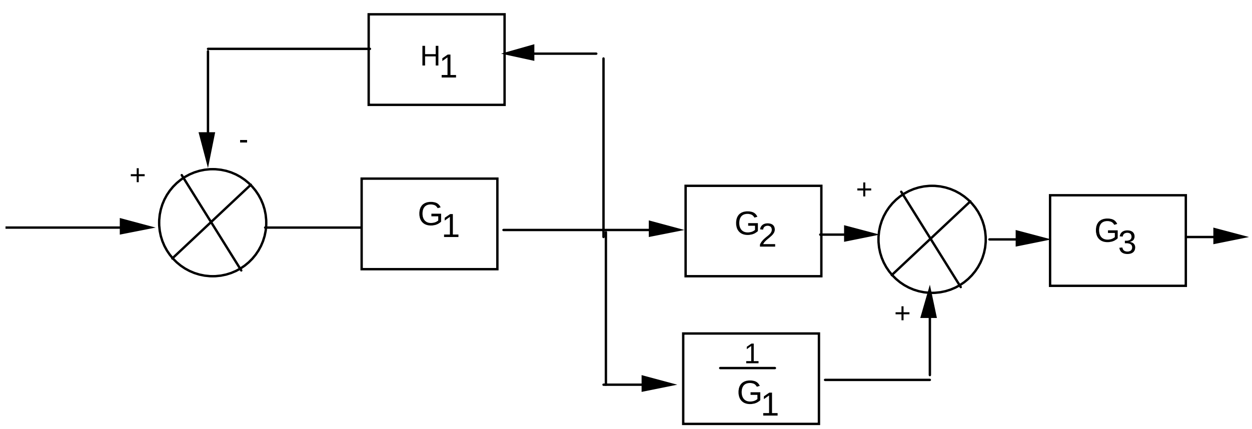
ans =

T(s)=Ge3(s)/(1+Ge3(s))

Transfer function:
      50 s - 100
-----------------------
s^3 + s^2 + 150 s - 100
```

2.

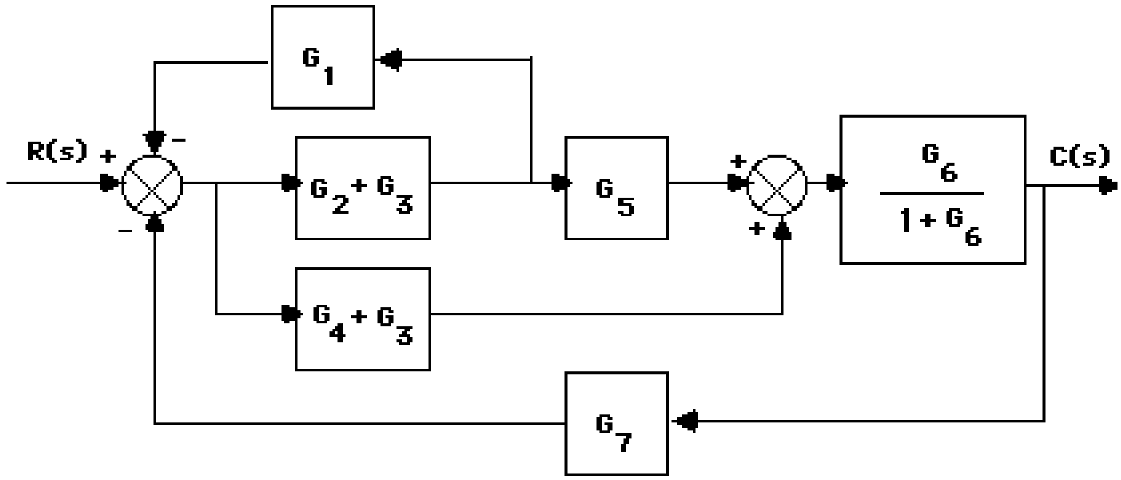
Push $G_1(s)$ to the left past the pickoff point.



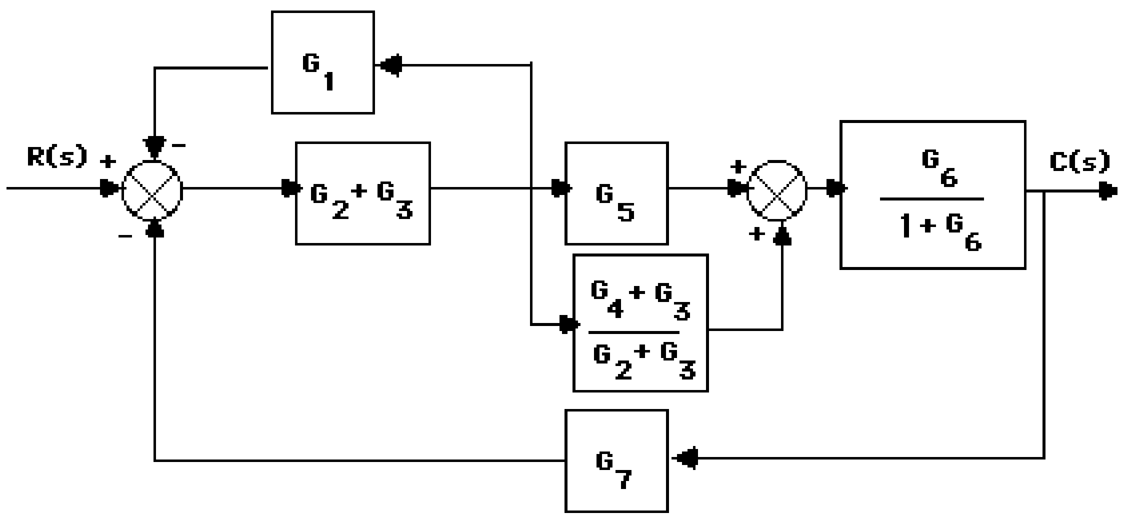
Thus, $T(s) = \left(\dfrac{G_1}{1+G_1H_1}\right)\left(G_2 + \dfrac{1}{G_1}\right)G_3 = \dfrac{(G_1G_2+1)G_3}{(1+G_1H_1)}$

3.

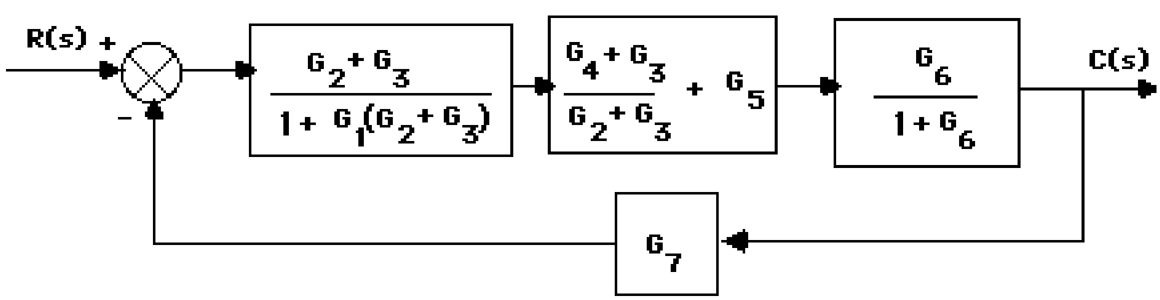
a. Split G_3 and combine with G_2 and G_4. Also use feedback formula on G_6 loop.



Push $G_2 + G_3$ to the left past the pickoff point.



Using the feedback formula and combining parallel blocks,

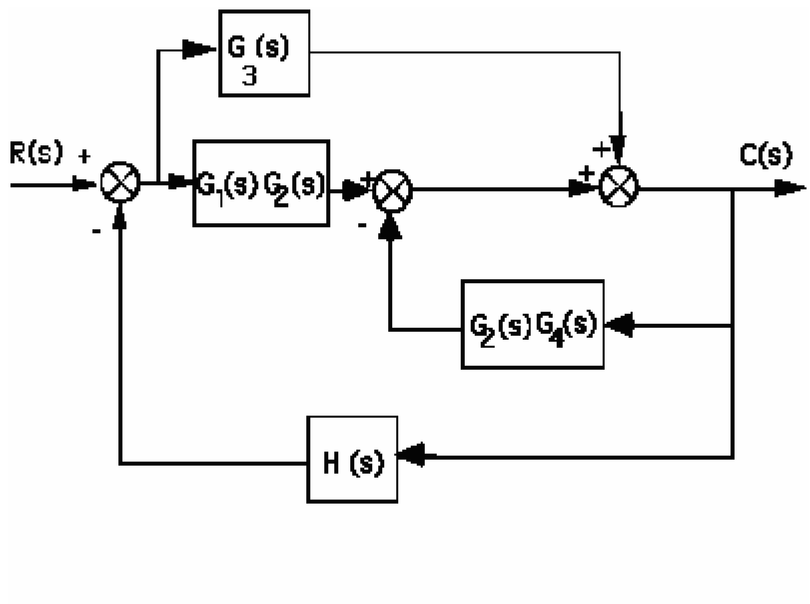


Multiplying the blocks of the forward path and applying the feedback formula,

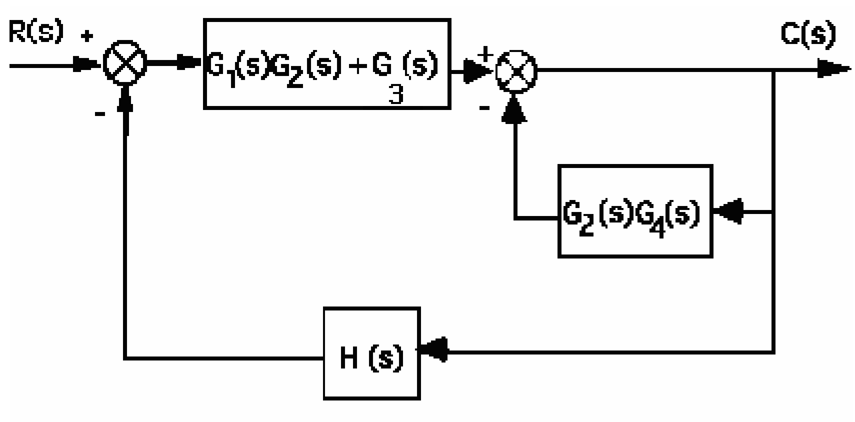
$$T(s) = \frac{G_6 G_4 + G_6 G_3 + G_6 G_5 G_3 + G_6 G_5 G_2}{1 + G_6 + G_3 G_1 + G_2 G_1 + G_7 G_6 G_4 + G_7 G_6 G_3 + G_7 G_6 G_5 G_3 + G_7 G_6 G_5 G_2 + G_6 G_3 G_1 + G_6 G_2 G_1}$$

4.

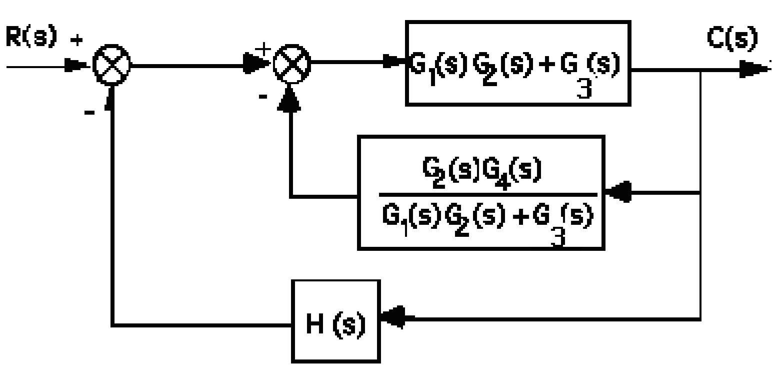
Push $G_2(s)$ to the left past the summing junction.



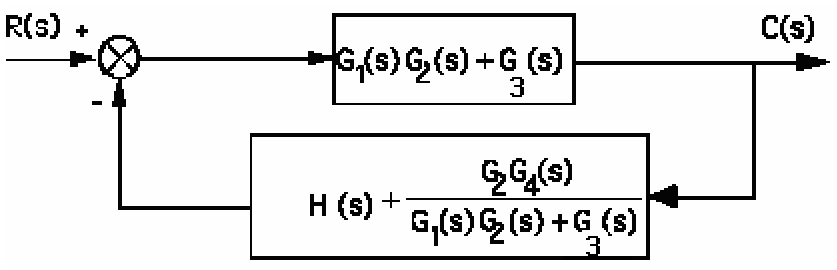
Collapse the summing junctions and add the parallel transfer functions.



Push $G_1(s)G_2(s) + G_5(s)$ to the right past the summing junction.



Collapse summing junctions and add feedback paths.



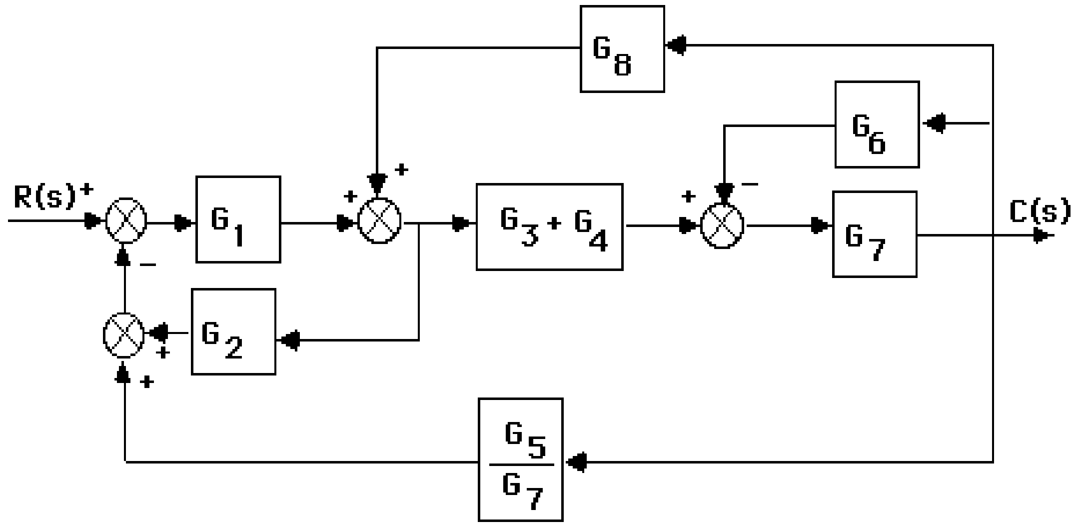
Applying the feedback formula,

$$T(s) = \frac{G_3(s)+G_1(s)G_2(s)}{1+[G_3(s)+G_1(s)G_2(s)]\left[H+\dfrac{G_2(s)G_4(s)}{G_3(s)+G_1(s)G_2(s)}\right]}$$

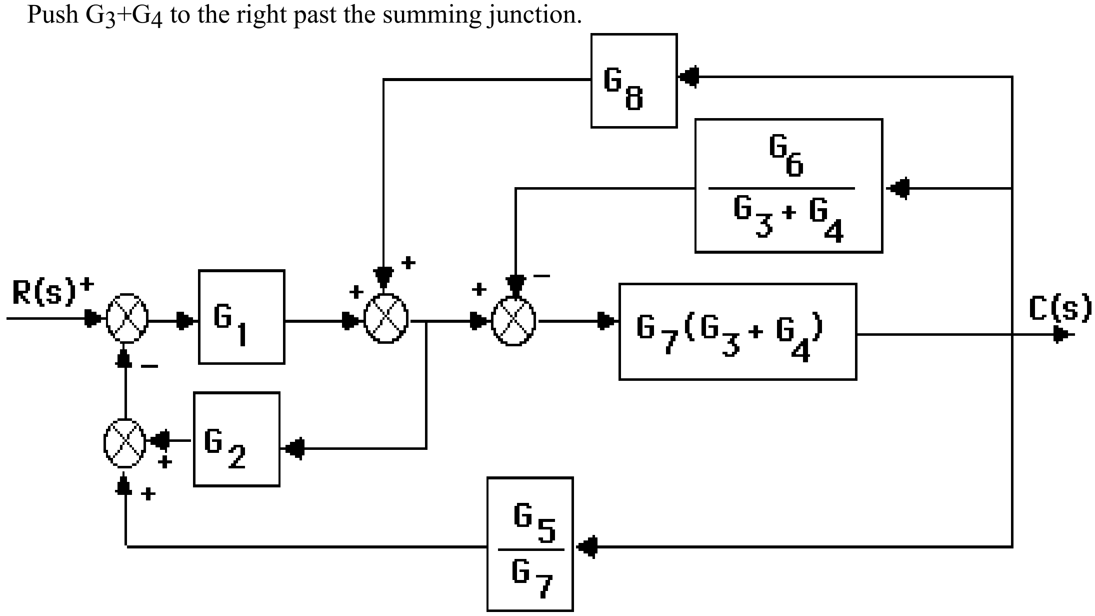
$$= \frac{G_3(s)+G_1(s)G_2(s)}{1+H[G_3(s)+G_1(s)G_2(s)]+G_2(s)G_4(s)}$$

5.

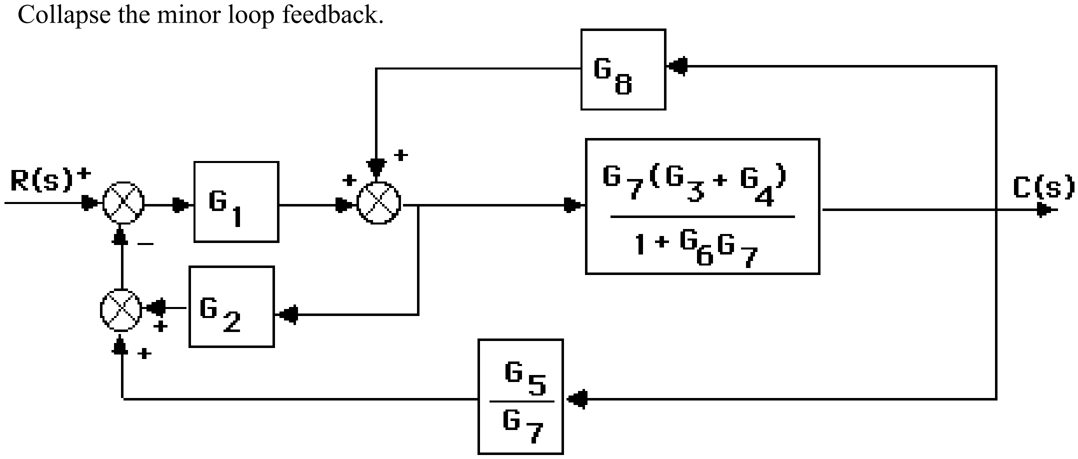
a. Push G_7 to the left past the pickoff point. Add the parallel blocks, G_3+G_4.



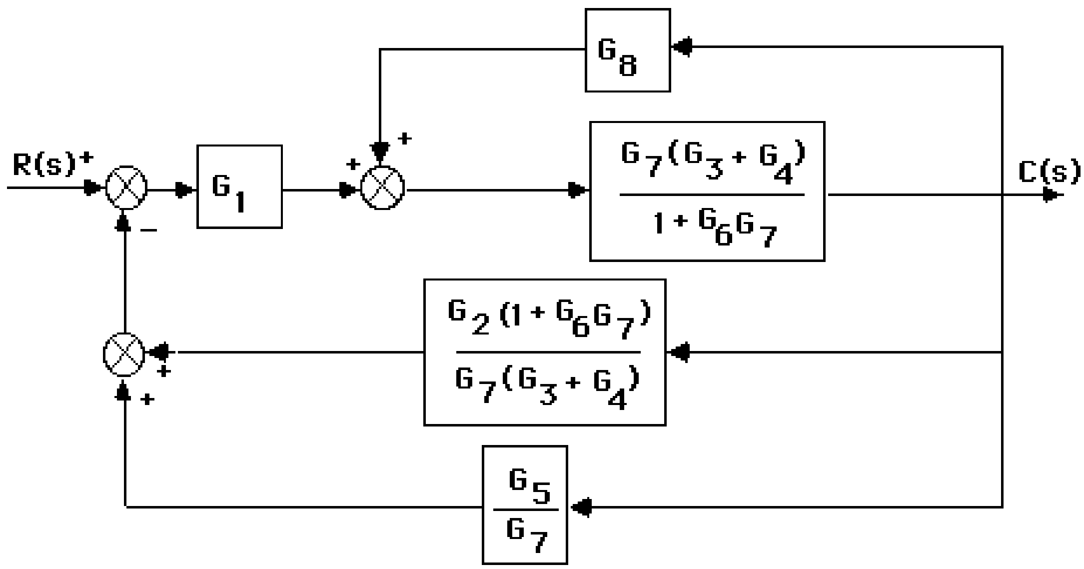
Push G_3+G_4 to the right past the summing junction.



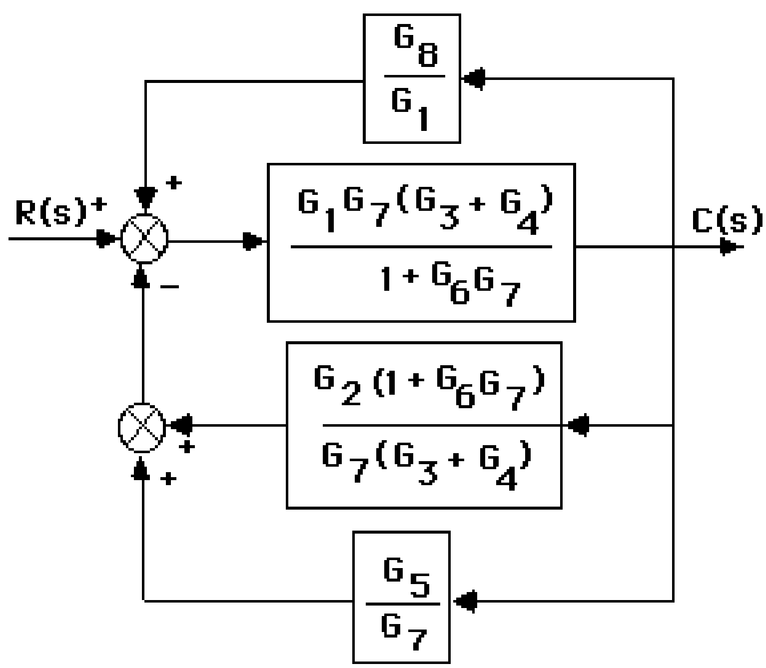
Collapse the minor loop feedback.



Push $\frac{G_7(G_3+G_4)}{1+G_6G_7}$ to the left past the pickoff point.



Push G_1 to the right past the summing junction.



Add the parallel feedback paths to get the single negative feedback,

$$H(s) = \frac{G_5}{G_7} + \frac{G_2(1+G_6G_7)}{G_7(G_3+G_4)} - \frac{G_8}{G_1}$$. Thus,

$$T(s) = \frac{G}{1+GH} = \frac{G_7 G_1 (G_4 + G_3)}{([G_7 G_6 + 1] G_2 G_1 + [G_4 + G_3][G_5 G_1 - G_8 G_7]) + (G_7 G_6 + 1)}$$

b.

Program:

```
G1=tf([0 1],[1 7]);                %G1=1/s+7 input transducer
G2=tf([0 0 1],[1 2 3]);            %G2=1/s^2+2s+3
G3=tf([0 1],[1 4]);                %G3=1/s+4
G4=tf([0 1],[1 0]);                %G4=1/s
G5=tf([0 5],[1 7]);                %G5=5/s+7
G6=tf([0 0 1],[1 5 10]);           %G6=1/s^2+5s+10
G7=tf([0 3],[1 2]);                %G7=3/s+2
G8=tf([0 1],[1 6]);                %G8=1/s+6
G9=tf([1],[1]);                    %Add G9=1 transducer at the input
T1=append(G1,G2,G3,G4,G5,G6,G7,G8,G9);
Q=[1 -2 -5 9
2 1 8 0
3 1 8 0
4 1 8 0
5 3 4 -6
6 7 0 0
7 3 4 -6
8 7 0 0];
inputs=9;
outputs=7;
Ts=connect(T1,Q,inputs,outputs);
T=tf(Ts)
```

Computer response:

```
Transfer function:

6 s^7 + 132 s^6 + 1176 s^5 + 5640 s^4 + 1.624e004 s^3

               + 2.857e004 s^2 + 2.988e004 s + 1.512e004

-----------------------------------------------------------

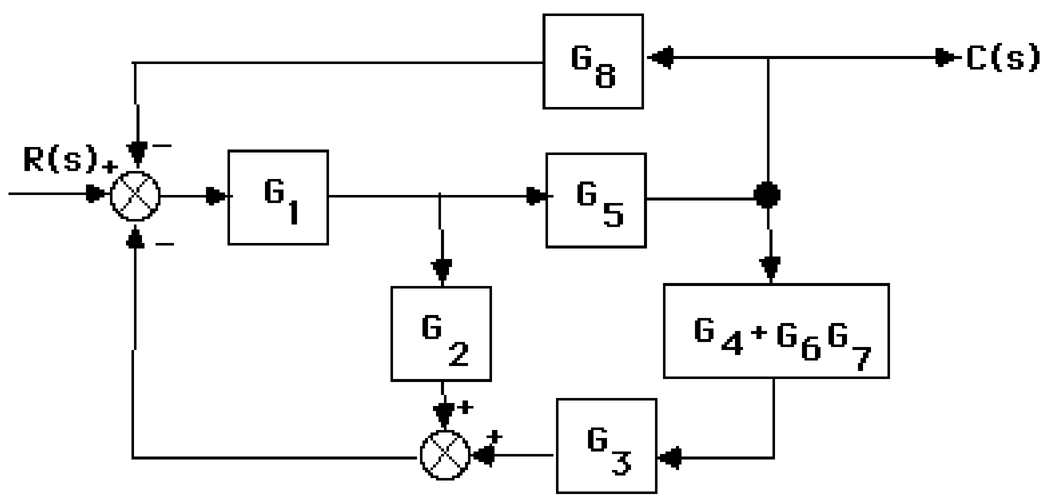
s^10 + 33 s^9 + 466 s^8 + 3720 s^7 + 1.867e004 s^6

        + 6.182e004 s^5 + 1.369e005 s^4 + 1.981e005 s^3

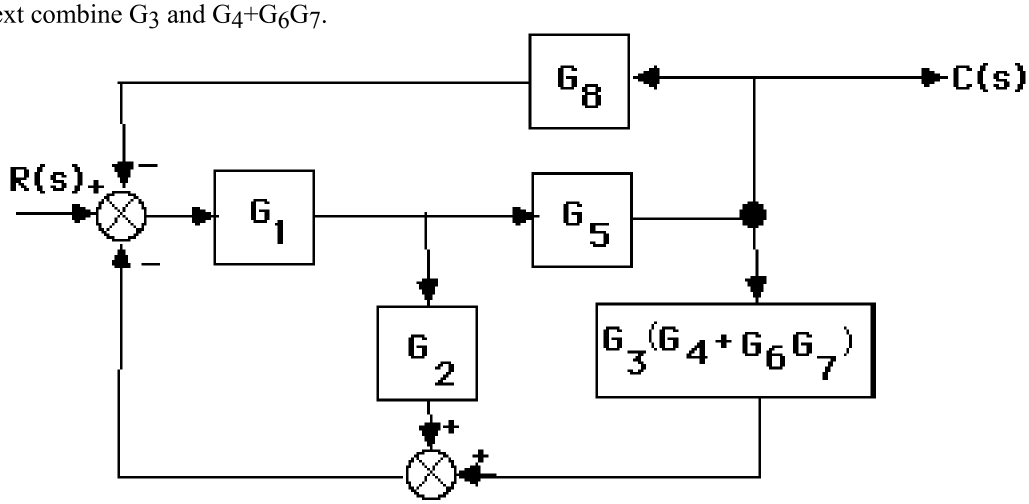
                  + 1.729e005 s^2 + 6.737e004 s - 1.044e004
```

6.

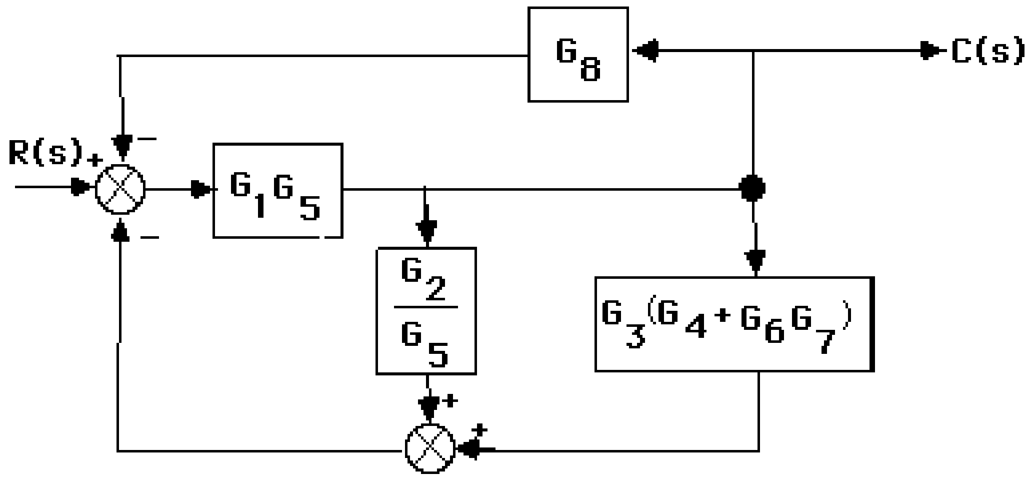
Combine G_6 and G_7 yielding G_6G_7. Add G_4 and obtain the following diagram:



Next combine G_3 and $G_4+G_6G_7$.



Push G_5 to the left past the pickoff point.



Notice that the feedback is in parallel form. Thus the equivalent feedback, $H_{eq}(s) = \frac{G_2}{G_5} + G_3(G_4+G_6G_7) + G_8$. Since the forward path transfer function is $G(s) = G_{eq}(s) = G_1G_5$, the closed-loop transfer function is

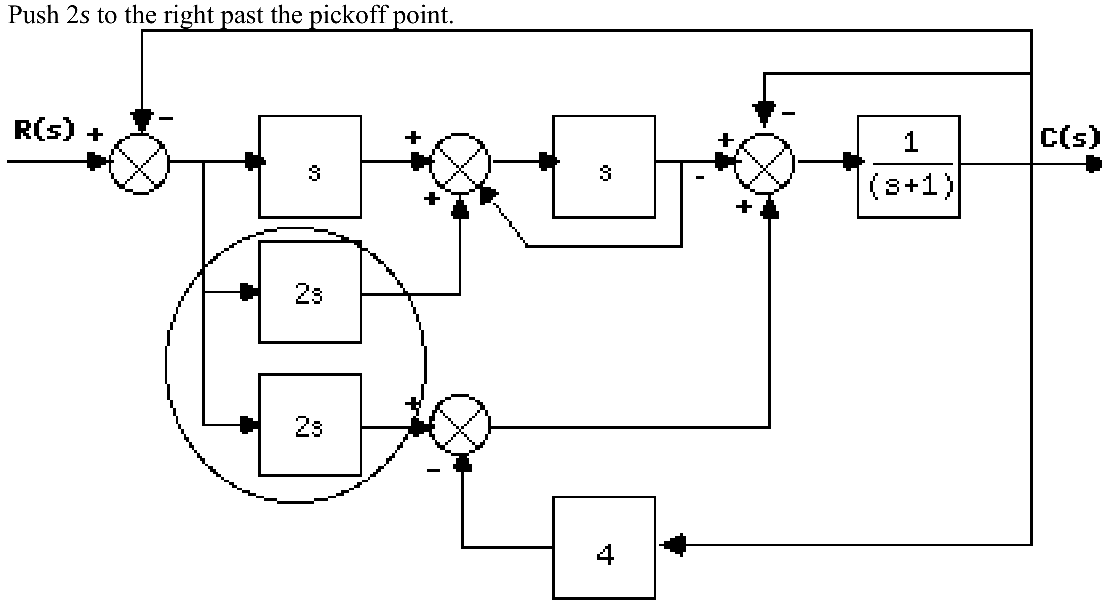
$$T(s) = \frac{G_{eq}(s)}{1+G_{eq}(s)H_{eq}(s)} \ .$$

Hence,

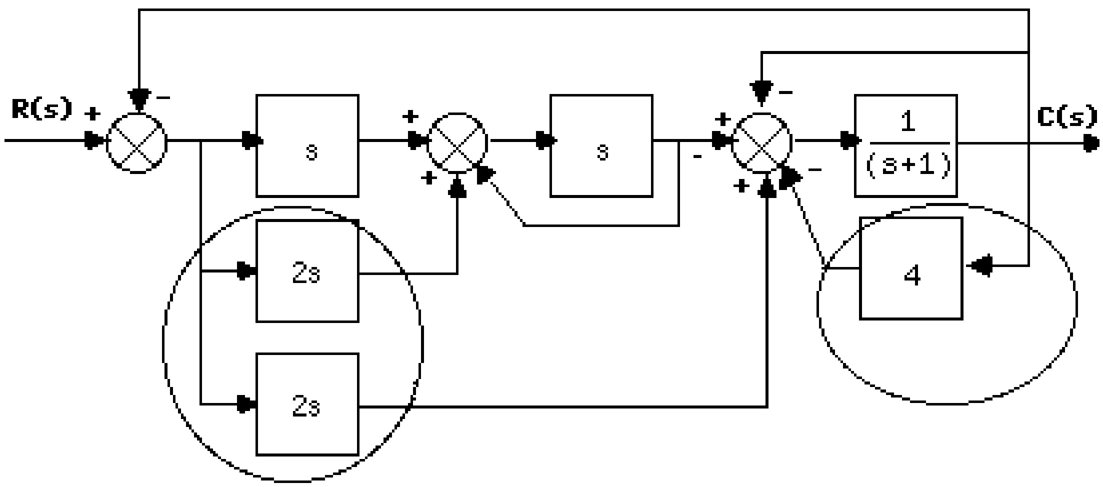
$$T(s) = \frac{G_5G_1}{1+G_1(G_8G_5+G_7G_6G_5G_3+G_5G_4G_3+G_2)}$$

7.

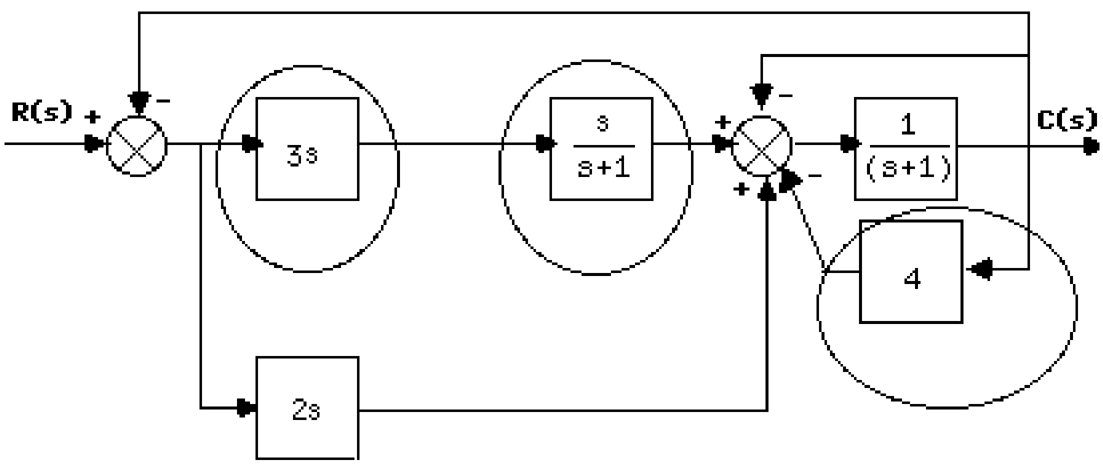
Push 2*s* to the right past the pickoff point.



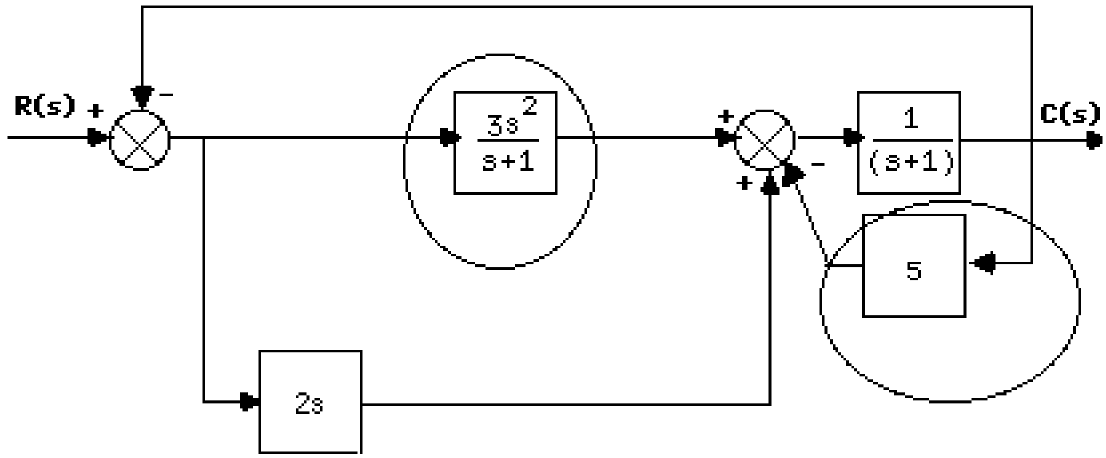
Combine summing junctions.



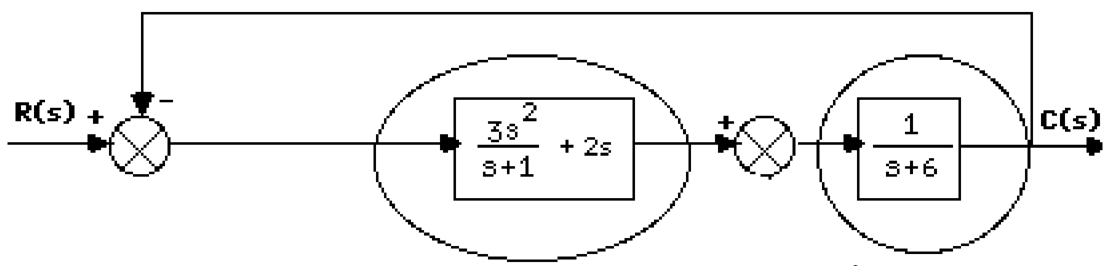
Combine parallel 2s and s. Apply feedback formula to unity feedback with G(s) = s.



Combine cascade pair and add feedback around 1/(s+1).



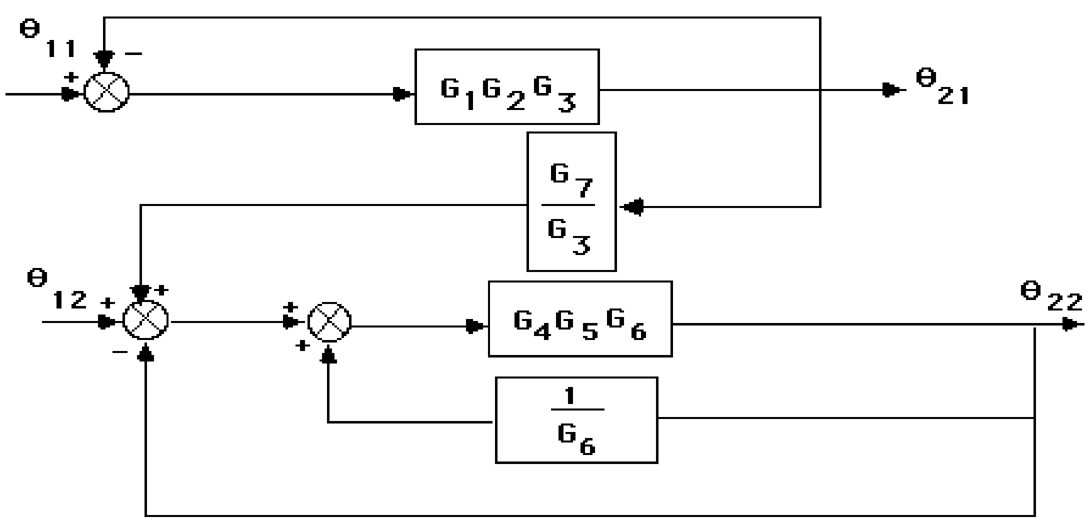
Combine parallel pair and feedback in forward path.



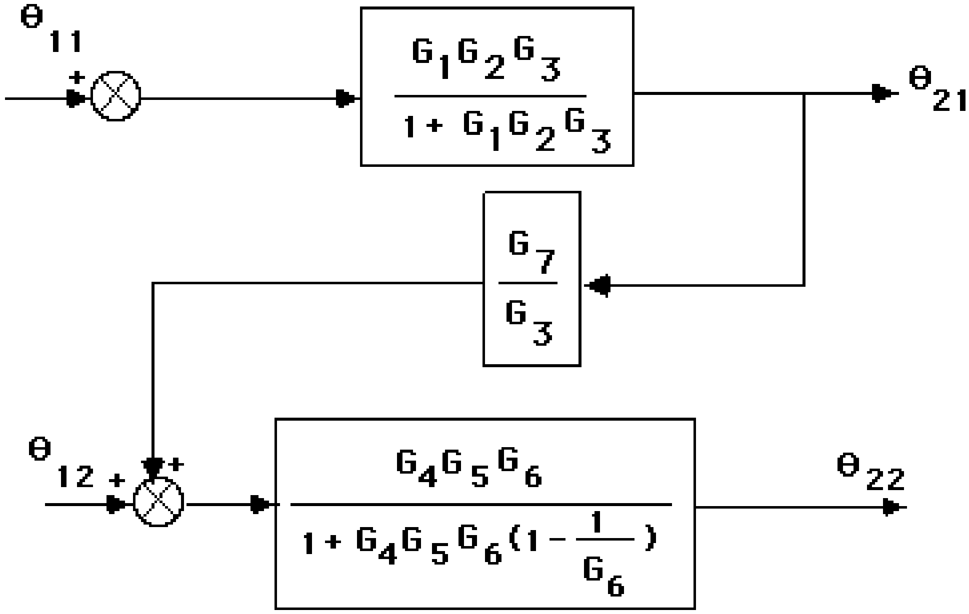
Combine cascade pair and apply final feedback formula yielding $T(s) = \dfrac{5s^2 + 2s}{6s^2 + 9s + 6}$.

8.

Push G_3 to the left past the pickoff point. Push G_6 to the left past the pickoff point.



Hence,

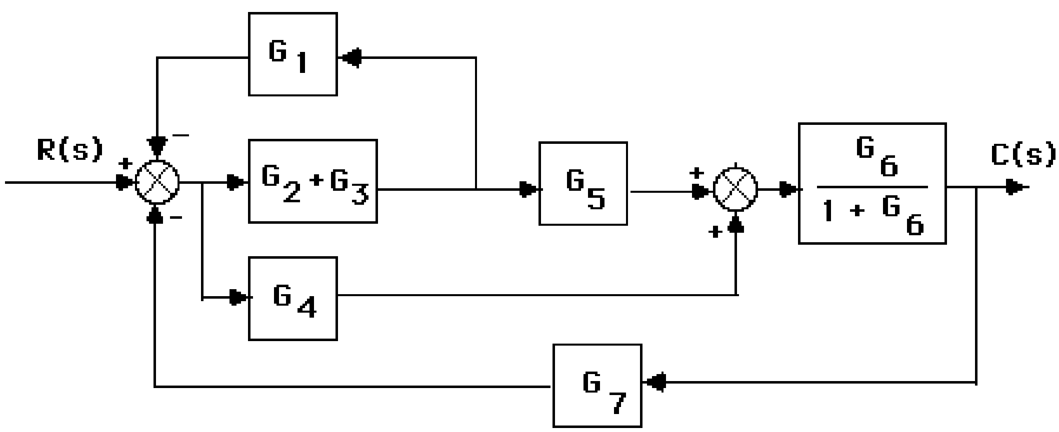


Thus the transfer function is the product of the functions, or

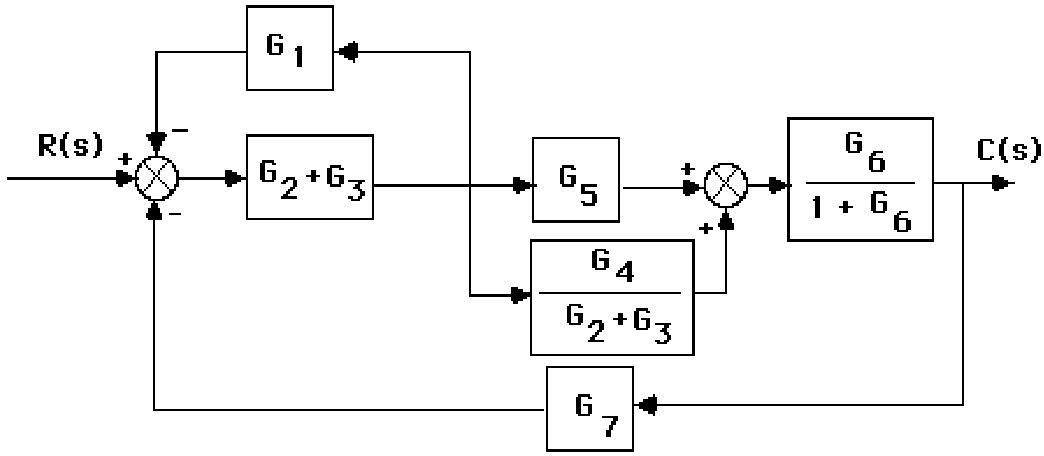
$$\frac{\theta_{22}(s)}{\theta_{11}(s)} = \frac{G_1G_2G_4G_5G_6G_7}{1 - G_4G_5 + G_4G_5G_6 + G_1G_2G_3 - G_1G_2G_3G_4G_5 + G_1G_2G_3G_4G_5G_6}$$

9.

Combine the feedback with G_6 and combine the parallel G_2 and G_3.



Move G_2+G_3 to the left past the pickoff point.



Combine feedback and parallel pair in the forward path yielding an equivalent forward-path transfer function of

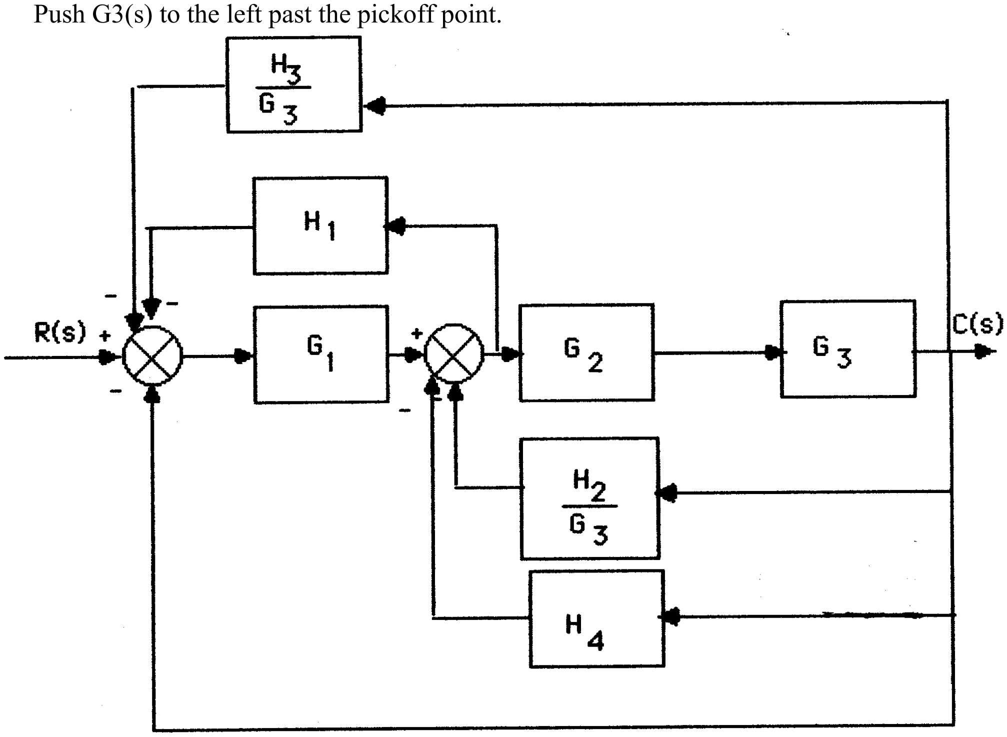
$$G_e(s) = \left(\frac{G_2+G_3}{1+G_1(G_2+G_3)}\right)\left(G_5+\frac{G_4}{G_2+G_3}\right)\left(\frac{G_6}{1+G_6}\right)$$

But, $T(s) = \frac{G_e(s)}{1+G_e(s)G_7(s)}$. Thus,

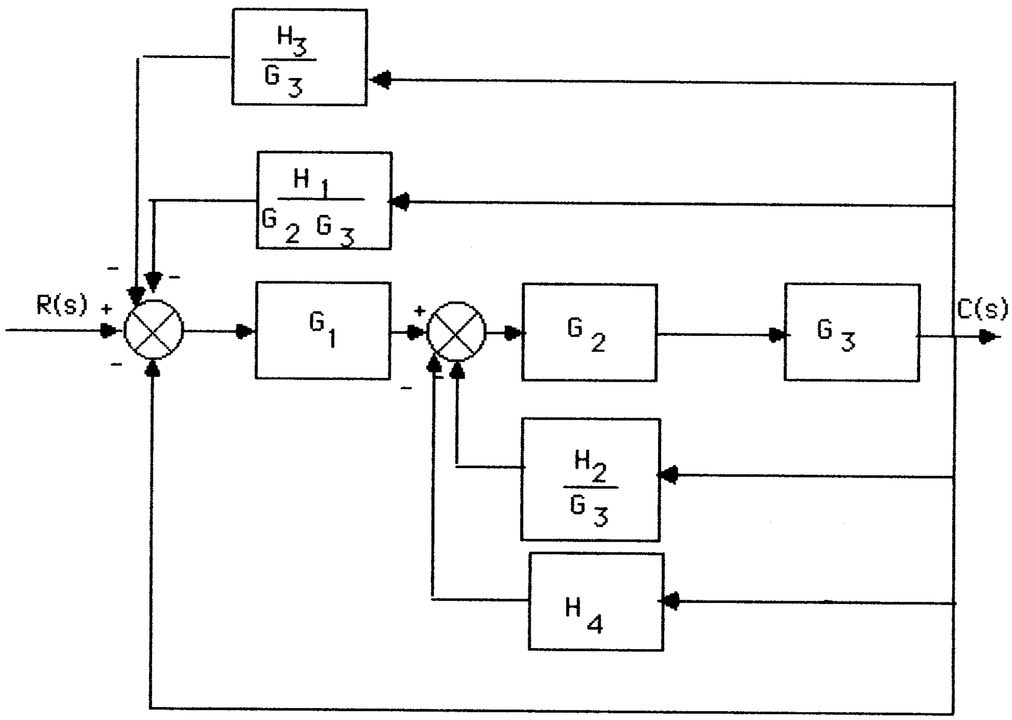
$$T(s) = \frac{G_6\,(G_4 + G_5\,G_3 + G_5\,G_2)}{G_6\,(G_7\,G_4 + G_7\,G_5\,G_3 + G_7\,G_5\,G_2 + G_3\,G_1 + G_2\,G_1 + 1) + G_1\,(G_3 + G_2) + 1}$$

10.

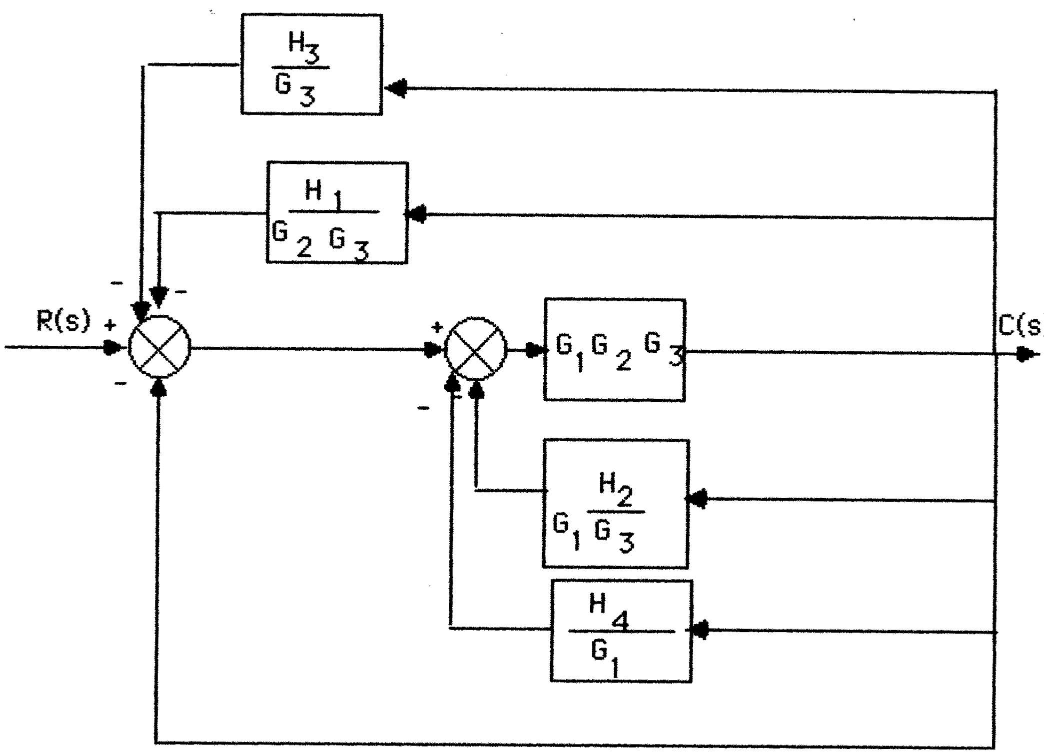
Push G3(s) to the left past the pickoff point.



Push G2(s)G3(s) to the left past the pickoff point.



Push G1(s) to the right past the summing junction.



Collapsing the summing junctions and adding the feedback transfer functions,

$$T(s) = \frac{G_1(s)G_2(s)G_3(s)}{1 + G_1(s)G_2(s)G_3(s)H_{eq}(s)}$$

where

$$H_{eq}(s) = \frac{H_3(s)}{G_3(s)} + \frac{H_1(s)}{G_2(s)G_3(s)} + \frac{H_2(s)}{G_1(s)G_3(s)} + \frac{H_4(s)}{G_1(s)} + 1$$

11.

$T(s) = \dfrac{225}{s^2 + 12s + 225}$. Therefore, $2\zeta\omega_n = 12$, and $\omega_n = 15$. Hence, $\zeta = 0.4$.

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 16.3\%$; $T_s = \dfrac{4}{\zeta\omega_n} = 0.667$; $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.229$.

12.

$$C(s) = \frac{5}{s(s^2 + 3s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 3s + 5}$$

$$A = 1$$

$$5 = s^2 + 3s + 5 + Bs^2 + Cs$$

$$\therefore B = -1,\ C = -3$$

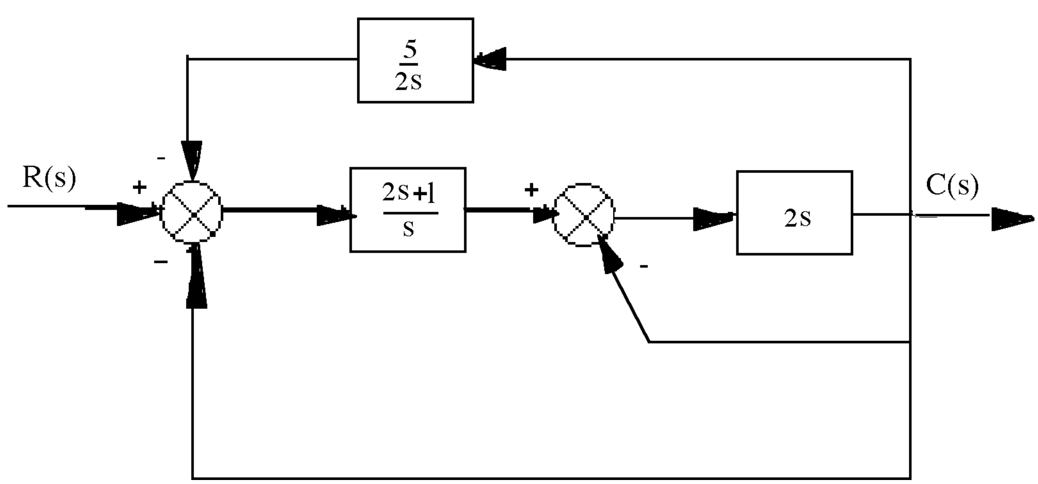
$$C(s) = \frac{1}{s} - \frac{s+3}{s^2 + 3s + 5} = \frac{1}{s} - \frac{s+3}{(s+1.5)^2 + 2.75}$$

$$= \frac{1}{s} - \frac{(s+1.5) + 1.5}{(s+1.5)^2 + 2.75} = \frac{1}{s} - \frac{(s+1.5) + \frac{1.5}{\sqrt{2.75}}\sqrt{2.75}}{(s+1.5)^2 + 2.75}$$

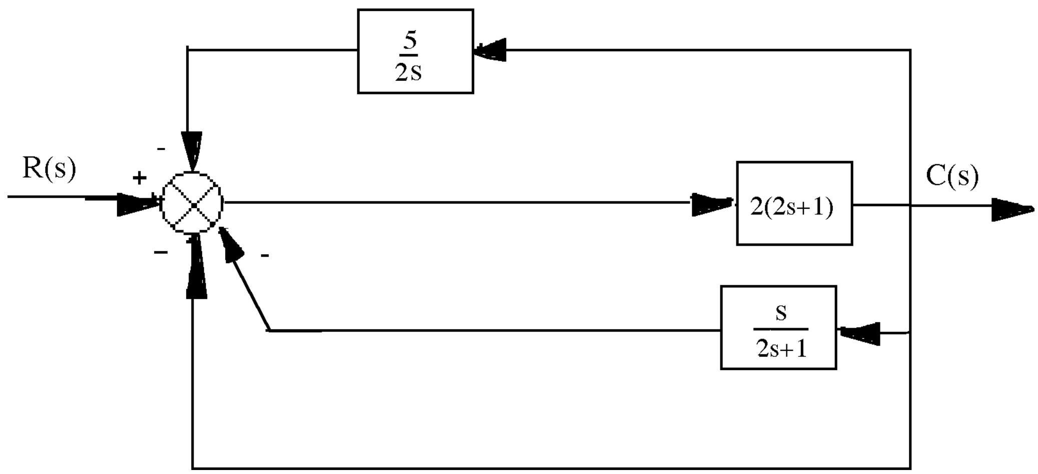
$$c(t) = 1 - e^{-1.5t}(\cos\sqrt{2.75}t + \frac{1.5}{\sqrt{2.75}}\sin\sqrt{2.75}t)$$

13.

Push 2s to the left past the pickoff point and combine the parallel combination of 2 and 1/s.



Push (2s+1)/s to the right past the summing junction and combine summing junctions.



Hence, $T(s) = \dfrac{2(2s+1)}{1+2(2s+1)H_{eq}(s)}$, where $H_{eq}(s) = 1 + \dfrac{s}{2s+1} + \dfrac{5}{2s}$.

14.

Since $G(s) = \dfrac{K}{s(s+30)}$, $T(s) = \dfrac{G(s)}{1+G(s)} = \dfrac{K}{s^2+30s+K}$. Therefore, $2\zeta\omega_n = 30$. Thus, $\zeta =$ $15/\omega_n = 0.456$ (i.e. 20% overshoot). Hence, $\omega_n = 32.89 = \sqrt{K}$. Therefore K = 1082.

15.

$T(s) = \dfrac{K}{s^2+\alpha s+K}$; $\zeta = \dfrac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2+\ln^2(\frac{\%OS}{100})}} = 0.358$; $T_s = \dfrac{4}{\zeta\omega_n} = 0.2$. Therefore, $\omega_n =$ 55.89. $K = \omega_n^2 = 3124$. $\alpha = 2\zeta\omega_n = 40$.

16.

The equivalent forward-path transfer function is $G(s) = \dfrac{10K_1}{s[s+(10K_2+2)]}$. Hence,

$T(s) = \dfrac{G(s)}{1+G(s)} = \dfrac{10K_1}{s^2+(10K_2+2)s+10K_1}$. Since

$T_s = \dfrac{4}{\text{Re}} = 2, \ \therefore \text{Re} = 2$; and $T_p = \dfrac{\pi}{\text{Im}} = 1, \ \therefore \text{Im} = \pi$. The poles are thus at $-2+j\pi$. Hence, $\omega_n = \sqrt{2^2+\pi^2} = \sqrt{10K_1}$. Thus, $K_1 = 1.387$. Also, $(10K_2 + 2)/2 = \text{Re} = 2$. Hence, $K_2 = 1/5$.

17.

a. For the inner loop, $G_e(s) = \dfrac{20}{s(s+12)}$, and $H_e(s) = 0.2s$. Therefore, $T_e(s) = \dfrac{G_e(s)}{1+G_e(s)H_e(s)}$ = $\dfrac{20}{s(s+16)}$. Combining with the equivalent transfer function of the parallel pair, $G_p(s) = 20$, the system

is reduced to an equivalent unity feedback system with $G(s) = G_p(s)\, T_e(s) = \frac{400}{s(s+16)}$. Hence, $T(s) = \frac{G(s)}{1+G(s)} = \frac{400}{s^2+16s+400}$.

b. $\omega_n^2 = 400$; $2\zeta\omega_n = 16$. Therefore, $\omega_n = 20$, and $\zeta = 0.4$. $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 25.38$;

$T_s = \frac{4}{\zeta\omega_n} = 0.5$; $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.171$. From Figure 4.16, $\omega_n T_r = 1.463$. Hence, $T_r = 0.0732$.

$\omega_d = Im = \omega_n\sqrt{1-\zeta^2} = 18.33$.

18.

$T(s) = \frac{28900}{s^2+200s+28900}$; from which, $2\zeta\omega_n = 200$ and $\omega_n = \sqrt{28900} = 170$. Hence,

$\zeta = 0.588$. $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 10.18\%$; $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.0229$ s.

Also, $T_s = \frac{4}{\zeta\omega_n} = 0.04$ s.

19.

For the generator, $E_g(s) = K_f I_f(s)$. But, $I_f(s) = \frac{E_i(s)}{R_f+L_f s}$. Therefore, $\frac{E_g(s)}{E_i(s)} = \frac{2}{s+1}$. For the motor, consider $R_a = 2\ \Omega$, the sum of both resistors. Also, $J_e = J_a+J_L(\frac{1}{2})^2 = 0.75+\frac{1}{4} = 1$; $D_e = D_L(\frac{1}{2})^2 = 1$.

Therefore,

$$\frac{\theta_m(s)}{E_g(s)} = \frac{\frac{K_t}{R_a J_e}}{s(s+\frac{1}{J_e}(D_e+\frac{K_t K_a}{R_a}))} = \frac{0.5}{s(s+1.5)} .$$

But, $\frac{\theta_o(s)}{\theta_m(s)} = \frac{1}{2}$. Thus, $\frac{\theta_o(s)}{E_g(s)} = \frac{0.25}{s(s+1.5)}$. Finally,

$$\frac{\theta_o(s)}{E_i(s)} = \frac{E_g(s)}{E_i(s)}\frac{\theta_o(s)}{E_g(s)} = \frac{0.5}{s(s+1)(s+1.5)} .$$

20.

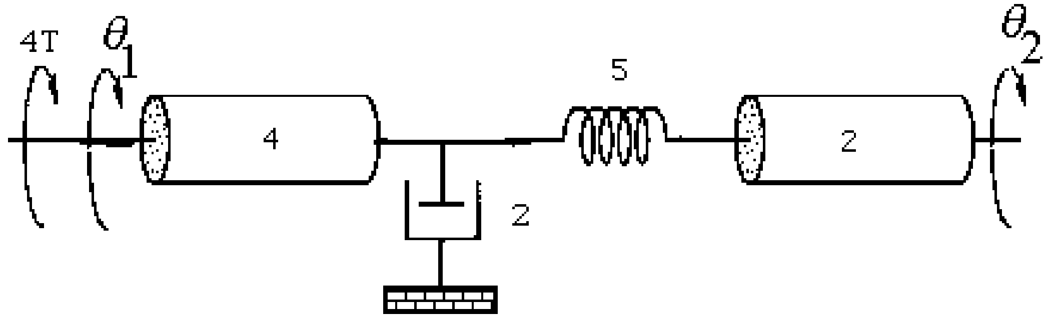
For the mechanical system, $J\left(\frac{N_2}{N_1}\right)^2 s^2\theta_2(s) = T\left(\frac{N_2}{N_1}\right)$. For the potentiometer, $E_i(s) = 10\frac{\theta_2(s)}{2\pi}$, or $\theta_2(s) = \frac{\pi}{5} E_i(s)$. For the network, $E_o(s) = E_i(s)\frac{R}{R+\frac{1}{Cs}} = E_i(s)\frac{s}{s+\frac{1}{RC}}$, or $E_i(s) = E_o(s)\frac{s+\frac{1}{RC}}{s}$.

Therefore, $\theta_2(s) = \frac{\pi}{5} E_o(s) \frac{s + \frac{1}{RC}}{s}$. Substitute into mechanical equation and obtain,

$$\frac{E_o(s)}{T(s)} = \frac{\frac{5N_1}{J\pi N_2}}{s\left(s + \frac{1}{RC}\right)}.$$

21.

The equivalent mechanical system is found by reflecting mechanical impedances to the spring.



Writing the equations of motion:

$$(4s^2 + 2s + 5)\theta_1(s) - 5\theta_2(s) = 4T(s)$$

$$-5\theta_1(s) + (2s^2 + 5)\theta_2(s) = 0$$

Solving for $\theta_2(s)$,

$$\theta_2(s) = \frac{\begin{vmatrix} (4s^2 + 2s + 5) & 4T(s) \\ -5 & 0 \end{vmatrix}}{\begin{vmatrix} (4s^2 + 2s + 5) & -5 \\ -5 & (2s^2 + 5) \end{vmatrix}} = \frac{20T(s)}{8s^4 + 4s^3 + 30s^2 + 10s}$$

The angular rotation of the pot is 0.2 that of θ_2, or

$$\frac{\theta_p(s)}{T(s)} = \frac{2}{s(4s^3 + 2s^2 + 15s + 5)}$$

For the pot:

$$\frac{E_p(s)}{\theta_p(s)} = \frac{50}{5(2\pi)} = \frac{5}{\pi}$$

For the electrical network: Using voltage division,

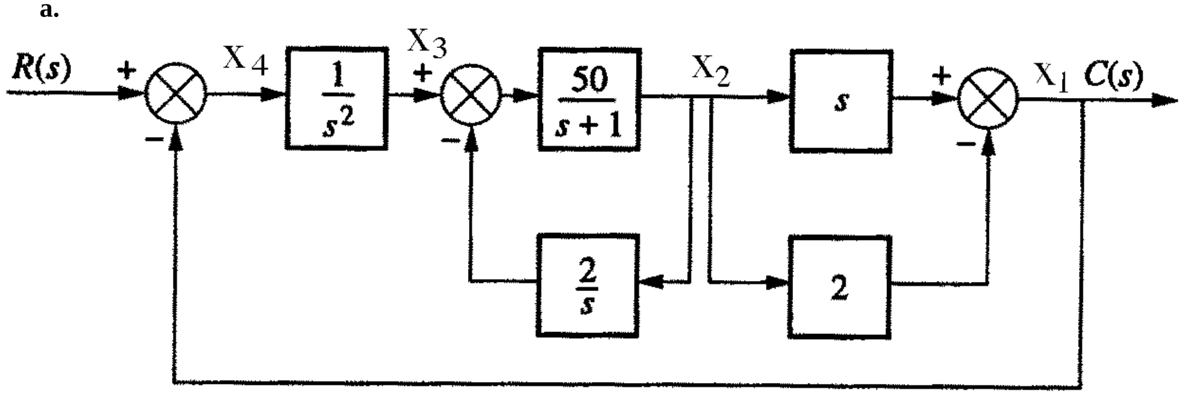
$$\frac{E_o(s)}{E_p(s)} = \frac{200{,}000}{\frac{1}{10^{-5}s} + 200{,}000} = \frac{s}{s + \frac{1}{2}}$$

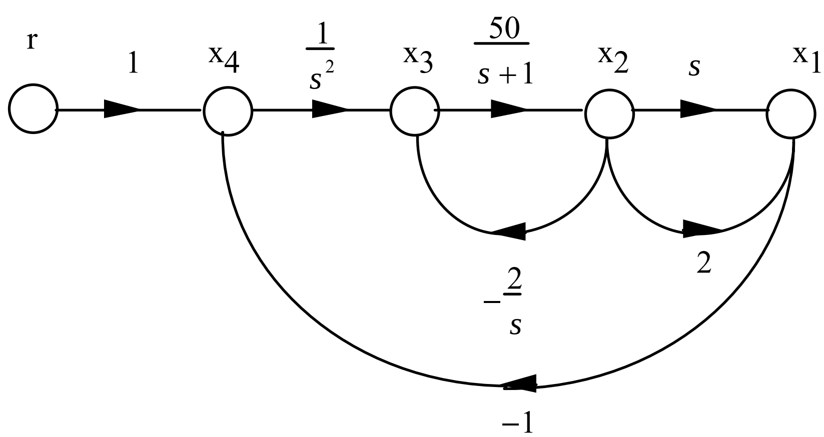
Substituting the previously obtained values,

$$\frac{E_o(s)}{T(s)} = \left(\frac{\theta_p(s)}{T(s)}\right)\left(\frac{E_p(s)}{\theta_p(s)}\right)\left(\frac{E_o(s)}{E_p(s)}\right) = \frac{\frac{10}{\pi}s}{s\left(s+\frac{1}{2}\right)\left(4s^3+2s^2+15s+5\right)}$$

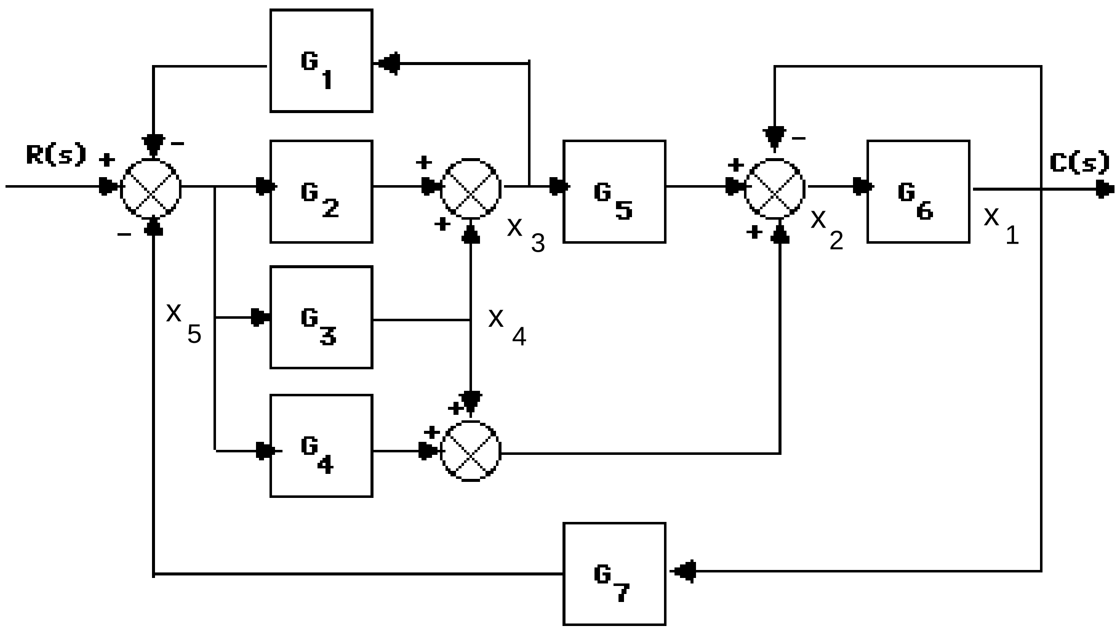
22.

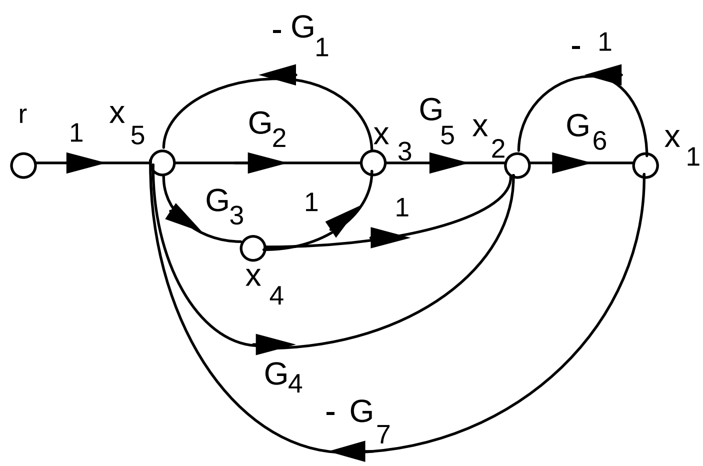
a.



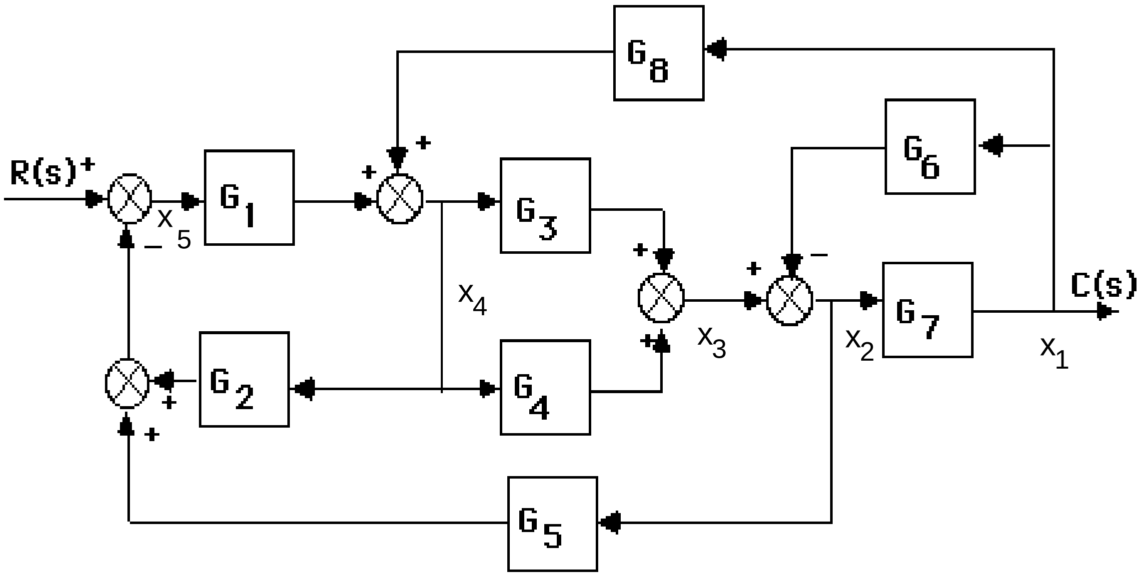


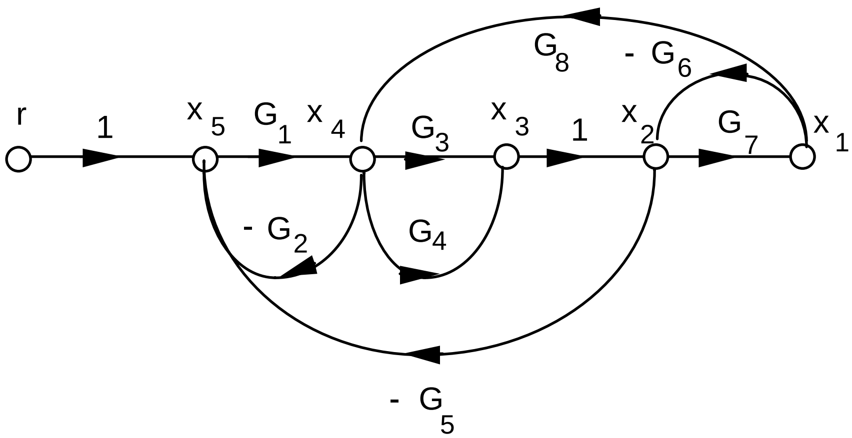
b.


R(s)
G1
G2
G3
G4
G5
G6
G7
C(s)
x3
x2
x1
x5
x4


r
1
x5
- G1
G2
x3
G5
x2
- 1
G6
x1
G3
1
1
x4
G4
- G7

c.

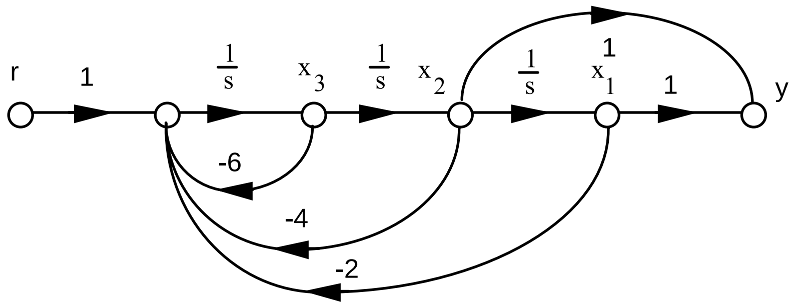




23.

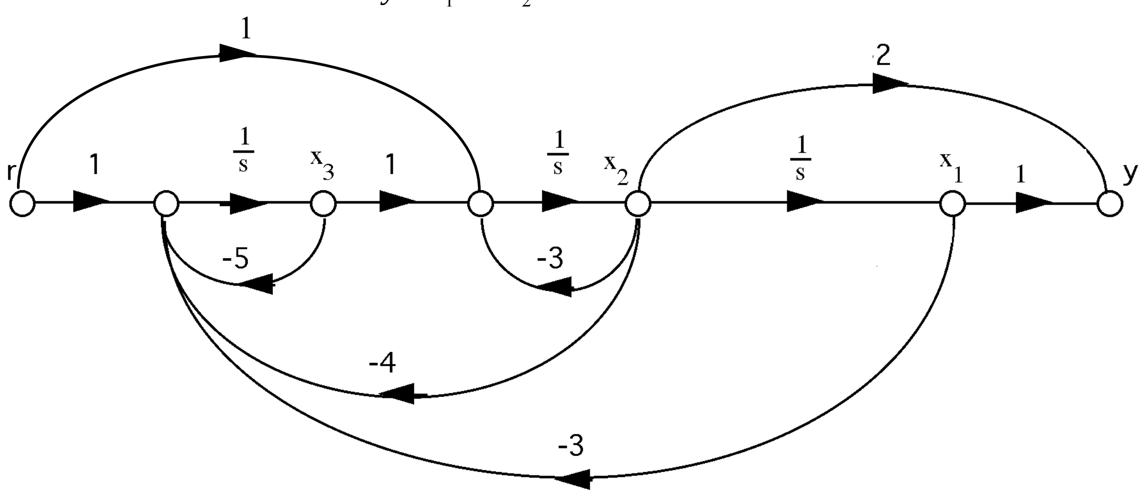
a.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -2x_1 - 4x_2 - 6x_3 + r$$
$$y = x_1 + x_2$$



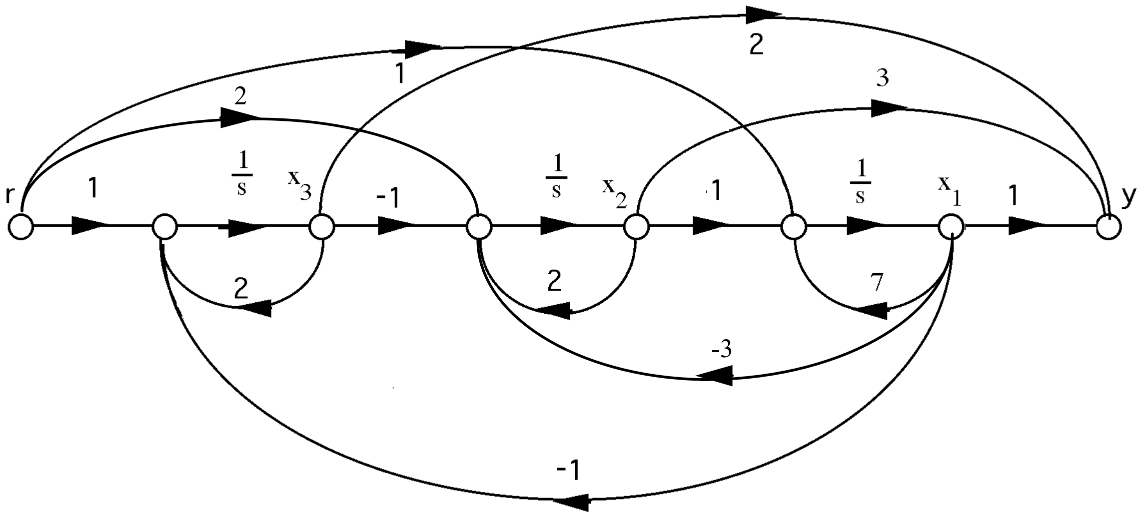
b.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -3x_2 + x_3 + r$$
$$\dot{x}_3 = -3x_1 - 4x_2 - 5x_3 + r$$
$$y = x_1 + 2x_2$$



c.

$$\dot{x}_1 = 7x_1 + x_2 + r$$
$$\dot{x}_2 = -3x_1 + 2x_2 - x_3 + 2r$$
$$\dot{x}_3 = -x_1 + 2x_3 + r$$
$$y = x_1 + 3x_2 + 2x_3$$



24.

a. Since $G(s) = \dfrac{10}{s^3 + 24s^2 + 191s + 504} = \dfrac{C(s)}{R(s)}$,

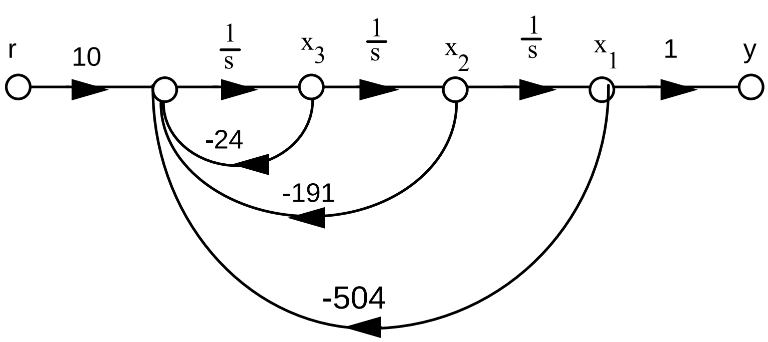
$$\dddot{c} + 24\ddot{c} + 191\dot{c} + 504c = 10r$$

Let,

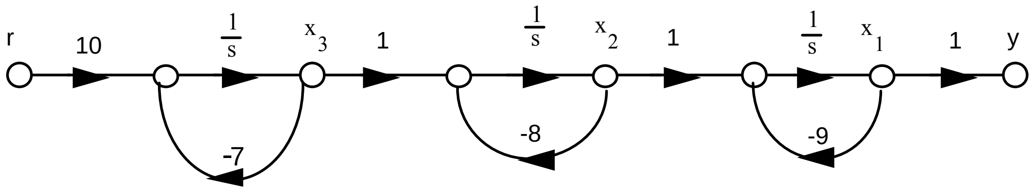
$$c = x_1$$
$$\dot{c} = x_2$$
$$\ddot{c} = x_3$$

Therefore,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -504x_1 - 191x_2 - 24x_3 + 10r$$
$$y = x_1$$



b. $G(s) = (\frac{10}{s+7})(\frac{1}{s+8})(\frac{1}{s+9})$



Therefore,

$$\dot{x}_1 = -9x_1 + x_2$$
$$\dot{x}_2 = -8x_2 + x_3$$
$$\dot{x}_3 = -7x_3 + 10r$$
$$y = x_1$$

25.

a. Since $G(s) = \frac{20}{s^4 + 15s^3 + 66s^2 + 80s} = \frac{C(s)}{R(s)}$,

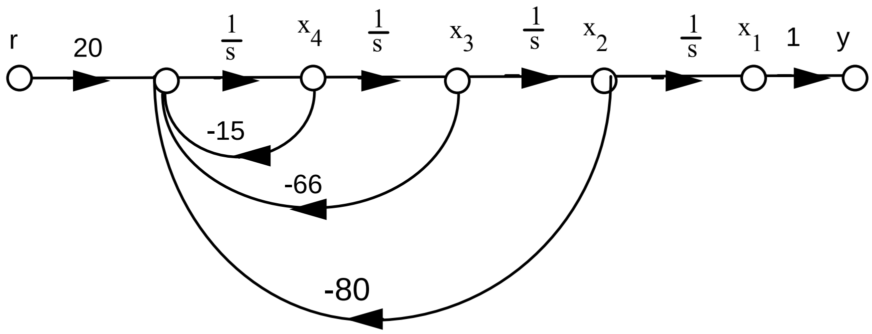
$$\ddddot{c} + 15\dddot{c} + 66\ddot{c} + 80\dot{c} = 20r$$

Let,

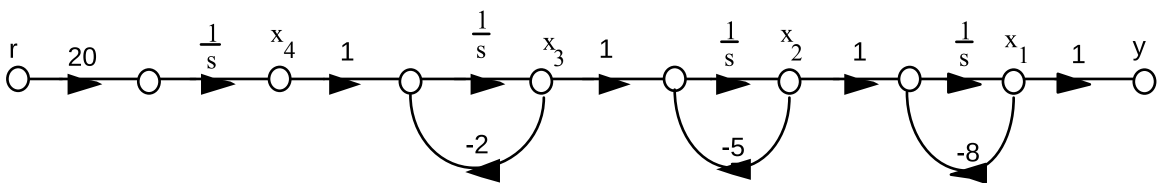
$$c = x_1$$
$$\dot{c} = x_2$$
$$\ddot{c} = x_3$$
$$\dddot{c} = x_4$$

Therefore,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -80x_2 - 66x_3 - 15x_4 + 20r$$
$$y = x_1$$



b. $G(s) = (\frac{20}{s})(\frac{1}{s+2})(\frac{1}{s+5})(\frac{1}{s+8})$. Hence,



From which,

$$\dot{x}_1 = -8x_1 + x_2$$
$$\dot{x}_2 = -5x_2 + x_3$$
$$\dot{x}_3 = -2x_3 + x_4$$
$$\dot{x}_4 = 20r$$
$$y = x_1$$

26.

$\Delta = 1 + [G_2G_3G_4 + G_3G_4 + G_4 + 1] + [G_3G_4 + G_4]$; $T_1 = G_1G_2G_3G_4$; $\Delta_1 = 1$. Therefore,

$$T(s) = \frac{T_1\Delta_1}{\Delta} = \frac{G_1G_2G_3G_4}{2 + G_2G_3G_4 + 2G_3G_4 + 2G_4}$$

27.

Closed-loop gains: $G_2G_4G_6G_7H_3$; $G_2G_5G_6G_7H_3$; $G_3G_4G_6G_7H_3$; $G_3G_5G_6G_7H_3$; G_6H_1; G_7H_2

Forward-path gains: $T_1 = G_1G_2G_4G_6G_7$; $T_2 = G_1G_2G_5G_6G_7$; $T_3 = G_1G_3G_4G_6G_7$; $T_4 =$

$G_1G_3G_5G_6G_7$

Nontouching loops 2 at a time: $G_6H_1G_7H_2$

$\Delta = 1 - [H_3G_6G_7(G_2G_4 + G_2G_5 + G_3G_4 + G_3G_5) + G_6H_1 + G_7H_2] + [G_6H_1G_7H_2]$

$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$

$$T(s) = \frac{T_1\Delta_1 + T_2\Delta_2 + T_3\Delta_3 + T_4\Delta_4}{\Delta}$$

$$= \frac{G_1G_2G_4G_6G_7 + G_1G_2G_5G_6G_7 + G_1G_3G_4G_6G_7 + G_1G_3G_5G_6G_7}{1 - H_3G_6G_7(G_2G_4 + G_2G_5 + G_3G_4 + G_3G_5) - G_6H_1 - G_7H_2 + G_6H_1G_7H_2}$$

28.

Closed-loop gains: $-s^2; -\frac{1}{s}\ ;\ -\frac{1}{s}\ ; -s^2$

Forward-path gains: $T_1 = s;\ T_2 = \frac{1}{s^2}$

Nontouching loops: None

$\Delta = 1 - (-s^2 - \frac{1}{s} - \frac{1}{s} - s^2)$

$\Delta_1 = \Delta_2 = 1$

$$G(s) = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta} = \frac{s + \frac{1}{s^2}}{1 + (s^2 + \frac{1}{s} + \frac{1}{s} + s^2)} = \frac{s^3+1}{2s^4+s^2+2s}$$

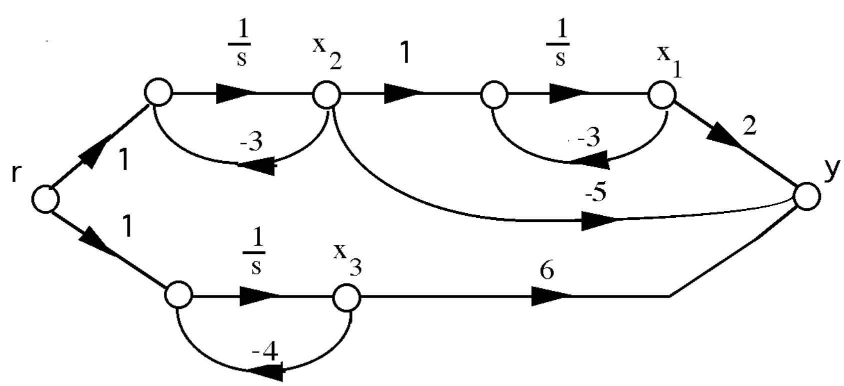
29.

$$T(s) = \frac{G_1\left(\frac{G_2G_3G_4G_5}{(1-G_2H_1)(1-G_4H_2)}\right)}{1 - \frac{G_2G_3G_4G_5G_6G_7G_8}{(1-G_2H_1)(1-G_4H_2)(1-G_7H_4)}} =$$

$$\frac{G_1G_2G_3G_4G_5(1-G_7H_4)}{1-G_2H_1-G_4H_2+G_2G_4H_1H_2-G_7H_4+G_2G_7H_1H_4+G_4G_7H_2H_4-G_2G_4G_7H_1H_2H_4-G_2G_3G_4G_5G_6G_7G_8}$$

30.

a. $G(s) = \dfrac{(s+1)(s+2)}{(s+3)^2(s+4)} = \dfrac{2}{(s+3)^2} - \dfrac{5}{s+3} + \dfrac{6}{s+4}$



Writing the state and output equations,

$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = -3x_2 + r$$

$$\dot{x}_3 = -4x_3 + r$$

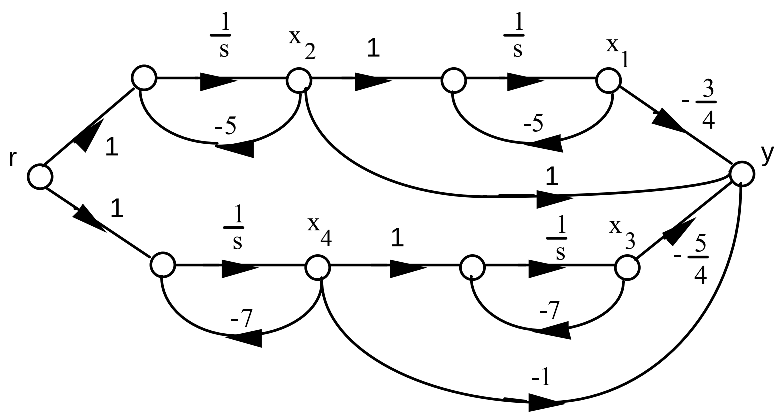
$$y = 2x_1 - 5x_2 + 6x_3$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & -5 & 6 \end{bmatrix}$$

b. G(s) = $G(s) = \frac{(s+2)}{(s+5)^2(s+7)^2} = -\frac{3/4}{(s+5)^2} + \frac{1}{s+5} - \frac{5/4}{(s+7)^2} - \frac{1}{s+7}$



Writing the state and output equations,

$$\dot{x}_1 = -5x_1 + x_2$$

$$\dot{x}_2 = -5x_2 + r$$

$$\dot{x}_3 = -7x_3 + x_4$$

$$\dot{x}_4 = -7x_4 + r$$

$$y = -\frac{3}{4}x_1 + x_2 - \frac{5}{4}x_3 - x_4$$

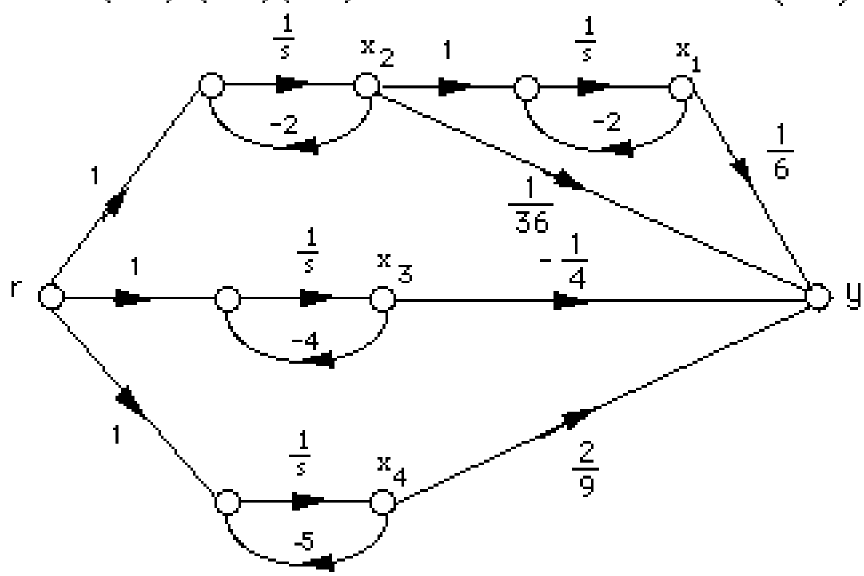
In vector matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -7 & 1 \\ 0 & 0 & 0 & -7 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} -\frac{3}{4} & 1 & -\frac{5}{4} & -1 \end{bmatrix} \mathbf{x}$$

c.

$$G(s) = \frac{s+3}{(s+2)^2(s+4)(s+5)} = \frac{2}{9}\frac{1}{s+5} - \frac{1}{4}\frac{1}{s+4} + \frac{1}{36}\frac{1}{s+2} + \frac{1}{6}\frac{1}{(s+2)^2}$$



Writing the state and output equations,

$$\dot{x}_1 = -2x_1 + x_2$$

$$\dot{x}_2 = -2x_2 + r$$

$$\dot{x}_3 = -4x_3 + r$$

$$\dot{x}_4 = -5x_4 + r$$

$$y = \frac{1}{6}x_1 + \frac{1}{36}x_2 - \frac{1}{4}x_3 + \frac{2}{9}x_4$$

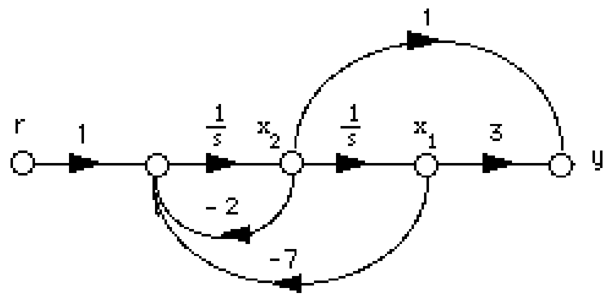
In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} \frac{1}{6} & \frac{1}{36} & -\frac{1}{4} & \frac{2}{9} \end{bmatrix} \mathbf{x}$$

31.

a.



Writing the state equations,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -7x_1 - 2x_2 + r$$

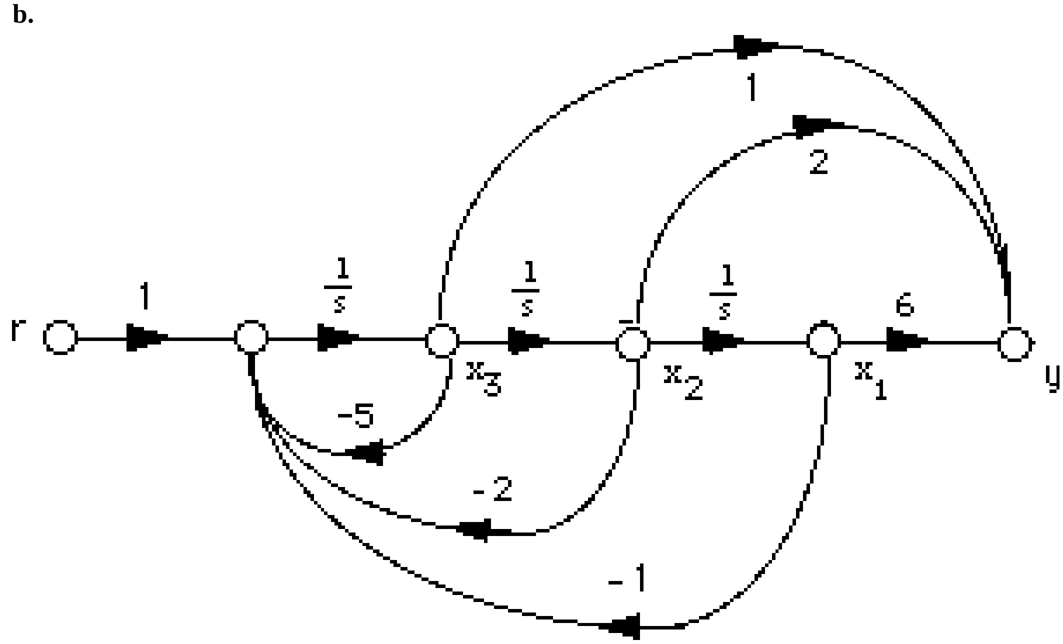
$$y = 3x_1 + x_2$$

In vector matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -7 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix}\mathbf{x}$$

b.



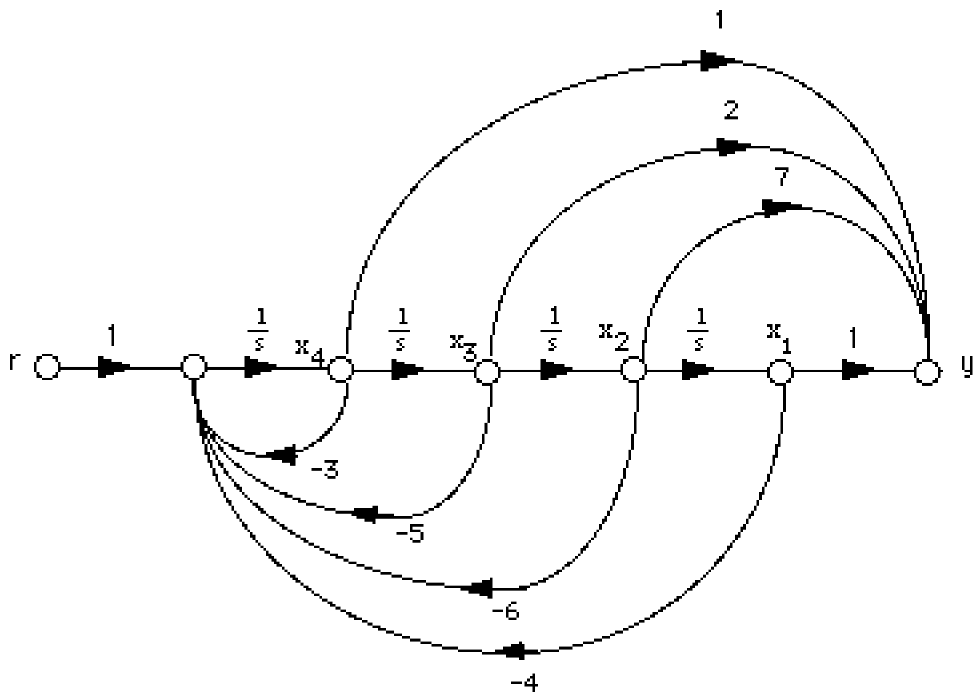
Writing the state equations,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -x_1 - 2x_2 - 5x_3 + r$$
$$y = 6x_1 + 2x_2 + x_3$$

In vector matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -5 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$
$$y = \begin{bmatrix} 6 & 2 & 1 \end{bmatrix}\mathbf{x}$$

c.



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -4x_1 - 6x_2 - 5x_3 - 3x_4 + r$$
$$y = x_1 + 7x_2 + 2x_3 + x_4$$

In vector matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -6 & -5 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 7 & 2 & 1 \end{bmatrix} \mathbf{x}$$

32.

a. Controller canonical form:

From the phase-variable form in Problem 5.31(a), reverse the order of the state variables and obtain,

$$\dot{x}_2 = x_1$$

$$\dot{x}_1 = -7x_2 - 2x_1 + r$$

$$y = 3x_2 + x_1$$

Putting the equations in order,

$$\dot{x}_1 = -2x_1 - 7x_2 + r$$

$$\dot{x}_2 = x_1$$

$$y = x_1 + 3x_2$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & -7 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 3 \end{bmatrix} \mathbf{x}$$

Observer canonical form:

$G(s) = \dfrac{s+3}{s^2+2s+7}$. Divide each term by $\dfrac{1}{s^2}$ and get

$$G(s) = \frac{\frac{1}{s} + \frac{3}{s^2}}{1 + \frac{2}{s} + \frac{7}{s^2}} = \frac{C(s)}{R(s)}$$

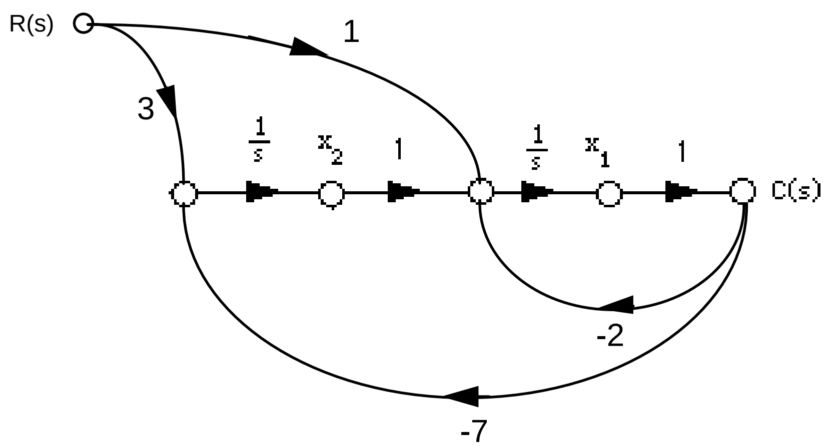
Cross multiplying,

$$\left(\frac{1}{s} + \frac{3}{s^2}\right) R(s) = \left(1 + \frac{2}{s} + \frac{7}{s^2}\right) C(s)$$

Thus,

$$\frac{1}{s}(R(s) - 2C(s)) + \frac{1}{s^2}(3R(s) - 7C(s)) = C(s)$$

Drawing the signal-flow graph,



Writing the state and output equations,

$$\dot{x}_1 = -2x_1 + x_2 + r$$

$$\dot{x}_2 = -7x_1 + 3r$$

$$y = x_1$$

In vector matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 1 \\ -7 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}r$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}$$

b. Controller canonical form:

From the phase-variable form in Problem 5.31(b), reverse the order of the state variables and obtain,

$$\dot{x}_3 = x_2$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_1 = -x_3 - 2x_2 - 5x_1$$

$$y = 6x_3 + 2x_2 + x_1$$

Putting the equations in order,

$$\dot{x}_1 = -5x_1 - 2x_2 - x_3$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = x_2$$

$$y = x_1 + 2x_2 + 6x_3$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 2 & 6 \end{bmatrix}\mathbf{x}$$

Observer canonical form:

$G(s) = \dfrac{s^2 + 2s + 6}{s^3 + 5s^2 + 2s + 1}$. Divide each term by $\dfrac{1}{s^3}$ and get

$$G(s) = \frac{\frac{1}{s} + \frac{2}{s^2} + \frac{6}{s^3}}{1 + \frac{5}{s} + \frac{2}{s^2} + \frac{1}{s^3}} = \frac{C(s)}{R(s)}$$

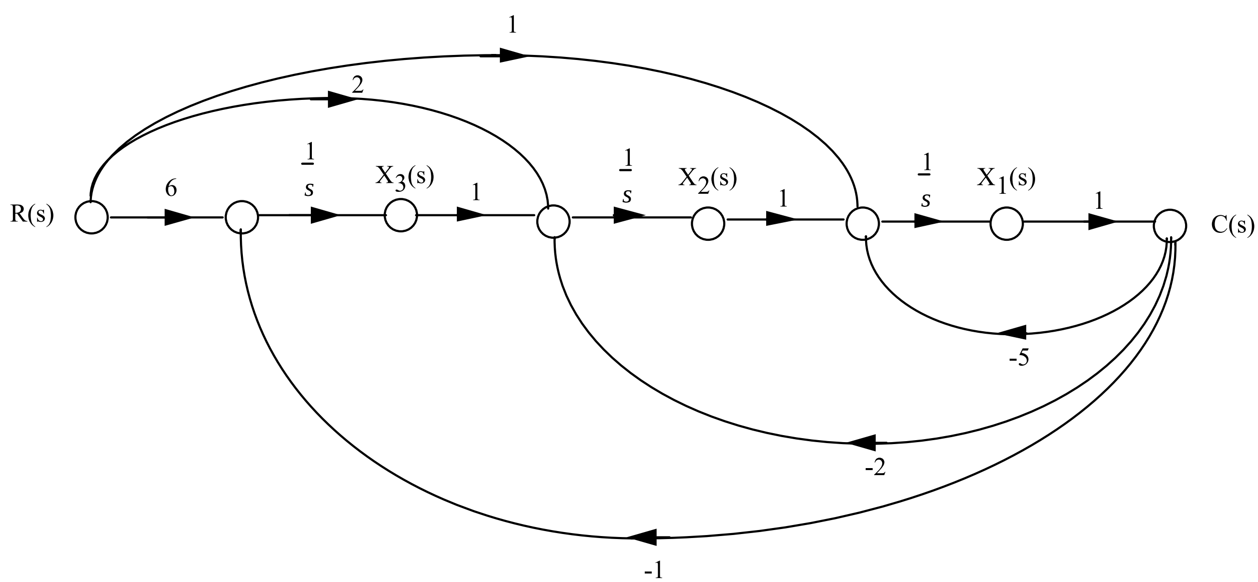
Cross-multiplying,

$$\left(\frac{1}{s} + \frac{2}{s^2} + \frac{6}{s^3}\right)R(s) = \left(1 + \frac{5}{s} + \frac{2}{s^2} + \frac{1}{s^3}\right)C(s)$$

Thus,

$$\frac{1}{s}(R(s) - 5c(s)) + \frac{1}{s^2}(2R(s) - 2C(s)) + \frac{1}{s^3}(6R(s) - C(s)) = C(s)$$

Drawing the signal-flow graph,



Writing the state and output equations,

$$\dot{x}_1 = -5x_1 + x_2 + r$$

$$\dot{x}_2 = -2x_1 + x_3 + 2r$$

$$\dot{x}_3 = -x_1 + 6r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

c. Controller canonical form:

From the phase-variable form in Problem 5.31(c), reverse the order of the state variables and obtain,

$$\dot{x}_4 = x_3$$

$$\dot{x}_3 = x_2$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_1 = -4x_4 - 6x_3 - 5x_2 - 3x_1 + r$$

$$y = x_4 + 7x_3 + 2x_2 + x_1$$

Putting the equations in order,

$$\dot{x}_1 = -3x_1 - 5x_2 - 6x_3 - 4x_4 + r$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = x_2$$

$$\dot{x}_4 = x_3$$

$$y = x_1 + 2x_2 + 7x_3 + x_4$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & -5 & -6 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 2 & 7 & 1 \end{bmatrix}\mathbf{x}$$

Observer canonical form:

$G(s) = \dfrac{s^3+2s^2+7s+1}{s^4+3s^3+5s^2+6s+4}$. Divide each term by $\dfrac{1}{s^2}$ and get

$$G(s) = \frac{\frac{1}{s} + \frac{2}{s^2} + \frac{7}{s^3} + \frac{1}{s^4}}{1 + \frac{3}{s} + \frac{5}{s^2} + \frac{6}{s^3} + \frac{4}{s^4}} = \frac{C(s)}{R(s)}$$

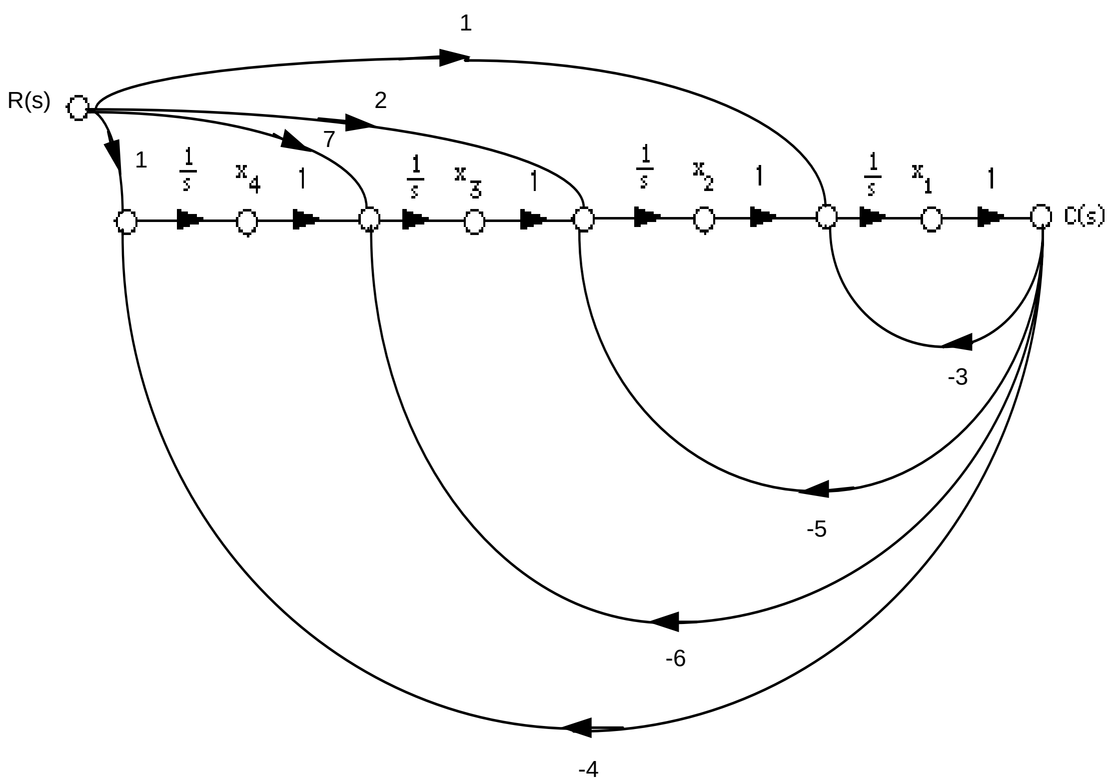
Cross multiplying,

$$\left(\frac{1}{s} + \frac{2}{s^2} + \frac{7}{s^3} + \frac{1}{s^4}\right) R(s) = \left(1 + \frac{3}{s} + \frac{5}{s^2} + \frac{6}{s^3} + \frac{4}{s^4}\right) C(s)$$

Thus,

$$\frac{1}{s}(R(s) - 3C(s)) + \frac{1}{s^2}(2R(s) - 5C(s)) + \frac{1}{s^3}(7R(s) - 6C(s)) + \frac{1}{s^4}(R(s) - 4C(s)) = C(s)$$

Drawing the signal-flow graph,



Writing the state and output equations,

$$\dot{x}_1 = -3x_1 + x_2 + r$$

$$\dot{x}_2 = -5x_1 + x_3 + 2r$$

$$\dot{x}_3 = -6x_1 + x_4 + 7r$$

$$\dot{x}_4 = -4x_1 + r$$

$$y = x_1$$

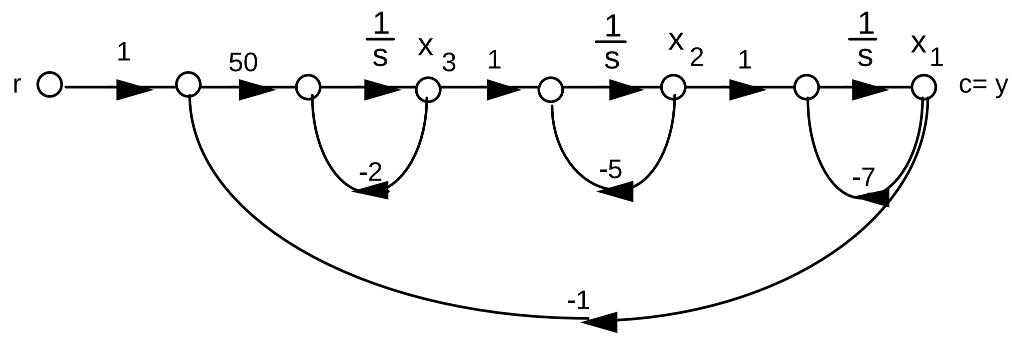
In vector matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -6 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 2 \\ 7 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

33.

a.



Writing the state equations,

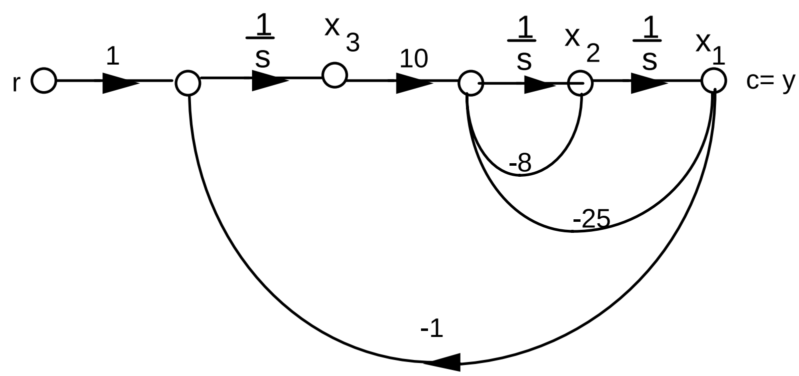
$$\dot{x}_1 = -7x_1 + x_2$$
$$\dot{x}_2 = -5x_2 + x_3$$
$$\dot{x}_3 = -50x_1 - 2x_3 + 50r$$
$$y = x_1$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -7 & 1 & 0 \\ 0 & -5 & 1 \\ -50 & 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

b.



Writing the state equations,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -25x_1 - 8x_2 + 10x_3$$

$$\dot{x}_3 = -x_1 + r$$

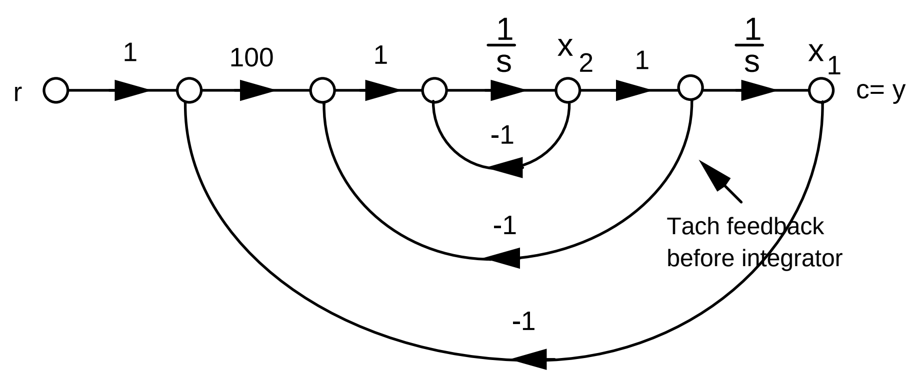
$$y = x_1$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -25 & -8 & 10 \\ -1 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

c.



$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - x_2 + 100(r-x_1) = -100x_1 - 2x_2 + 100r$$

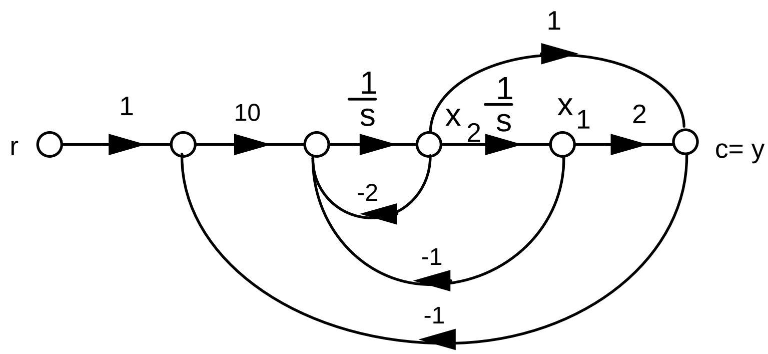
$$y = x_1$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -100 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}$$

d. Since $\frac{1}{(s+1)^2} = \frac{1}{s^2+2s+1}$, we draw the signal-flow as follows:



Writing the state equations,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - 2x_2 + 10(r-c) = -x_1 - 2x_2 + 10(r - (2x_1+x_2)) = -21x_1 - 12x_2 + 10r$$

$$y = 2x_1 + x_2$$

In vector-matrix form,

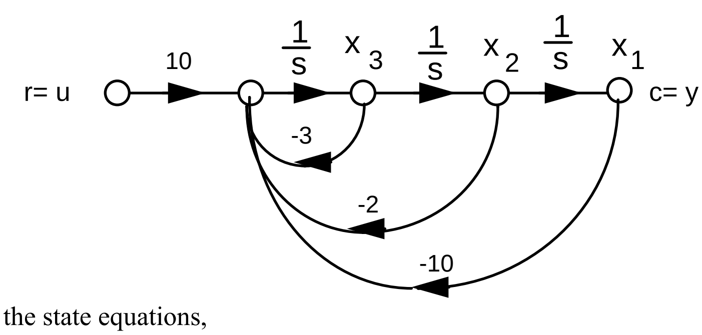
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -21 & -12 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix}\mathbf{x}$$

34.

a. Phase-variable form:

$$T(s) = \frac{10}{s^3+3s^2+2s+10}$$



Writing the state equations,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -10x_1 - 2x_2 - 3x_3 + 10u$$

$$y = x_1$$

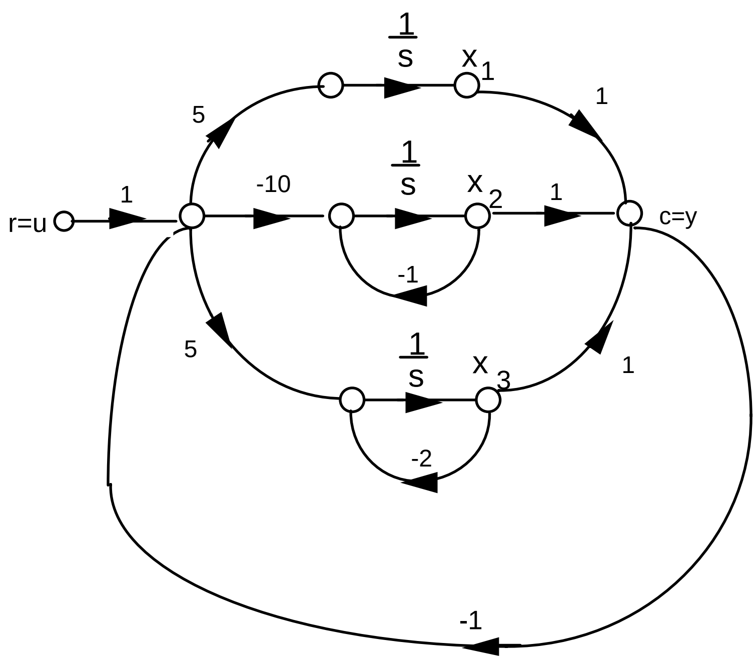
In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

b. Parallel form:

$$G(s) = \frac{5}{s} + \frac{-10}{s+1} + \frac{5}{s+2}$$



Writing the state equations,

$$\dot{x}_1 = 5(u - x_1 - x_2 - x_3) = -5x_1 - 5x_2 - 5x_3 + 5u$$

$$\dot{x}_2 = -10(u - x_1 - x_2 - x_3) - x_2 = 10x_1 + 9x_2 + 10x_3 - 10u$$

$$\dot{x}_3 = 5(u - x_1 - x_2 - x_3) - 2x_3 = -5x_1 - 5x_2 - 7x_3 + 5u$$

$$y = x_1 + x_2 + x_3$$

In vector-matrix form,

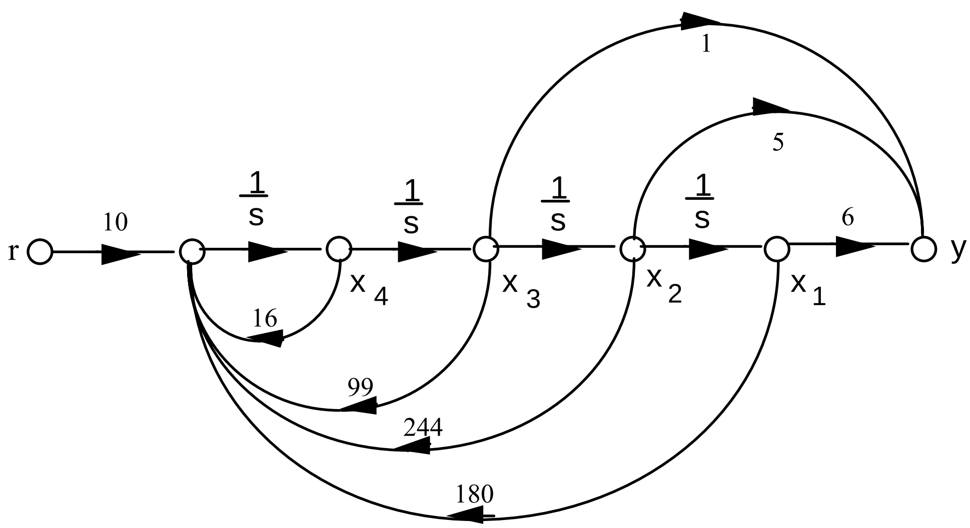
$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & -5 & -5 \\ 10 & 9 & 10 \\ -5 & -5 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 5 \\ -10 \\ 5 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\mathbf{x}$$

35.

a. $T(s) = \dfrac{10(s^2 + 5s + 6)}{s^4 + 16s^3 + 99s^2 + 244s + 180}$

Drawing the signal-flow diagram,



Writing the state and output equations,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_1 = -180x_1 - 244x_2 - 99x_3 - 16x_4 + 10r$$

$$y = 6x_1 + 5x_2 + x_3$$

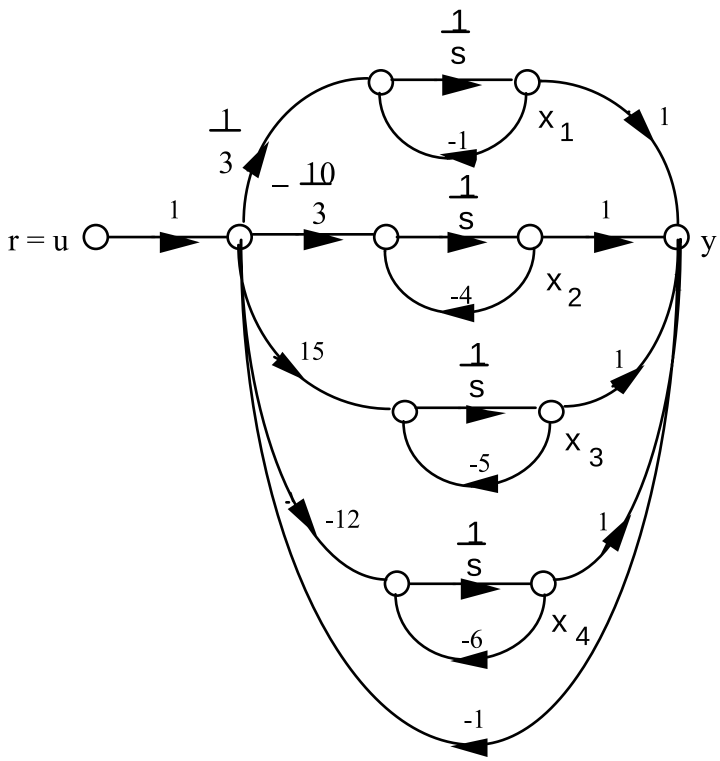
In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -180 & -244 & -99 & -16 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}r$$

$$y = \begin{bmatrix} 6 & 5 & 1 & 0 \end{bmatrix}\mathbf{x}$$

b. $G(s) = \dfrac{10(s+2)(s+3)}{(s+1)(s+4)(s+5)(s+6)} = \dfrac{1/3}{s+1} - \dfrac{10/3}{s+4} + \dfrac{15}{s+5} - \dfrac{12}{s+6}$

Drawing the signal-flow diagram and including the unity-feedback path,



Writing the state and output equations,

$$\dot{x}_1 = \frac{1}{3}(u - x_1 - x_2 - x_3 - x_4) - x_1$$
$$\dot{x}_2 = \frac{-10}{3}(u - x_1 - x_2 - x_3 - x_4) - 4x_2$$
$$\dot{x}_3 = 15(u - x_1 - x_2 - x_3 - x_4) - 5x_3$$
$$\dot{x}_4 = -12(u - x_1 - x_2 - x_3 - x_4) - 12x_4$$
$$y = x_1 + x_2 + x_3 + x_4$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{10}{3} & -\frac{2}{3} & \frac{10}{3} & \frac{10}{3} \\ -15 & -15 & -20 & -15 \\ 12 & 12 & 12 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{3} \\ -\frac{10}{3} \\ 15 \\ -12 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\mathbf{x}$$

36.

Program:

```
'(a)'
'G(s)'
G=zpk([-2 -3],[-1 -4 -5 -6],10)
'T(s)'
T=feedback(G,1,-1)
[numt,dent]=tfdata(T,'v');
'Find controller canonical form'
[Acc,Bcc,Ccc,Dcc]=tf2ss(numt,dent)
A1=flipud(Acc);
'Transform to phase-variable form'
Apv=fliplr(A1)
Bpv=flipud(Bcc)
Cpv=fliplr(Ccc)
'(b)'
'G(s)'
G=zpk([-2 -3],[-1 -4 -5 -6],10)
'T(s)'
T=feedback(G,1,-1)
[numt,dent]=tfdata(T,'v');
'Find controller canonical form'
[Acc,Bcc,Ccc,Dcc]=tf2ss(numt,dent)
'Transform to modal form'
[A,B,C,D]=canon(Acc,Bcc,Ccc,Dcc,'modal')
```

Computer response:

```
ans =

(a)

ans =

G(s)

Zero/pole/gain:
    10 (s+2) (s+3)
-----------------------
(s+1) (s+4) (s+5) (s+6)

ans =

T(s)

Zero/pole/gain:

              10 (s+2) (s+3)
-----------------------------------------
(s+1.264) (s+3.412) (s^2 + 11.32s + 41.73)

ans =
```

```
Find controller canonical form

Acc =

  -16.0000  -99.0000 -244.0000 -180.0000
    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0
Bcc =

     1
     0
     0
     0

Ccc =

         0   10.0000   50.0000   60.0000

Dcc =

     0

ans =

Transform to phase-variable form

Apv =

         0    1.0000         0         0
         0         0    1.0000         0
         0         0         0    1.0000
 -180.0000 -244.0000  -99.0000  -16.0000

Bpv =

     0
     0
     0
     1

Cpv =

   60.0000   50.0000   10.0000         0

ans =

(b)

ans =

G(s)

Zero/pole/gain:
    10 (s+2) (s+3)
-----------------------
(s+1) (s+4) (s+5) (s+6)

ans =

T(s)

Zero/pole/gain:
```

```
              10 (s+2) (s+3)
------------------------------------------
(s+1.264) (s+3.412) (s^2 + 11.32s + 41.73)

ans =

Find controller canonical form

Acc =

  -16.0000  -99.0000 -244.0000 -180.0000
    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0

Bcc =

     1
     0
     0
     0

Ccc =

         0   10.0000   50.0000   60.0000

Dcc =

     0

ans =

Transform to modal form

A =

   -5.6618    3.1109         0         0
   -3.1109   -5.6618         0         0
         0         0   -3.4124         0
         0         0         0   -1.2639

B =

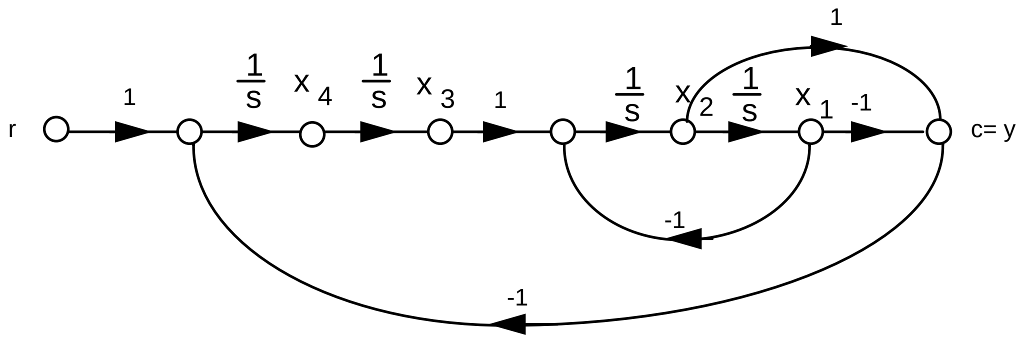
   -4.1108
    1.0468
    1.3125
    0.0487
C =

    0.1827    0.6973   -0.1401    4.2067

D =

     0
```

37.



Writing the state equations,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = x_1 - x_2 + r$$

$$y = -x_1 + x_2$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{r}$$

$$y = c = [-1 \quad 1 \quad 0 \quad 0]\,\mathbf{x}$$

38.

a.

$$\ddot{\theta}_1 + 5\dot{\theta}_1 + 6\theta_1 - 3\dot{\theta}_2 - 4\theta_2 = 0$$

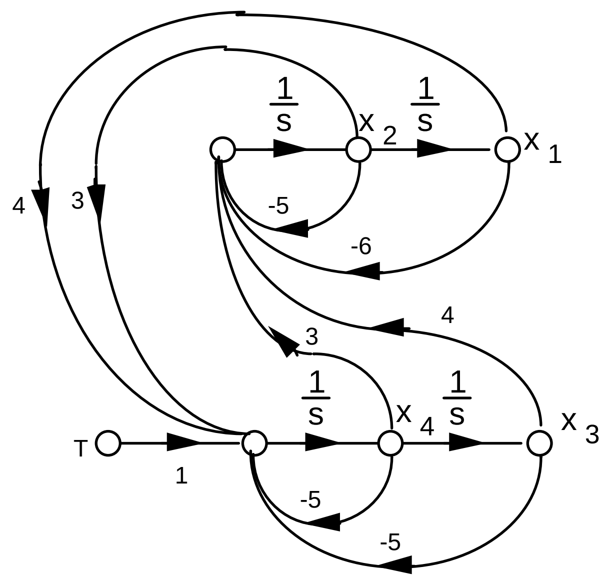
$$-3\dot{\theta}_1 - 4\theta_1 + \ddot{\theta}_2 + 5\dot{\theta}_2 + 5\theta_2 = T$$

or

$$\ddot{\theta}_1 = -5\dot{\theta}_1 - 6\theta_1 + 3\dot{\theta}_2 + 4\theta_2$$

$$\ddot{\theta}_2 = 3\dot{\theta}_1 + 4\theta_1 - 5\dot{\theta}_2 - 5\theta_2 + T$$

Letting, $\theta_1 = x_1$; $\dot{\theta}_1 = x_2$; $\theta_2 = x_3$; $\dot{\theta}_2 = x_4$,



where x = θ.

b. Using the signal-flow diagram,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -6x_1 - 5x_2 + 4x_3 + 3x_4$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = 4x_1 + 3x_2 - 5x_3 - 5x_4 + T$$

$$y = x_3$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -5 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 4 & 3 & -5 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} T$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}$$

39.

Program:

```
numg=7;
deng=poly([0 -9 -12]);
G=tf(numg,deng);
T=feedback(G,1)
[numt,dent]=tfdata(T,'v')
[A,B,C,D]=tf2ss(numt,dent); %Obtain controller canonical form
'(a)'                       %Display label
A=flipud(A);                %Convert to phase-variable form
A=fliplr(A)                 %Convert to phase-variable form
B=flipud(B)                 %Convert to phase-variable form
C=fliplr(C)                 %Convert to phase-variable form
```

```
'(b)'                        %Display label
[a,b,c,d]=canon(A,B,C,D)     %Convert to parallel form
```

Computer response:

```
Transfer function:
           7
------------------------
s^3 + 21 s^2 + 108 s + 7

numt =

     0     0     0     7

dent =

     1    21   108     7

ans =

(a)

A =

     0     1     0
     0     0     1
    -7  -108   -21

B =

     0
     0
     1

C =

     7     0     0

ans =
```

```
(b)

a =

   -0.0657         0         0
         0  -12.1807         0
         0         0   -8.7537


b =

   -0.0095
   -3.5857
    2.5906


c =

   -6.9849   -0.0470   -0.0908


d =

     0
```

40.

$$\dot{\mathbf{x}}_1 = \mathbf{A_1x_1} + \mathbf{B_1}r \quad (1)$$

$$y_1 = \mathbf{C_1x_1} \quad (2)$$

$$\dot{\mathbf{x}}_2 = \mathbf{A_2x_2} + \mathbf{B_2}y_1 \quad (3)$$

$$y_2 = \mathbf{C_2x_2} \quad (4)$$

Substituting Eq. (2) into Eq. (3),

$$\dot{\mathbf{x}}_1 = \mathbf{A_1x_1} + \mathbf{B_1}r$$

$$\dot{\mathbf{x}}_2 = \mathbf{B_2C_1x_1} + \mathbf{A_2x_2}$$

$$y_2 = \mathbf{C_2x_2}$$

In vector-matrix notation,

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A_1} & \mathbf{O} \\ \mathbf{B_2C_1} & \mathbf{A_2} \end{bmatrix} \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} + \begin{bmatrix} \mathbf{B_1} \\ \mathbf{O} \end{bmatrix} r$$

$$\mathbf{y}_2 = \begin{bmatrix} \mathbf{O} & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

41.

$$\dot{\mathbf{x}}_1 = \mathbf{A_1x_1} + \mathbf{B_1}r \quad (1)$$

$$y_1 = \mathbf{C_1x_1} \quad (2)$$

$$\dot{\mathbf{x}}_2 = \mathbf{A_2 x_2} + \mathbf{B_2} r \quad (3)$$

$$y_2 = \mathbf{C_2 x_2} \quad (4)$$

In vector-matrix form,

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} r$$

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 = [\mathbf{C}_1 \;\; \mathbf{C}_2] \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

42.

$$\dot{\mathbf{x}}_1 = \mathbf{A_1 x_1} + \mathbf{B_1} e \quad (1)$$

$$y = \mathbf{C_1 x_1} \quad (2)$$

$$\dot{\mathbf{x}}_2 = \mathbf{A_2 x_2} + \mathbf{B_2} y \quad (3)$$

$$p = \mathbf{C_2 x_2} \quad (4)$$

Substituting e = r - p into Eq. (1) and substituting Eq. (2) into (3), we obtain,

$$\dot{\mathbf{x}}_1 = \mathbf{A_1 x_1} + \mathbf{B_1}(r - p) \quad (5)$$

$$y = \mathbf{C_1 x_1} \quad (6)$$

$$\dot{\mathbf{x}}_2 = \mathbf{A_2 x_2} + \mathbf{B_2 C_1 x_1} \quad (7)$$

$$p = \mathbf{C_2 x_2} \quad (8)$$

Substituting Eq. (8) into Eq. (5),

$$\dot{\mathbf{x}}_1 = \mathbf{A_1 x_1 - B_1 C_2 x_2 + B_1 r}$$

$$\dot{\mathbf{x}}_2 = \mathbf{B_2 C_1 x_1 + A_2 x_2}$$

$$\mathbf{y = C_1 x_1}$$

In vector-matrix form,

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & -\mathbf{B}_1\mathbf{C}_2 \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{r}$$

$$y = [\mathbf{C}_1 \;\; \mathbf{0}] \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

43.

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{B}u$$

$$y = \mathbf{CPz}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 2 & 1 & -4 \\ 1 & -2 & 0 \\ 4 & 6 & 2 \end{bmatrix}; \therefore \mathbf{P} = \begin{bmatrix} 0.0606 & 0.3939 & 0.1212 \\ 0.0303 & -0.3030 & 0.0606 \\ -0.2121 & 0.1212 & 0.0758 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} -1.6 & 1.23 & 3.81 \\ 3.33 & 1.33 & -2.33 \\ 1.63 & -1.79 & 1.26 \end{bmatrix}; \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} -3 \\ 4 \\ 4 \end{bmatrix}; \mathbf{CP} = \begin{bmatrix} -0.544 & -0.0702 & 0.912 \end{bmatrix}$$

44.

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{B}u$$

$$y = \mathbf{CPz}$$

$$\boldsymbol{P}^{-1} = \begin{pmatrix} 4 & -1 & 0 \\ 2 & 3 & -2 \\ 8 & 5 & 1 \end{pmatrix} \quad \boldsymbol{P} = \begin{pmatrix} \frac{13}{70} & \frac{1}{70} & \frac{1}{35} \\ -\frac{9}{35} & \frac{2}{35} & \frac{4}{35} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 1.46 & -0.657 & -0.314 \\ -1.94 & -6.46 & 4.09 \\ 0 & -8 & 4 \end{pmatrix} \quad \boldsymbol{P}^{-1}\boldsymbol{B} = \begin{pmatrix} 27 \\ 21 \\ 59 \end{pmatrix} \quad \boldsymbol{CP} = \left(\frac{11}{70}, -\frac{123}{70}, \frac{17}{35}\right)$$

45.

Eigenvalues are -1, -2, and -3 since,

$$|\lambda I - A| = (\lambda + 3)(\lambda + 2)(\lambda + 1)$$

Solving for the eigenvectors, $\mathbf{Ax} = \lambda\mathbf{x}$

or,

$$4x_3 - 5x_2 - (\lambda + 5)x_1 = 0$$
$$-2x_3 - x_2\lambda + 2x_1 = 0$$
$$-2x_2 - (\lambda + 1)x_3 = 0$$

For $\lambda = -1$, $x_2 = 0$, $x_1 = x_3$. For $\lambda = -2$, $x_1 = x_2 = \frac{x_3}{2}$. For $\lambda = -3$, $x_1 = -\frac{x_2}{2}$, $x_2 = x_3$. Thus,

$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{B}u$; $y = \mathbf{CPz}$, where

$$\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}; \quad \boldsymbol{P}^{-1}\boldsymbol{B} = \begin{pmatrix} -12 \\ 8 \\ -3 \end{pmatrix}; \quad \boldsymbol{CP} = (1, 4, 7)$$

46.

Eigenvalues are 1, -2, and 3 since,

$$|\lambda\mathbf{I} - \mathbf{A}| = (\lambda - 3)(\lambda + 2)(\lambda - 1)$$

Solving for the eigenvectors, $\mathbf{Ax} = \lambda\mathbf{x}$

or,

$$(-\lambda - 10)x_1 + 7x_3 - 3x_2 = 0$$
$$\frac{73}{4}x_1 + \left(-\lambda + \frac{25}{4}\right)x_2 - \frac{47}{4}x_3 = 0$$
$$-\frac{29}{4}x_1 - \frac{9}{4}x_2 + \left(-\lambda + \frac{23}{4}\right)x_3 = 0$$

For $\lambda = 1$, $x_1 = x_2 = \frac{x_3}{2}$. For $\lambda = -2$, $x_1 = 2x_3$, $x_2 = -3x_3$. For $\lambda = 3$, $x_1 = x_3$, $x_2 = -2x_3$. Thus,

$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{B}u$; $y = \mathbf{CPz}$, where

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} ; \quad \mathbf{P}^{-1}\mathbf{B} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} ; \quad \mathbf{CP} = (7, 12, 9)$$

47.

Program:

```
A=[-10 -3 7;18.25 6.25 -11.75;-7.25 -2.25 5.75];
B=[1;3;2];
C=[1 -2 4];
[P,d]=eig(A);
Ad=inv(P)*A*P
Bd=inv(P)*B
Cd=C*P
```

Computer response:

```
Ad =

   -2.0000    0.0000    0.0000
   -0.0000    3.0000   -0.0000
    0.0000    0.0000    1.0000


Bd =

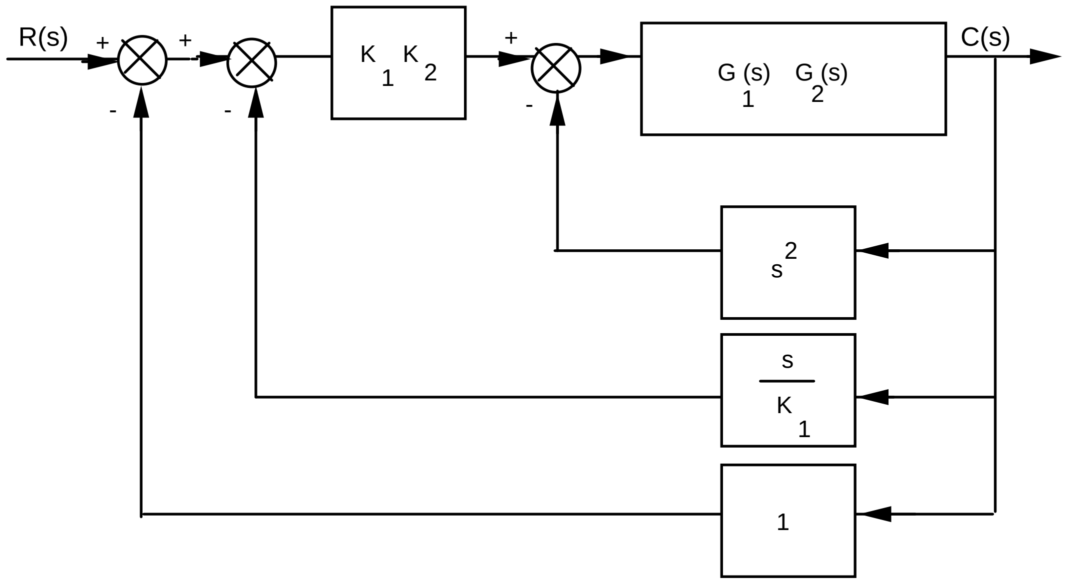
    1.8708
   -3.6742
    3.6742


Cd =

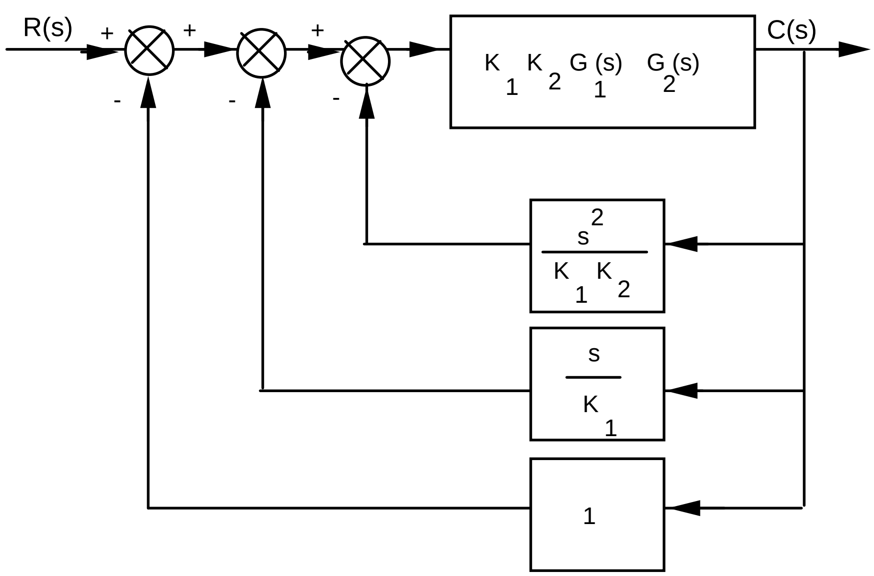
    3.2071    3.6742    2.8577
```

48.

a. Combine $G_1(s)$ and $G_2(s)$. Then push K_1 to the right past the summing junction:

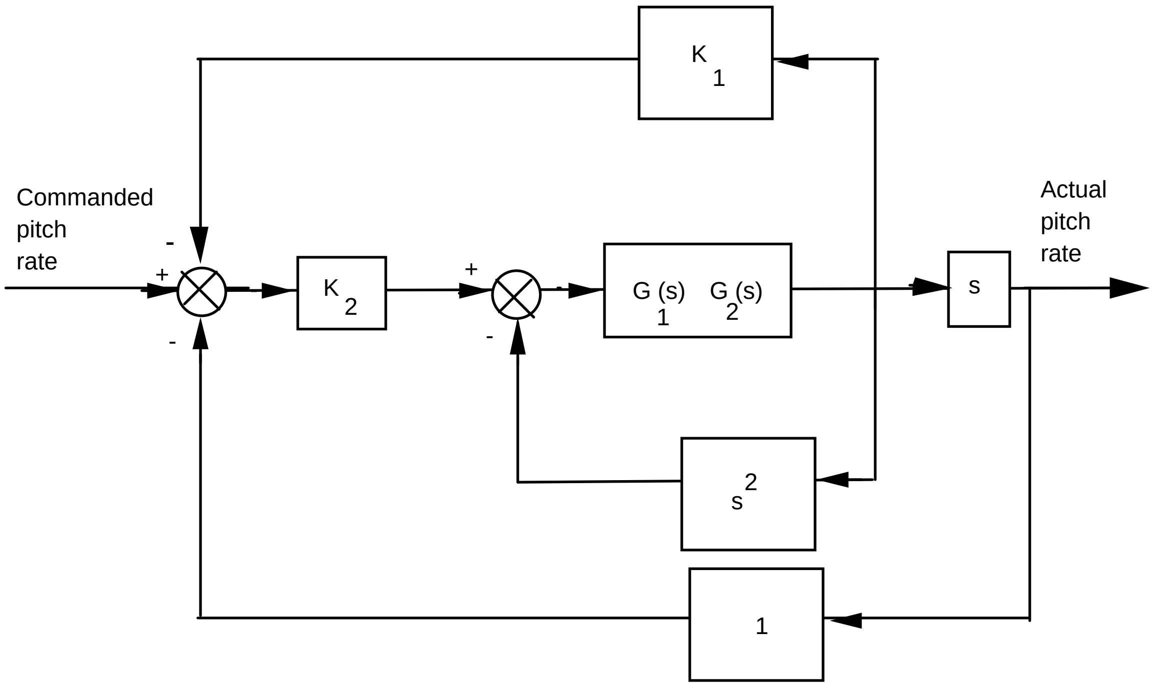


Push K_1K_2 to the right past the summing junction:

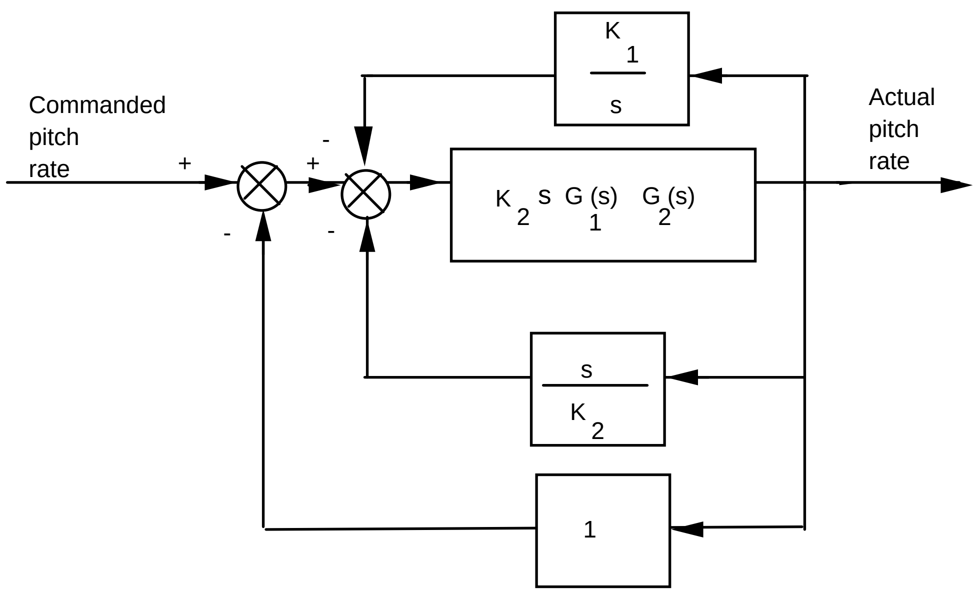


$$\text{Hence, } T(s) = \frac{K_1K_2G_1(s)G_2(s)}{1 + K_1K_2G_1(s)G_2(s)\left(1 + \frac{s}{K_1} + \frac{s^2}{K_1K_2}\right)}$$

b. Rearranging the block diagram to show commanded pitch rate as the input and actual pitch rate as the output:



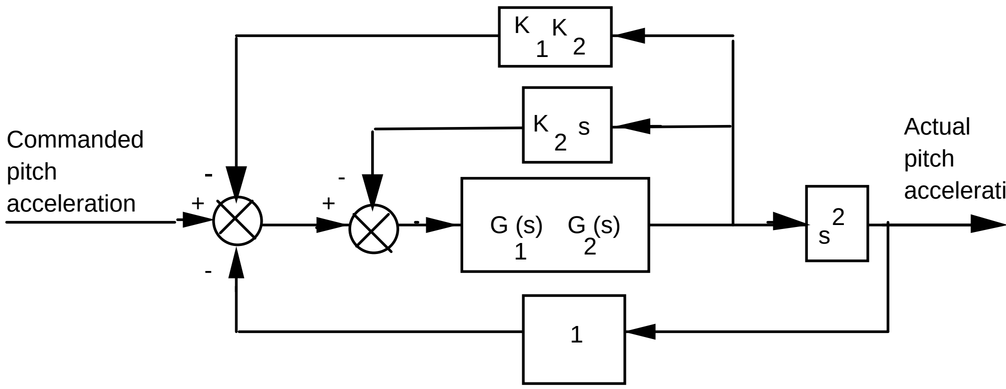
Pushing K_2 to the right past the summing junction; and pushing s to the left past the pick-off point yields,



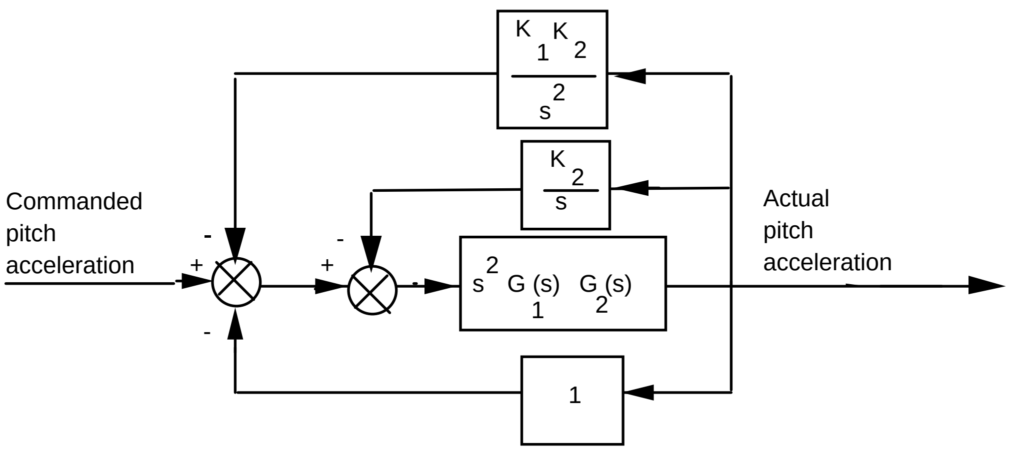
Finding the closed-loop transfer function:

$$T(s) = \frac{K_2 s G_1(s) G_2(s)}{1 + K_2 s G_1(s) G_2(s)\left(1 + \frac{s}{K_2} + \frac{K_1}{s}\right)} = \frac{K_2 s G_1(s) G_2(s)}{1 + G_1(s) G_2(s)(s^2 + K_2 s + K_1 K_2)}$$

c. Rearranging the block diagram to show commanded pitch acceleration as the input and actual pitch acceleration as the output:



Pushing s^2 to the left past the pick-off point yields,

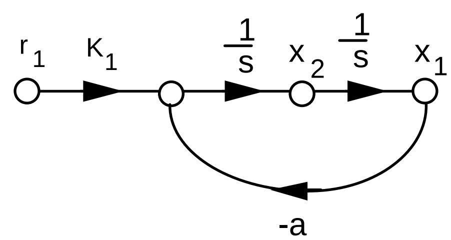


Finding the closed-loop transfer function:

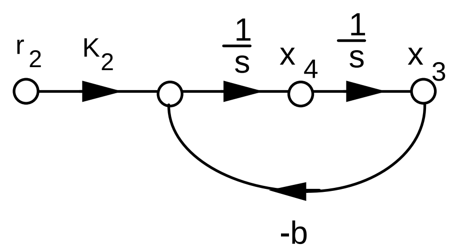
$$T(s) = \frac{s^2 G_1(s)G_2(s)}{1 + s^2 G_1(s)G_2(s)\left(1 + \frac{K_1K_2}{s^2} + \frac{K_2}{s}\right)} = \frac{s^2 G_1(s)G_2(s)}{1 + G_1(s)G_2(s)(s^2 + K_2 s + K_1K_2)}$$

49.

Establish a sinusoidal model for the carrier: $T(s) = \frac{K_1}{s^2+a^2}$



Establish a sinusoidal model for the message: $T(s) = \frac{K_2}{s^2+b^2}$



Writing the state equations,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - a^2 x_1 + K_1 r$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = - b^2 x_3 + K_2 r$$

$$y = x_1 x_3$$

50.

The equivalent forward transfer function is $G(s) = \frac{K_1K_2}{s(s+a_1)}$. The equivalent feedback transfer function is

$H(s) = K_3 + \frac{K_4 s}{s+a_2}$. Hence, the closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K_1K_2(s+a_2)}{s^3 + (a_1+a_2)s^2 + (a_1a_2+K_1K_2K_3+K_1K_2K_4)s + K_1K_2K_3a_2}$$

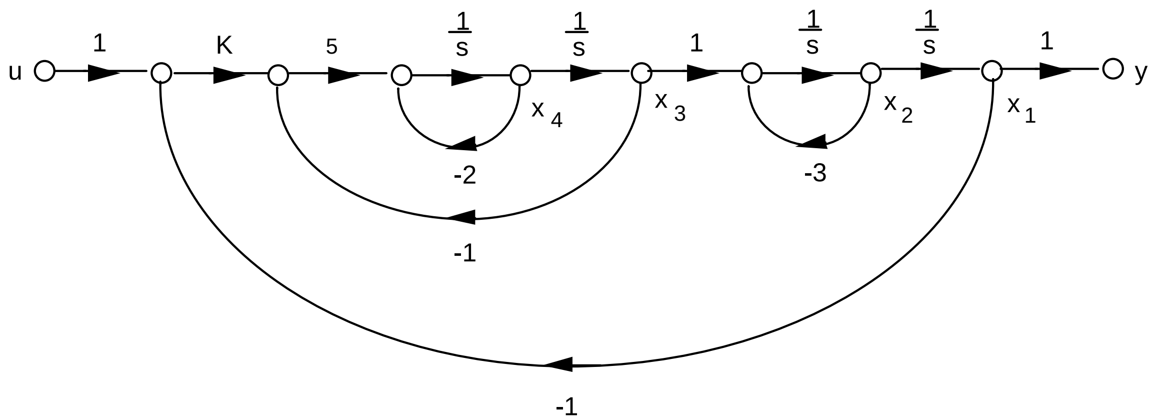
51.

a. The equivalent forward transfer function is

$$G_e(s) = K \frac{\frac{5}{s(s+2)}}{1+\frac{5}{s(s+2)}} \frac{1}{s(s+3)} = \frac{5K}{s(s+3)(s^2+2s+5)}$$

$$T(s) = \frac{G_e}{1+G_e} = \frac{5K}{s^4+5s^3+11s^2+15s+5K}$$

b. Draw the signal-flow diagram:



Writing the state and output equations from the signal-flow diagram:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -3x_2 + x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -5Kx_1 - 5x_3 - 2x_4 + 5Ku$$

$$y = x_1$$

In vector-matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5K & 0 & -5 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5\mathrm{K} \end{bmatrix}u$$

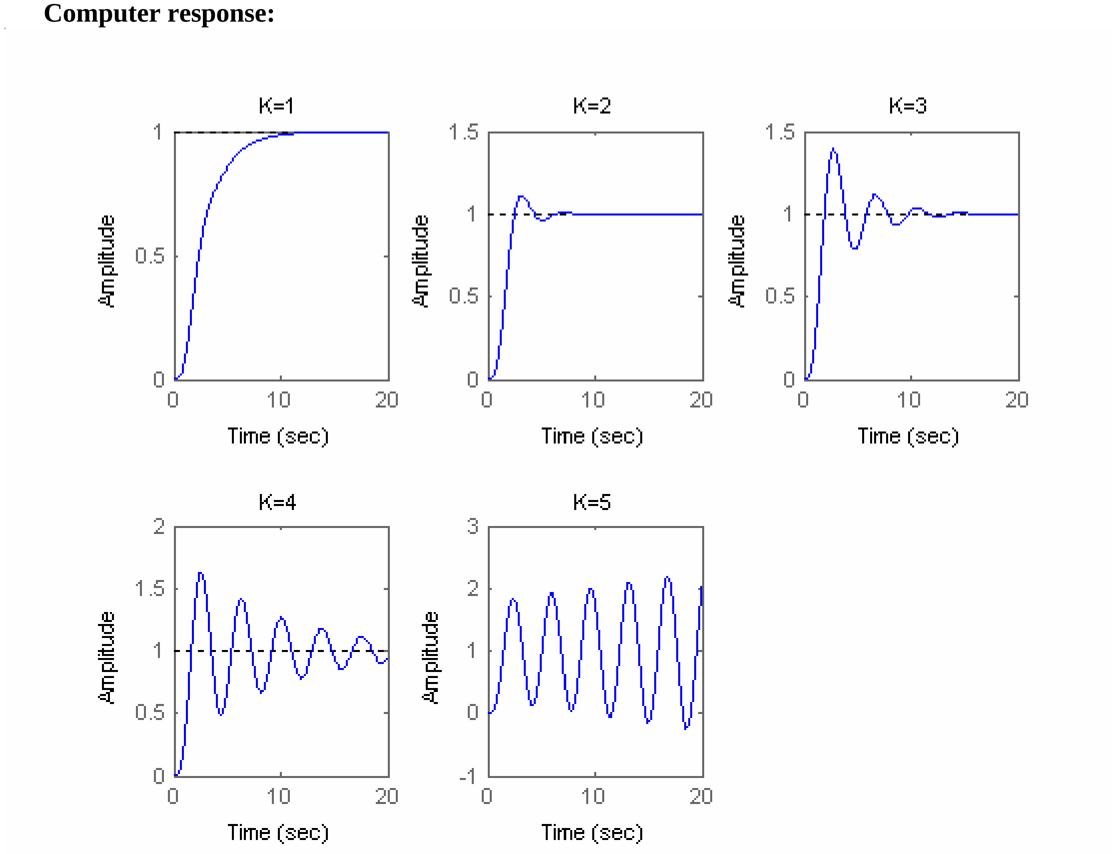
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

c.

Program:

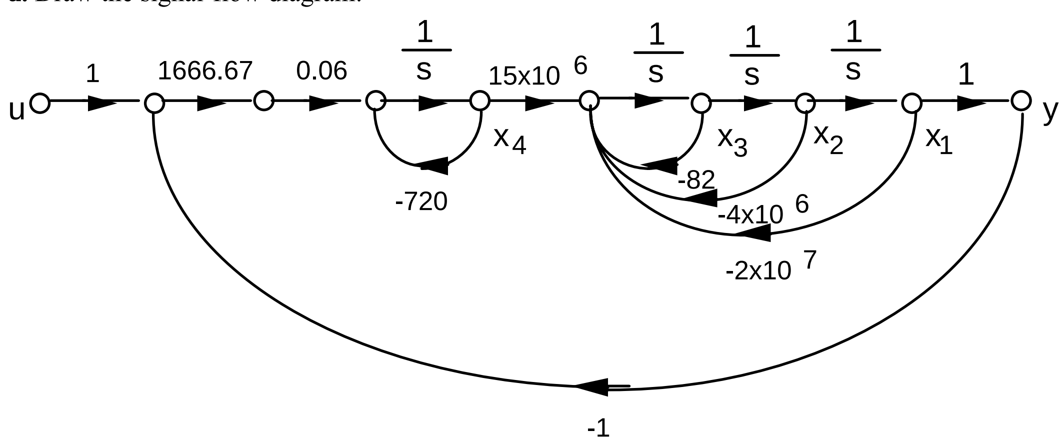
```
for K=1:1:5
numt=5*K;
dent=[1 5 11 15 5*K];
T=tf(numt,dent);
hold on;
subplot(2,3,K);
step(T,0:0.01:20)
title(['K=',int2str(K)])
end
```

Computer response:



52.

a. Draw the signal-flow diagram:



Write state and output equations from the signal-flow diagram:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -2*10^7 x_1 - 4*10^6 x_2 - 82x_3 + 15*10^6 x_4$$

$$\dot{x}_4 = -100x_1 - 720x_4 + 100u$$

$$y = x_1$$

In vector-matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2*10^7 & -4*10^6 & -82 & 15*10^6 \\ -100 & 0 & 0 & -720 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

b.

Program:

```
numg=1666.67*0.06*15e6;
deng=conv([1 720],[1 82 4e6 2e7]);
'G(s)'
G=tf(numg,deng)
'T(s)'
T=feedback(G,1)
step(T)
```

Computer response:

```
ans =

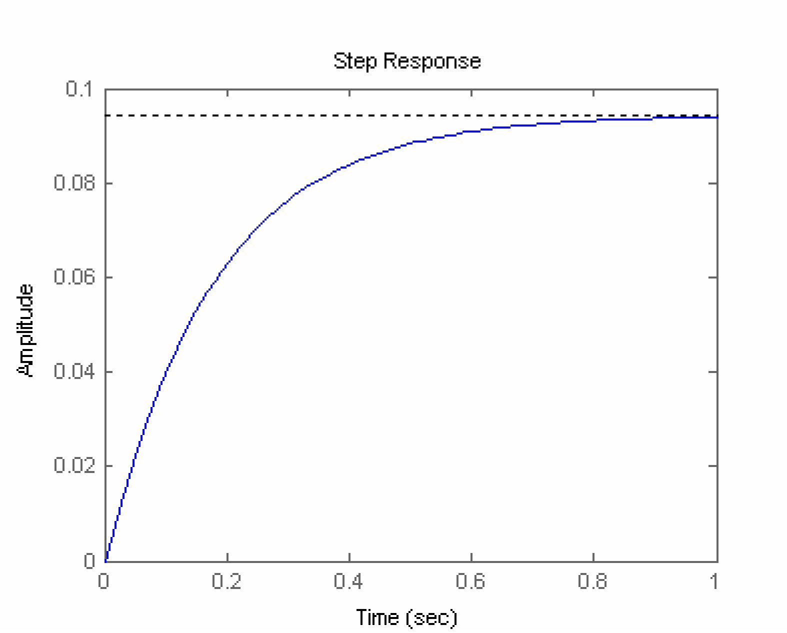
G(s)

Transfer function:
                      1.5e009
----------------------------------------------------
s^4 + 802 s^3 + 4.059e006 s^2 + 2.9e009 s + 1.44e010

ans =

T(s)

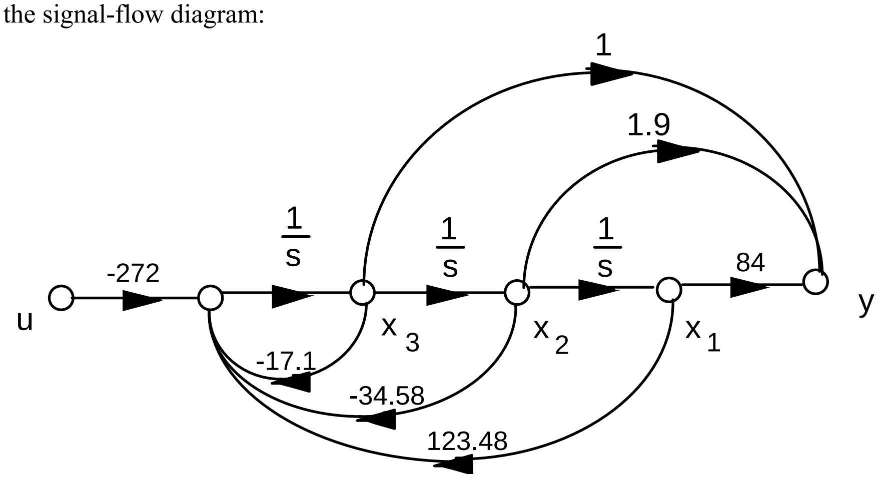
Transfer function:
                      1.5e009
----------------------------------------------------
s^4 + 802 s^3 + 4.059e006 s^2 + 2.9e009 s + 1.59e010
```



53.

a. Phase-variable from: $G(s) = \frac{-272(s^2+1.9s+84)}{s^3+17.1s^2+34.58s-123.48}$

Drawing the signal-flow diagram:



Writing the state and output equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = 123.48x_1 - 34.58x_2 - 17.1x_3 - 272u$$
$$y = 84x_1 + 1.9x_2 + x_3$$

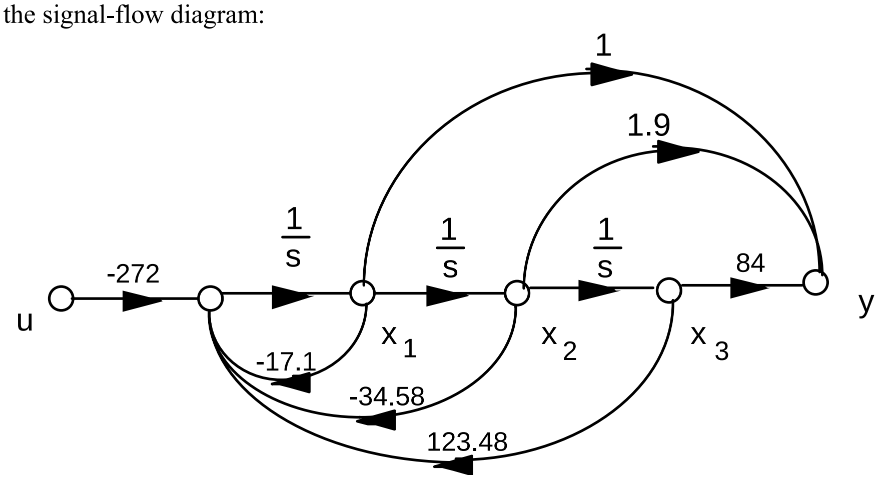
In vector-matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 123.48 & -34.58 & -17.1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ -272 \end{bmatrix}u$$

$$y = \begin{bmatrix} 84 & 1.9 & 1 \end{bmatrix}\mathbf{x}$$

b. Controller canonical form: $G(s) = \dfrac{-272(s^2+1.9s+84)}{s^3+17.1s^2+34.58s-123.48}$

Drawing the signal-flow diagram:



Writing the state and output equations:

$$\dot{x}_1 = -17.1x_1 - 34.58x_2 + 123.48x_3 - 272u$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = x_2$$

$$y = x_1 + 1.9x_2 + 84x_3$$

In vector-matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} -17.1 & -34.58 & 123.48 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -272 \\ 0 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 1.9 & 84 \end{bmatrix}\mathbf{x}$$

c. Observer canonical form: Divide by highest power of s and obtain

$$G(s) = \frac{\dfrac{-272}{s} - \dfrac{516.8}{s^2} - \dfrac{22848}{s^3}}{1 + \dfrac{17.1}{s} + \dfrac{34.58}{s^2} - \dfrac{123.48}{s^3}}$$

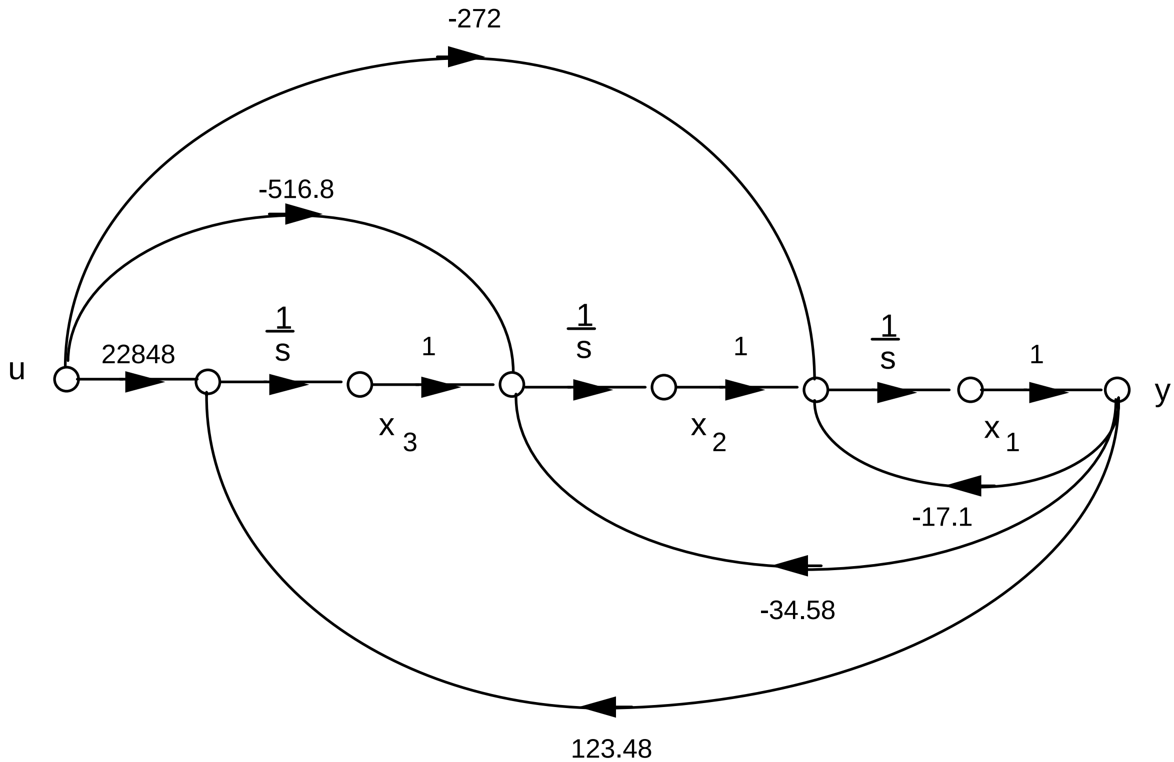
Cross multiplying,

$$\left[\frac{-272}{s} - \frac{516.8}{s^2} - \frac{22848}{s^3}\right]R(s) = \left[1 + \frac{17.1}{s} + \frac{34.58}{s^2} - \frac{123.48}{s^3}\right]C(s)$$

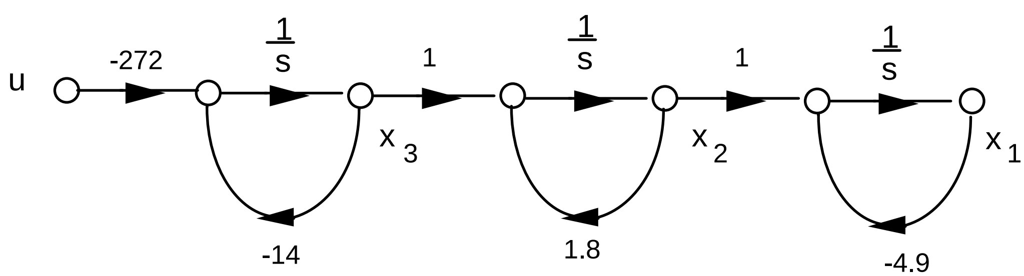
Rearranging,

$$C(s) = \frac{1}{s}\,[-272R(s) - 17.1C(s)] + \frac{1}{s^2}\,[-516.8R(s) - 34.58C(s)] + \frac{1}{s^3}\,[-22848R(s) + 123.48C(s)]$$

Drawing the signal-flow diagram, where r = u and y = c:



d. Draw signal-flow ignoring the polynomial in the numerator:



Write the state equations:

$$\dot{x}_1 = -4.9x_1 + x_2$$

$$\dot{x}_2 = 1.8x_2 + x_3$$

$$\dot{x}_3 = -14x_3 - 272u$$

The output equation is

$$y = \ddot{x}_1 + 1.9\dot{x}_1 + 84x_1 \tag{1}$$

But,

$$\dot{x}_1 = -4.9x_1 + x_2 \tag{2}$$

and

$$\ddot{x}_1 = -4.9\dot{x}_1 + \dot{x}_2 = -4.9(-4.9x_1 + x_2) + 1.8x_2 + x_3 \tag{3}$$

Substituting Eqs. (2) and (3) into (1) yields,

$$y = 98.7x_1 - 1.2x_2 + x_3$$

In vector-matrix form:

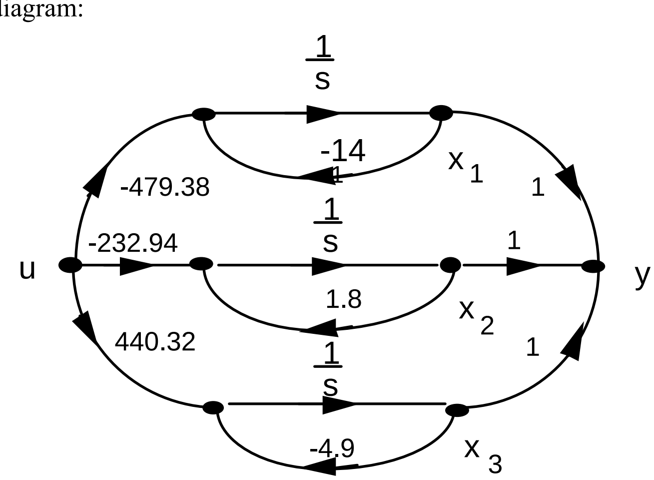
$$\dot{\mathbf{x}} = \begin{bmatrix} -4.9 & 1 & 0 \\ 0 & 1.8 & 1 \\ 0 & 0 & -14 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ -272 \end{bmatrix}u$$

$$y = \begin{bmatrix} 98.7 & -1.2 & 1 \end{bmatrix}\mathbf{x}$$

e. Expand as partial fractions:

$$G(s) = -479.38\frac{1}{s+14} - 232.94\frac{1}{s-1.8} + 440.32\frac{1}{s+4.9}$$

Draw signal-flow diagram:



Write state and output equations:

$$\dot{x}_1 = -14x_1 + -479.38u$$

$$\dot{x}_2 = 1.8x_2 - 232.94u$$

$$\dot{x}_3 = -4.9x_3 + 440.32u$$

$$y = x_1 + x_2 + x_3$$

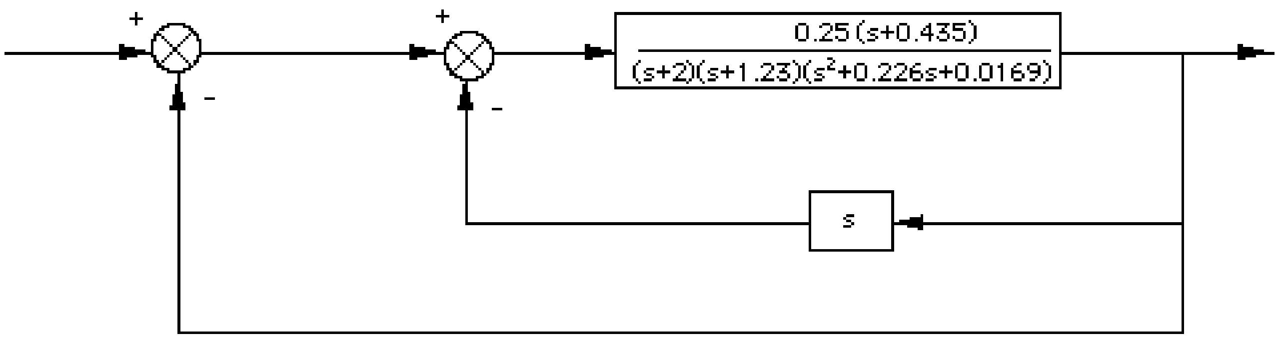
In vector-matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} -14 & 0 & 0 \\ 0 & 1.8 & 0 \\ 0 & 0 & -4.9 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -479.38 \\ -232.94 \\ 440.32 \end{bmatrix}u$$

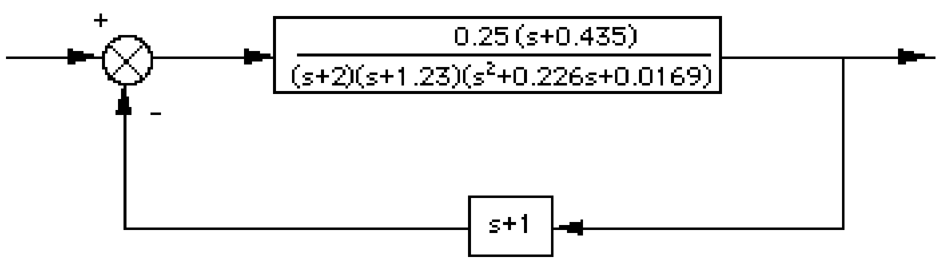
$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\mathbf{x}$$

54.

Push Pitch Gain to the right past the pickoff point.



Collapse the summing junctions and add the feedback transfer functions.



Apply the feedback formula and obtain,

$$T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{0.25(s+0.435)}{s^4 + 3.4586s^3 + 3.4569s^2 + 0.9693s + 0.15032}$$

55.

Program:

```
numg1=-0.125*[1 0.435]
deng1=conv([1 1.23],[1 0.226 0.0169])
'G1'
G1=tf(numg1,deng1)
'G2'
G2=tf(2,[1 2])
G3=-1
'H1'
H1=tf([-1 0],1)
'Inner Loop'
Ge=feedback(G1*G2,H1)
'Closed-Loop'
T=feedback(G3*Ge,1)
```

Computer response:

```
numg1 =

   -0.1250   -0.0544


deng1 =

    1.0000    1.4560    0.2949    0.0208


ans =

G1


Transfer function:
         -0.125 s - 0.05438
------------------------------------
s^3 + 1.456 s^2 + 0.2949 s + 0.02079


ans =

G2


Transfer function:
  2
-----
s + 2


G3 =

    -1


ans =

H1


Transfer function:
-s


ans =

Inner Loop

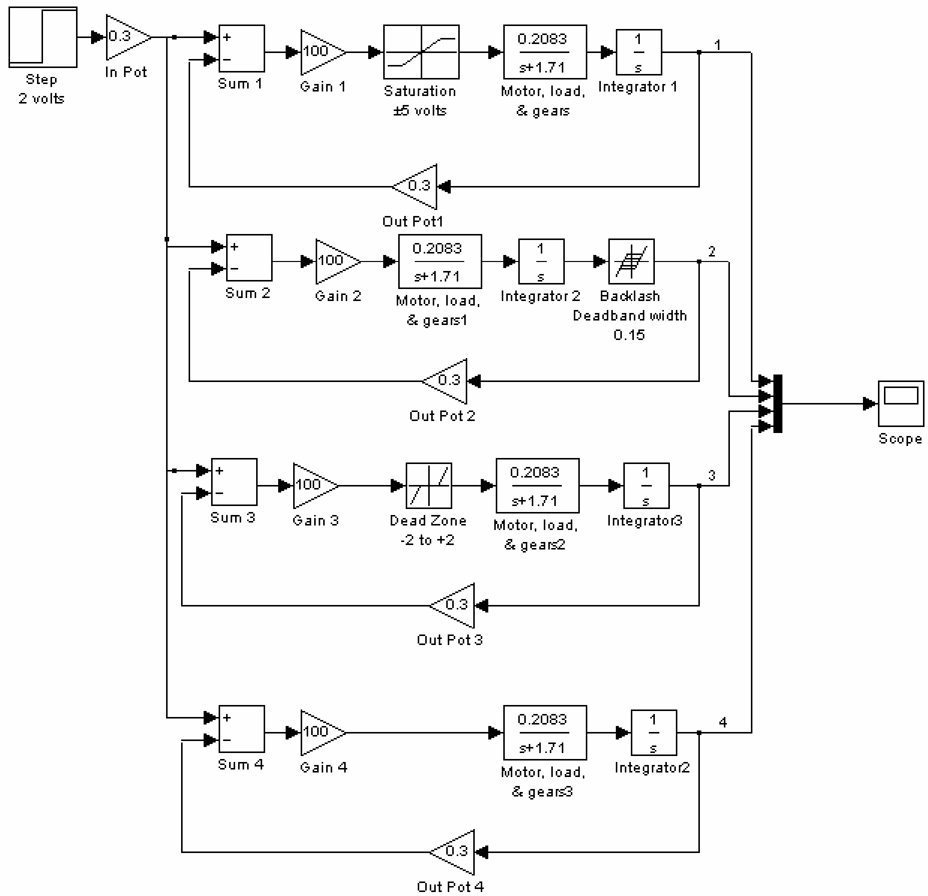

Transfer function:
                -0.25 s - 0.1088
------------------------------------------------
s^4 + 3.456 s^3 + 3.457 s^2 + 0.7193 s + 0.04157

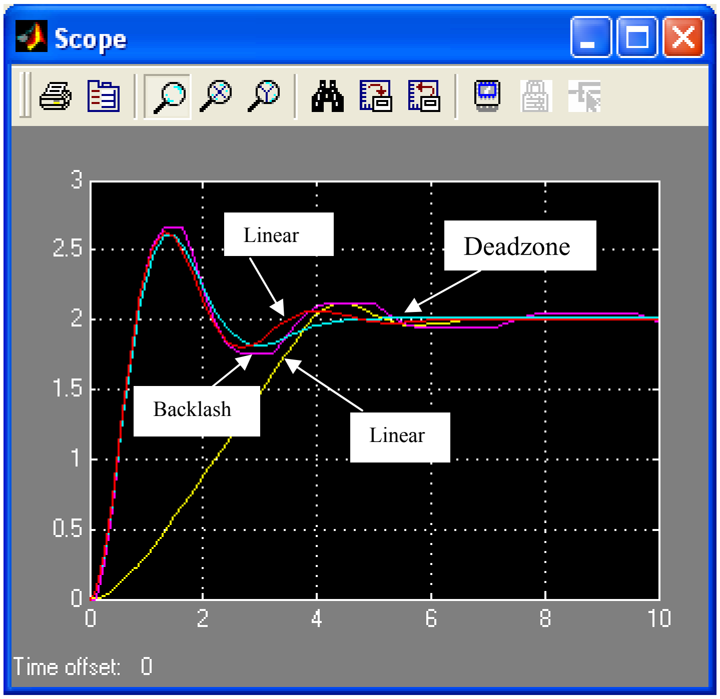

ans =

Closed-Loop
```

```
Transfer function:
                0.25 s + 0.1088
-----------------------------------------------
s^4 + 3.456 s^3 + 3.457 s^2 + 0.9693 s + 0.1503
```

56.





57.

a. Since $V_L(s) = V_g(s) - V_R(s)$, the summing junction has $V_g(s)$ as the positive input and $V_R(s)$ as the negative input, and $V_L(s)$ as the error. Since $I(s) = V_L(s)\,(1/(Ls))$, $G(s) = 1/(Ls)$. Also, since $V_R(s) = I(s)R$, the feedback is $H(s) = R$. Summarizing, the circuit can be modeled as a negative feedback system, where $G(s) = 1/(Ls)$, $H(s) = R$, input = $V_g(s)$, output = $I(s)$, and error = $V_L(s)$, where the negative input to the summing junction is $V_R(s)$.

b. $T(s) = \dfrac{I(s)}{V_g(s)} = \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{\frac{1}{Ls}}{1+\frac{1}{Ls}R} = \dfrac{1}{Ls+R}$. Hence, $I(s) = V_g(s)\dfrac{1}{Ls+R}$.

c. Using circuit analysis, $I(s) = \dfrac{V_g(s)}{Ls+R}$.

SOLUTIONS TO DESIGN PROBLEMS

58.

$J_e = J_a + J_L(\frac{1}{20})^2 = 2+2 = 4$; $D_e = D_a + D_L(\frac{1}{20})^2 = 2 + D_L(\frac{1}{20})^2$. Therefore, the forward-path transfer function is,

$$G(s) = (1000)\left(\frac{\frac{1}{4}}{s(s+\frac{1}{4}(D_e+2))}\right)\left(\frac{1}{20}\right). \text{ Thus, } T(s) = \frac{G}{1+G} = \frac{\frac{25}{2}}{s^2+\frac{1}{4}(D_e+2)s+\frac{25}{2}}.$$

Hence, $\zeta = \dfrac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} = 0.456$; $\omega_n = \sqrt{\dfrac{25}{2}}$; $2\zeta\omega_n = \dfrac{D_e+2}{4}$. Therefore $D_e = 10.9$; from

which $D_L = 3560$.

59.

a. $T(s) = \dfrac{25}{s^2+s+25}$; from which, $2\zeta\omega_n = 1$ and $\omega_n = 5$. Hence, $\zeta = 0.1$. Therefore,

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 72.92\%$; $T_s = \dfrac{4}{\zeta\omega_n} = 8$.

b. $T(s) = \dfrac{25K_1}{s^2+(1+25K_2)s+25K_1}$; from which, $2\zeta\omega_n = 1+25K_2$ and $\omega_n = 5\sqrt{K_1}$. Hence,

$\zeta = \dfrac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} = 0.404$. Also, $T_s = \dfrac{4}{\zeta\omega_n} = 0.2$, Thus, $\zeta\omega_n = 20$; from which $K_2 = \dfrac{39}{25}$ and

$\omega_n = 49.5$. Hence, $K_1 = 98.01$.

60.

The equivalent forward path transfer function is $G_e(s) = \dfrac{K}{s(1+(1+K_2))}$. Thus, $T(s) = \dfrac{G_e(s)}{1+G_e(s)} =$

$\dfrac{K}{s^2+(1+K_2)s+K}$. Prior to tachometer compensation ($K_2 = 0$), $T(s) = \dfrac{K}{s^2+s+K}$. Therefore $K = \omega_n^2 =$

100. Thus, after tachometer compensation, $T(s) = \dfrac{100}{s^2+(1+K_2)s+100}$. Hence, $\omega_n = 10$; $2\zeta\omega_n = 1+K_2$.

Therefore, $K_2 = 2\zeta\omega_n - 1 = 2(0.5)(10) - 1 = 9$.

61.

At the N_2 shaft, with rotation, $\theta_L(s)$

$$(J_{eq}s^2 + D_{eq}s)\theta_L(s) + F(s)r = T_{eq}(s)$$
$$F(s) = (Ms^2 + f_v s)X(s)$$

Thus,

$$(J_{eq}s^2 + D_{eq}s)\theta_L(s) + (Ms^2 + f_v s)X(s)r = T_{eq}(s)$$

But, $X(s) = r\theta_L(s)$. Hence,

$$\left[(J_{eq} + Mr^2)s^2 + (D_{eq} + f_v r^2)s\right]\theta_L(s) = T_{eq}(s)$$

where

$$J_{eq} = J_a(2)^2 + J = 5$$
$$D_{eq} = D_a(2)^2 + D = 4 + D$$
$$r = 2$$

Thus, the total load inertia and load damping is

$$J_L = J_{eq} + Mr^2 = 5 + 4M$$
$$D_L = D_{eq} + f_v r^2 = 4 + D + (1)(2)^2 = 8 + D$$

Reflecting J_L and D_L to the motor yields,

$$J_m = \frac{(5+4M)}{4};\ D_m = \frac{(8+D)}{4}$$

Thus, the motor transfer function is

$$\frac{\theta_m(s)}{E_a(s)} \frac{\frac{K_t}{R_a J_m}}{s(s+\frac{1}{J_m}(D_m + \frac{K_t K_a}{R_a}))} = \frac{\frac{1}{J_m}}{s(s+\frac{1}{J_m}(D_m+1))}$$

The gears are (10/20)(1) = 1/2. Thus, the forward-path transfer function is

$$G_e(s) = (500)\left(\frac{\frac{1}{J_m}}{s(s+\frac{1}{J_m}(D_m+1))}\right)\frac{1}{2}$$

Finding the closed-loop transfer function yields,

$$T(s) = \frac{G_e(s)}{1+G_e(s)} = \frac{250/J_m}{s^2 + \frac{D_m+1}{J_m}s + \frac{250}{J_m}}$$

For $T_s = 2$, $\frac{D_m+1}{J_m} = 4$. For 20% overshoot, $\zeta = 0.456$. Thus,

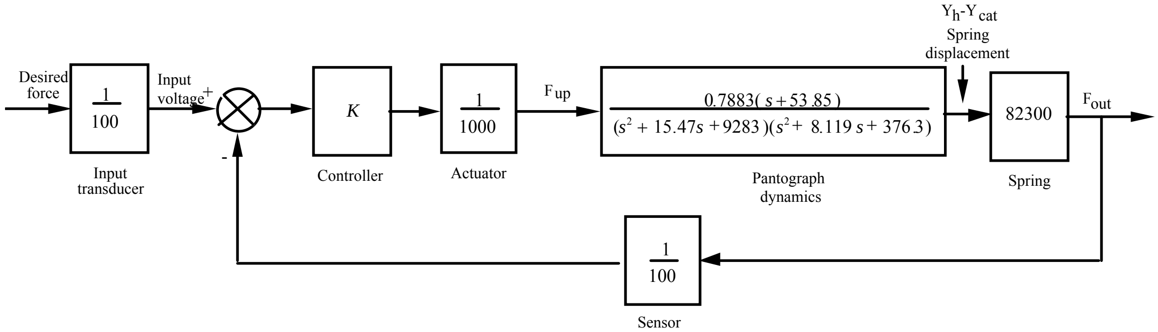
$$2\zeta\omega_n = 2(0.456)\omega_n = \frac{D_m+1}{J_m} = 4$$

Or, $\omega_n = 4.386 = \sqrt{\frac{250}{J_m}}$; from which $J_m = 13$ and hence, $D_m = 51$. But,

$J_m = \frac{(5+4M)}{4};\ D_m = \frac{(8+D)}{4}$. Thus, M = 11.75 and D = 196.

62.

a.



b. $G(s) = \dfrac{Y_h(s) - Y_{cat}(s)}{F_{up}(s)} = \dfrac{0.7883(s+53.85)}{(s^2+15.47s+9283)(s^2+8.119s+376.3)}$

$$G_e(s) = (K/100)*(1/1000)*G(s)*82.3e3 = \frac{648.7709\ (s+53.85)}{(s^2 + 8.119s + 376.3)\ (s^2 + 15.47s + 9283)}$$

$$T(s) = Ge/(1+Ge) = \frac{648.7709\ (s+53.85)}{(s^2 + 8.189s + 380.2)\ (s^2 + 15.4s + 9279)}$$

$$= \frac{648.8\ s + 3.494e04}{s^4 + 23.59\ s^3 + 9785\ s^2 + 8.184e04\ s + 3.528e06}$$

c.

For G(s) = (yh-ycat)/Fup

Phase-variable form

Ap =

0	1	0	0
0	0	1	0
0	0	0	1
-3.493e6	-81190	- 9785	-23.59

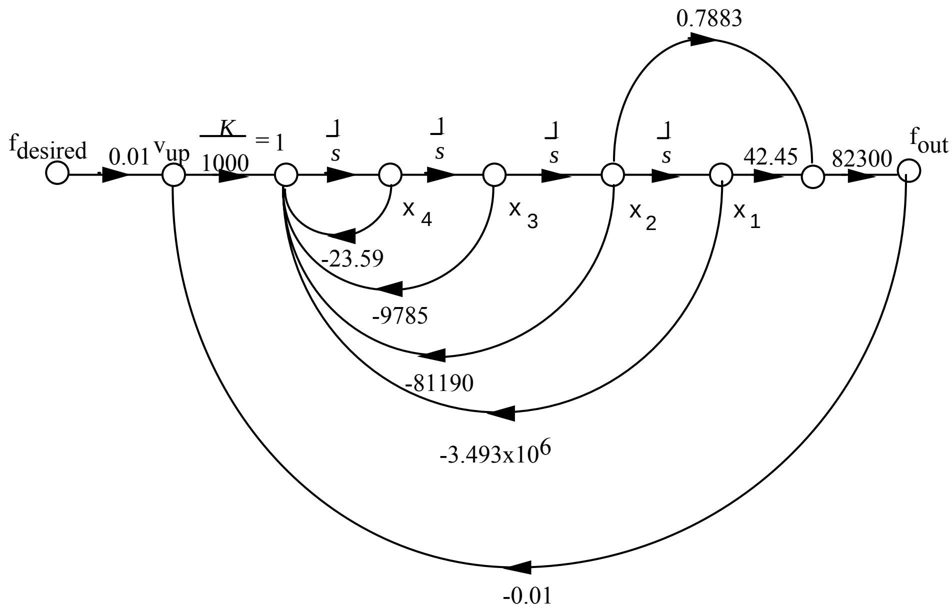
Bp =

0
0
0
1

Cp =

42.45	0.7883	0	0

Using this result to draw the signal-flow diagram,



Writing the state and output equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -23.59x_4 - 9785x_3 - 81190x_2 - 3493000x_1 + 0.01f_{desired} - 0.01\,f_{out}$$

But,

$$f_{out} = 42.45 * 82300x_1 + 0.7883 * 82300x_2$$

Substituting f_{out} into the state equations yields

$$\dot{x}_4 = -3527936.35x_1 - 81838.7709x_2 - 9785x_3 - 23.59x_4 + 0.01f_{desired}$$

Putting the state and output equations into vector-matrix form.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3.528x10^6 & -81840 & -9785 & -23.59 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.01 \end{bmatrix} f_{desired}$$

$$y = f_{out} = \begin{bmatrix} 3494000 & 64880 & 0 & 0 \end{bmatrix}\mathbf{x}$$

S I X

Stability

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Stability Design via Gain

From the antenna control challenge of Chapter 5,

$$T(s) = \frac{76.39K}{s^3+151.32s^2+198s+76.39K}$$

Make a Routh table:

s^3	1	198
s^2	151.32	76.39K
s^1	$\frac{29961.36-76.39K}{151.32}$	0
s^0	76.39K	0

From the s^1 row, K<392.2. From the s^0 row, 0<K. Therefore, 0<K<392.2.

UFSS Vehicle: Stability Design via Gain

$$G_1 = \frac{-0.125\,(s+0.437)\cdot 2}{s\,(s+2)\,(s+1.29)\,(s+0.193)}$$

$$G_2 = \frac{G_1}{1+G_1\,(-s)}$$

$$G_2 = \frac{-0.25s-0.10925}{s^4+3.483s^3+3.465s^2+0.60719s}$$

$$G_3 = -K_1 G_2$$

$$G_3 = \frac{(0.25s+0.10925)K_1}{s^4+3.483s^3+3.465s^2+0.60719s}$$

$$T(s) = \frac{G_3(s)}{1+G_3(s)} = \frac{(0.25s+0.10925)K_1}{s^4+3.483s^3+3.465s^2+0.25(K_1+2.4288)s+0.10925K_1}$$

s^4	1	3.465	$0.10925K_1$
s^3	3.483	$0.25(K_1+2.4288)$	0
s^2	$\dfrac{-\frac{1}{4}(K_1 - 45.84)}{3.483}$	$0.10925K_1$	0
s^1	$0.25\dfrac{(K_1 + 4.2141)(K_1 - 26.42)}{K_1 - 45.84}$	0	0
s^0	$0.10925K_1$	0	0

For stability : $0 < K_1 < 26.42$

ANSWERS TO REVIEW QUESTIONS

1. Natural response

2. It grows without bound

3. It would destroy itself or hit limit stops

4. Sinusoidal inputs of the same frequency as the natural response yield unbounded responses even though the sinusoidal input is bounded.

5. Poles must be in the left-half-plane or on the $j\omega$ axis.

6. The number of poles of the closed-loop transfer function that are in the left-half-plane, the right-half-plane, and on the $j\omega$ axis.

7. If there is an even polynomial of second order and the original polynomial is of fourth order, the original polynomial can be easily factored.

8. Just the way the arithmetic works out

9. The presence of an even polynomial that is a factor of the original polynomial

10. For the ease of finding coefficients below that row

11. It would affect the number of sign changes

12. Seven

13. No; it could have quadrantal poles.

14. None; the even polynomial has 2 right-half-plane poles and two left-half-plane poles.

15. Yes

16. Det $(s\mathbf{I}-\mathbf{A}) = 0$

SOLUTIONS TO PROBLEMS

1.

s^5	1	5	1
s^4	3	4	3
s^3	3.667	0	0
s^2	4	3	0
s^1	-2.75	0	0
s^0	3	0	0

2 rhp; 3 lhp

2.

s^5	1	4	3
s^4	-1	-4	-2
s^3	ε	1	0
s^2	$\frac{1-4\varepsilon}{\varepsilon}$	-2	0
s^1	$\frac{2\varepsilon^2+1-4\varepsilon}{1-4\varepsilon}$	0	0
s^0	-2	0	0

3 rhp, 2 lhp

3.

s^5	1	3	2
s^4	-1	-3	-2
s^3	-2	-3	ROZ
s^2	-3	-4	
s^1	-1/3		
s^0	-4		

Even (4): 4 jω; Rest(1): 1 rhp; Total (5): 1 rhp; 4 jω

4.

s^4	1	8	15	
s^3	4	20	0	
s^2	3	15	0	
s^1	6	0	0	ROZ
s^0	15	0	0	

Even (2): 2 jω; Rest (2): 2 lhp; Total: 2 jω; 2 lhp

5.

s^6	1	-6	1	-6
s^5	1	0	1	
s^4	-6	0	-6	
s^3	-24	0	0	ROZ
s^2	ε	-6		
s^1	-144/ε	0		
s^0	-6			

Even (4): 2 rhp; 2 lhp; Rest (2): 1 rhp; 1 lhp; Total: 3 rhp; 3 lhp

6.

Program:

```
den=[1 1 -6 0 1 1 -6]
A=roots(den)
```

Computer response:

```
den =

     1     1    -6     0     1     1    -6


A =

  -3.0000
   2.0000
  -0.7071 + 0.7071i
  -0.7071 - 0.7071i
   0.7071 + 0.7071i
   0.7071 - 0.7071i
```

7.

Program:

```
%-det([si() si();sj() sj()])/sj()
                              %Template for use in each cell.
syms e                        %Construct a symbolic object for
                              %epsilon.
%%%%%%%%%%%%%%%%%%%%%%%%%%%$$$$$$$$$$%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
s5=[1  4   3   0   0]         %Create s^5 row of Routh table.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%$$$$$$$$$%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
s4=[-1   -4   -2   0   0]     %Create s^4 row of Routh table.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if -det([s5(1) s5(2);s4(1) s4(2)])/s4(1)==0
                              s3=[e...
 -det([s5(1) s5(3);s4(1) s4(3)])/s4(1)   0   0];
                              %Create s^3 row of Routh table
                              %if 1st element is 0.
else
    s3=[-det([s5(1) s5(2);s4(1) s4(2)])/s4(1)...
 -det([s5(1) s5(3);s4(1) s4(3)])/s4(1)   0   0];
                              %Create s^3 row of Routh table
                              %if 1st element is not zero.
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if -det([s4(1) s4(2);s3(1) s3(2)])/s3(1)==0
                              s2=[e ...
 -det([s4(1) s4(3);s3(1) s3(3)])/s3(1)   0   0];
                              %Create s^2 row of Routh table
                              %if 1st element is 0.
else
    s2=[-det([s4(1) s4(2);s3(1) s3(2)])/s3(1) ...
 -det([s4(1) s4(3);s3(1) s3(3)])/s3(1)   0   0];
                             %Create s^2 row of Routh table
                             %if 1st element is not zero.
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if -det([s3(1) s3(2);s2(1) s2(2)])/s2(1)==0
                              s1=[e ...
 -det([s3(1) s3(3);s2(1) s2(3)])/s2(1)   0   0];
                              %Create s^1 row of Routh table
                              %if 1st element is 0.
else
s1=[-det([s3(1) s3(2);s2(1) s2(2)])/s2(1) ...
 -det([s3(1) s3(3);s2(1) s2(3)])/s2(1)   0   0];
                              %Create s^1 row of Routh table
                              %if 1st element is not zero
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

s0=[-det([s2(1) s2(2);s1(1) s1(2)])/s1(1) ...
 -det([s2(1) s2(3);s1(1) s1(3)])/s1(1)   0   0];
                              %Create s^0 row of Routh table.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

's3'                          %Display label.
s3=simplify(s3);              %Simplify terms in s^3 row.
pretty(s3)                    %Pretty print s^3 row.
's2'                          %Display label.
s2=simplify(s2);              %Simplify terms in s^2 row.
pretty(s2)                    %Pretty print s^2 row.
's1'                          %Display label.
s1=simplify(s1);              %Simplify terms in s^1 row.
pretty(s1)                    %Pretty print s^1 row.
's0'                          %Display label.
s0=simplify(s0);              %Simplify terms in s^0 row.
pretty(s0)                    %Pretty print s^0 row.
```

Computer response:

```
s5 =

     1     4     3     0     0

s4 =

    -1    -4    -2     0     0

ans =

s3


                              [e    1    0    0]
ans =

s2


                         [  -1 + 4 e          ]
                         [- --------    -2   0   0]
                         [     e              ]

ans =

s1


                       [      2                    ]
                       [  2 e  + 1 - 4 e           ]
                       [- --------------    0    0    0]
                       [      -1 + 4 e             ]

ans =

s0


                              [-2    0    0    0]
```

8.

$$T(s) = \frac{240}{s^4 + 10s^3 + 35s^2 + 50s + 264}$$

s4	1	35	264
s3	10	50	0
s2	30	264	0
s1	-38	0	0
s0	264	0	0

2 rhp, 2 lhp

9.

Program:

```
numg=240;
deng=poly([-1 -2 -3 -4]);
'G(s)'
G=tf(numg,deng)
'Poles of G(s)'
pole(G)
'T(s)'
```

```
T=feedback(G,1)
'Poles of T(s)'
pole(T)
```

Computer response:

```
ans =

G(s)

Transfer function:
               240
---------------------------------
s^4 + 10 s^3 + 35 s^2 + 50 s + 24

ans =

Poles of G(s)

ans =

   -4.0000
   -3.0000
   -2.0000
   -1.0000

ans =

T(s)

Transfer function:
               240
----------------------------------
s^4 + 10 s^3 + 35 s^2 + 50 s + 264

ans =

Poles of T(s)

ans =

  -5.3948 + 2.6702i
  -5.3948 - 2.6702i
   0.3948 + 2.6702i
   0.3948 - 2.6702i
```

System is unstable, since two closed-loop poles are in the right half-plane.

10.

$$T(s) = \frac{1}{4s^4 + 4s^2 + 1}$$

s4	4	4	1	
s3	16	8	0	ROZ
s2	2	1	0	
s1	4	0	0	ROZ
s0	1	0	0	

Even (4): 4 jω

11.

$$T(s) = \frac{84}{s^8 + 5s^7 + 12s^6 + 25s^5 + 45s^4 + 50s^3 + 82s^2 + 60s + 84}$$

s^8	1	12	45	82	84
s^7	1	5	10	12	
s^6	1	5	10	12	
s^5	3	10	10		ROZ
s^4	5	20	36		
s^3	-5	-29			
s^2	-1	4			
s^1	-49				
s^0	4				

Even (6): 2 rhp, 2 lhp, 2 jω; Rest (2): 0 rhp, 2 lhp, 0 jω ; Total: 2 rhp, 4 lhp, 2 jω

12.

$$T(s) = \frac{1}{2s^4 + 5s^3 + s^2 + 2s + 1}$$

s^4	2	1	1
s^3	5	2	0
s^2	1	5	
s^1	-23	0	
s^0	5		

Total: 2 lhp, 2 rhp

13.

$$T(s) = \frac{8}{s^7 - 2s^6 - s^5 + 2s^4 + 4s^3 - 8s^2 - 4s + 8}$$

s^7	1	-1	4	-4	
s^6	-2	2	-8	8	
s^5	-12	8	-16	0	ROZ
s^4	0.6667	-5.333	8	0	
s^3	-88	128	0	0	
s^2	-4.364	8	0	0	
s^1	-33.33	0	0	0	
s^0	8	0	0	0	

Even (6): 3 rhp, 3 lhp; Rest (1): 1 rhp; Total: 4 rhp, 3 lhp

14.

Program:

```
numg=8;
deng=[1 -2 -1 2 4 -8 -4 0];
'G(s)'
G=tf(numg,deng)
'T(s)'
T=feedback(G,1)
'Poles of T(s)'
pole(T)
```

Computer response:

```
ans =

G(s)


Transfer function:
                       8
-----------------------------------------------
s^7 - 2 s^6 - s^5 + 2 s^4 + 4 s^3 - 8 s^2 - 4 s


ans =

T(s)





Transfer function:
                         8
---------------------------------------------------
s^7 - 2 s^6 - s^5 + 2 s^4 + 4 s^3 - 8 s^2 - 4 s + 8


ans =
```

```
Poles of T(s)

ans =

  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
  -1.0000
   2.0000
   1.0000 + 1.0000i
   1.0000 - 1.0000i
   1.0000
```

Thus, there are 4 rhp poles and 3 lhp poles.

15.

Even (6): 1 rhp, 1 lhp, 4 jω; Rest (1): 1 lhp; Total: 1 rhp, 2 lhp, 4 jω

16.

$$T(s) = \frac{18}{s^5 + s^4 - 7s^3 - 7s^2 - 18s - 18}$$

s^5	1	-7	-18	
s^4	1	-7	-18	
s^3	4	-14	0	ROZ
s^2	-3.5	-18	0	
s^1	-34.57	0	0	
s^0	-18	0	0	

Even (4): 1 rhp, 1 lhp, 2 jω; Rest (1): 1 lhp; Total: 1 rhp, 2 lhp, 2 jω

17.

$G(s) = \dfrac{507}{s^4 + 3s^3 + 10s^2 + 30s + 169}$; $H(s) = \dfrac{1}{s}$. Therefore,

$$T(s) = \frac{G}{1 + GH} = \frac{507s}{s^5 + 3s^4 + 10s^3 + 30s^2 + 169s + 507}$$

s^5	1	10	169	
s^4	3	30	507	
s^3	12	60	0	ROZ
s^2	15	507	0	
s^1	-345.6	0	0	

s^0	507	0	0	

Even (4): 2 rhp, 2 lhp, 0 jω; Rest (1): 0 rhp, 1 lhp, 0 jω; Total (5): 2 rhp, 3 lhp, 0 jω

18.

$$T(s) = \frac{K(s^2+1)}{(1+K)s^2 + 3s + (2+K)}$$

. For a second-order system, if all coefficients are positive, the roots will be in the lhp. Thus, $K > -1$.

19.

$$T(s) = \frac{K(s+6)}{s^3 + 4s^2 + (K+3)s + 6K}$$

s^3	1	$3 + K$
s^2	4	$6K$
s^1	$3 - \frac{1}{2}K$	0
s^0	$6K$	0

Stable for $0 < K < 6$

20.

$$T(s) = \frac{K(s+3)(s+5)}{(1+K)s^2 + (8K-6)s + (8+15K)}$$

			For 1st column negative	For 1st column positive
s^2	$1+K$	$8+15K$	$K < -1$	$K > -1$
s^1	$8K-6$	0	$K < 6/8$	$K > 6/8$
s^0	$8+15K$	0	$K < -8/15$	$K > -8/15$

Stable for $K > 6/8$

21.

Program:

```
K=[-6:0.00005:0];
for i=1:length(K);
dent=[(1+K(i)) (8*K(i)-6) (8+15*K(i))];
R=roots(dent);
A=real(R);
B=max(A);
if B>0
R
K=K(i)
break
end
end
K=[6:-0.00005:0];
for i=1:length(K);
dent=[(1+K(i)) (8*K(i)-6) (8+15*K(i))];
R=roots(dent);
```

```
A=real(R);
B=max(A);
if B>0
R
K=K(i)
break
end
end
```

Computer response:

```
R =

  1.0e+005 *

    2.7999
   -0.0000


K =

   -1.0000


R =

   0.0001 + 3.3166i
   0.0001 - 3.3166i


K =

    0.7500
```

22.

Program:

```
%-det([si() si();sj() sj()])/sj()
                              %Template for use in each cell.
syms K                        %Construct a symbolic object for
                              %gain, K.
s2=[(1+K)   (8+15*K) 0];      %Create s^2 row of Routh table.
s1=[(8*K-6)  0 0];            %Create s^1 row of Routh table.
s0=[-det([s2(1) s2(2);s1(1) s1(2)])/s1(1)...
 -det([s2(1) s2(3);s1(1) s1(3)])/s1(1)   0   0];
                              %Create s^0 row of Routh table.
's2'                          %Display label.
s2=simplify(s2);              %Simplify terms in s^1 row.
pretty(s2)                    %Pretty print s^1 row.
's1'                          %Display label.
s1=simplify(s1);              %Simplify terms in s^1 row.
pretty(s1)                    %Pretty print s^1 row.
's0'                          %Display label.
s0=simplify(s0);              %Simplify terms in s^0 row.
pretty(s0)                    %Pretty print s^0 row.
```

Computer response:

```
ans =

s2


                           [1 + K    8 + 15 K    0]


ans =
```

```
s1

                    [8 K - 6    0    0]

ans =

s0

                  [8 + 15 K    0    0    0]
```

23.

$T(s)=\dfrac{K(s+2)(s-2)}{(K+1)s^2+(3-4K)}$. For positive coefficients in the denominator, $-1<K<\dfrac{3}{4}$. Hence marginal stability only for this range of K.

24.

$T(s)=\dfrac{K(s+1)}{s^5+2s^4+Ks+K}$. Always unstable since s^3 and s^2 terms are missing.

25.

$$T(s)=\frac{K(s-2)(s+4)(s+5)}{Ks^3+(7K+1)s^2+2Ks+(3-40K)}$$

s^3	K	2K
s^1	$\dfrac{54K^2-K}{7K+1}$	0
s^0	3-40K	0

For stability, $\dfrac{1}{54}<K<\dfrac{3}{40}$

26.

$T(s)=\dfrac{K(s+2)}{s^4+3s^3-3s^2+(K+3)s+(2K-4)}$

s^4	1	- 3	2K - 4
s^3	3	K+ 3	0
s^2	$\dfrac{-(K+12)}{3}$	2K - 4	0
s^1	$\dfrac{K(K+33)}{K+12}$	0	0
s^0	2K - 4	0	0

Conditions state that K < -12, K > 2, and K > -33. These conditions cannot be met simultaneously. System is not stable for any value of K.

27.

$$T(s)=\frac{K}{s^3+80s^2+2001s+(K+15390)}$$

s^3	1	2001
s^2	80	K+15390
s^1	$-\frac{1}{80}K+\frac{14469}{8}$	0
s^0	K+15390	0

There will be a row of zeros at s^1 row if K = 144690. The previous row, s^2, yields the auxiliary equation, $80s^2+(144690+15390)=0$. Thus, s = ± j44.73. Hence, K = 144690 yields an oscillation of 44.73 rad/s.

28.

$$T(s)=\frac{Ks^4-Ks^2+2Ks+2K}{(K+1)s^2+2(1-K)s+(2K+1)}$$

Since all coefficients must be positive for stability in a second-order polynomial, $-1 < K < \infty$; $-\infty < K < 1$; $-1 < 2K < \infty$. Hence, $-\frac{1}{2} < K < 1$.

29.

$$T(s)=\frac{(s+2)(s+7)}{s^4+11s^3+(K+31)s^2+(8K+21)s+12K}$$

Making a Routh table,

s^4	1	$K+31$	$12K$
s^3	11	$8K+21$	0
s^2	$\frac{3K+320}{11}$	$12K$	0
s^1	$\frac{24K^2+1171K+6720}{3K+320}$	0	0
s^0	$12K$	0	0

s^2 row says $-106.7 < K$. s^1 row says $K < -42.15$ and $-6.64 < K$. s^0 row says $0 < K$.

30.

$$T(s) = \frac{K(s+4)}{s^3 + 3s^2 + (2+K)s + 4K}$$

Making a Routh table,

s^3	1	$2+K$
s^2	3	$4K$
s^1	$6 - K$	0
s^0	$4K$	0

a. For stability, $0 < K < 6$.

b. Oscillation for K = 6.

c. From previous row with $K = 6$, $3s^2 + 24 = 0$. Thus $s = \pm j\sqrt{8}$, or $\omega = \sqrt{8}$ rad/s.

31.

a. $G(s) = \frac{K(s-1)(s-2)}{(s+2)(s^2+2s+2)}$. Therefore, $T(s) = \frac{(s-2)(s-1)K}{s^3+(K+4)s^2+(6-3K)s+2(K+2)}$.

Making a Routh table,

s^3	1	6-3K
s^2	4+K	4+2K
s^1	$\frac{-(3K^2+8K-20)}{K+4}$	0
s^0	4+2K	0

From s^1 row: K = 1.57, -4.24; From s^2 row: $-4 < K$; From s^0 row: $-2 < K$. Therefore, $-2 < K < 1.57$.

b. If K = 1.57, the previous row is $5.57s^2 + 7.14$. Thus, $s = \pm j1.13$.

c. From part b, ω = 1.13 rad/s.

32.

Applying the feedback formula on the inner loop and multiplying by K yields

$$G_e(s) = \frac{K}{s(s^2+5s+7)}$$

Thus,

$$T(s) = \frac{K}{s^3+5s^2+7s+K}$$

Making a Routh table:

s^3	1	7
s^2	5	K
s^1	$\frac{35 - K}{5}$	0
s^0	K	0

For oscillation, the s^1 row must be a row of zeros. Thus, K = 35 will make the system oscillate. The previous row now becomes, $5s^2 + 35$. Thus, $s^2 + 7 = 0$, or $s = \pm j\sqrt{7}$. Hence, the frequency of oscillation is $\sqrt{7}$ rad/s.

33.

$$T(s) = \frac{Ks^2 + 2Ks}{s^3 + (K-1)s^2 + (2K-4)s + 24}$$

s^3	1	2K-4
s^2	K-1	24
s^1	$\frac{2K^2 - 6K - 20}{K - 1}$	0
s^0	24	0

For stability, K > 5; Row of zeros if K = 5. Therefore, $4s^2 + 24 = 0$. Hence, $\omega = \sqrt{6}$ for oscillation.

34.

Program:

```
K=[0:0.001:200];
for i=1:length(K);
deng=conv([1 -4 8],[1 3]);
numg=[0 K(i) 2*K(i) 0];
dent=numg+deng;
R=roots(dent);
A=real(R);
B=max(A);
if B<0
R
K=K(i)
break
end
end
```

Computer response:

```
R =

  -4.0000
  -0.0000 + 2.4495i
```

```
 -0.0000 - 2.4495i


K =

     5
```

a. From the computer response, (a) the range of K for stability is $0 < K < 5$.

b. The system oscillates at K = 5 at a frequency of 2.4494 rad/s as seen from R, the poles of the closed-loop system.

35.

$$T(s) = \frac{K(s+2)}{s^4 + 3s^3 - 3s^2 + (K+3)s + (2K-4)}$$

s^4	1	- 3	2K-4
s^3	3	K+3	0
s^2	$-\frac{K+12}{3}$	2K-4	0
s^1	$\frac{K(K+33)}{K+12}$	0	0
s^0	2K-4	0	0

For $K < -33$: 1 sign change; For $-33 < K < -12$: 1 sign change; For $-12 < K < 0$: 1 sign change; For $0 < K < 2$: 3 sign changes; For $K > 2$: 2 sign changes. Therefore, $K > 2$ yields two right-half-plane poles.

36.

$$T(s) = \frac{K}{s^4 + 7s^3 + 15s^2 + 13s + (4+K)}$$

s^4	1	15	K+4
s^3	7	13	0
s^2	$\frac{92}{7}$	K+4	0
s^1	$\frac{1000-49K}{92}$	0	0
s^0	K+4	0	0

a. System is stable for $-4 < K < 20.41$.

b. Row of zeros when K = 20.41. Therefore, $\frac{92}{7}\,s^2 + 24.41$. Thus, $s = \pm j1.3628$, or $\omega = 1.3628$ rad/s.

37.

$$T(s) = \frac{K}{s^3 + 14s^2 + 45s + (K+50)}$$

s^3	1	45
s^2	14	K+50
s^1	$\frac{580-K}{14}$	0
s^0	K+50	0

a. System is stable for $-50 < K < 580$.

b. Row of zeros when K = 580. Therefore, $14s^2 + 630$. Thus, $s = \pm j\sqrt{45}$, or $\omega = 6.71$ rad/s.

38.

$$T(s) = \frac{K}{s^4 + 8s^3 + 17s^2 + 10s + K}$$

s^4	1	17	K
s^3	8	10	0
s^2	$\frac{126}{8}$	K	0
s^1	$-\frac{32}{63}K + 10$	0	0
s^0	K	0	0

a. For stability $0 < K < 19.69$.

b. Row of zeros when K = 19.69. Therefore, $\frac{126}{8}s^2 + 19.69$. Thus, $s = \pm j\sqrt{1.25}$, or

$\omega = 1.118$ rad/s.

c. Denominator of closed-loop transfer function is $s^4 + 8s^3 + 17s^2 + 10s + K$. Substituting K = 19.69 and solving for the roots yield $s = \pm j1.118$, -4.5, and -3.5.

39.

$$T(s) = \frac{K(s^2 + 2s + 1)}{s^3 + 2s^2 + (K+1)s - K}$$

s^3	1	K+1
s^2	2	- K
s^1	$\frac{3K+2}{2}$	0
s^0	- K	0

Stability if $-\frac{2}{3} < K < 0$.

40.

$$T(s) = \frac{2s^4 + (K+2)s^3 + Ks^2}{s^3 + s^2 + 2s + K}$$

s^3	1	2
s^2	1	K
s^1	2 - K	0
s^0	K	0

Row of zeros when K = 2. Therefore $s^2 + 2$ and $s = \pm j\sqrt{2}$, or ω = 1.414 rad/s. Thus K = 2 will yield the even polynomial with 2 jω roots and no sign changes.

41.

	1	K_2	1
s^3	K_1	5	0
s^2	$\frac{K_1K_2 - 5}{K_1}$	1	0
s^1	$\frac{K_1^2 - 5K_1K_2 + 25}{5 - K_1K_2}$	0	0
s^0	1	0	0

For stability, $K_1K_2 > 5$; $K_1^2 + 25 < 5K_1K_2$; and $K_1 > 0$. Thus $0 < K_1^2 < 5K_1K_2 - 25$, or $0 < K_1 < \sqrt{5K_1K_2 - 25}$.

42.

s^4	1	1	1
s^3	K_1	K_2	0
s^2	$\frac{K_1-K_2}{K_1}$	1	0
s^1	$\frac{K_1^2 - K_1K_2 + K_2^2}{K_2 - K_1}$	0	0
s^0	1	0	0

For two jω poles, $K_1^2 - K_1K_2 + K_2^2 = 0$. However, there are no real roots. Therefore, there is no relationship between K_1 and K_2 that will yield just two jω poles.

43.

s^8	1	1.18E+03	2.15E+03	-1.06E+04	-415
s^7	103	4.04E+03	-8.96E+03	-1.55E+03	0
s^6	1140.7767	2236.99029	-10584.951	-415	0
s^5	3838.02357	-8004.2915	-1512.5299	0	0
s^4	4616.10784	-10135.382	-415	0	0
s^3	422.685462	-1167.4817	0	0	0
s^2	2614.57505	-415	0	0	0
s^1	-1100.3907	0	0	0	0
s^0	-415	0	0	0	0

a. From the first column, 1 rhp, 7 lhp, 0 jω.

b. G(s) is not stable because of 1 rhp.

44.

Eigenvalues are the roots of the following equation:

$$|s\mathbf{I}-\mathbf{A}| = \left\| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 3 \\ 2 & 2 & -4 \\ 1 & -4 & 3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} s & -1 & -3 \\ -2 & s-2 & 4 \\ -1 & 4 & s-3 \end{bmatrix} \right\| = s^3 - 5s^2 - 15s + 40$$

Hence, eigenvalues are -3.2824, 1.9133, 6.3691. Therefore, 1 rhp, 2 lhp, 0 jω.

45.

Program:

```
A=[0 1 0;0 1 -4;-1 1 3];
eig(A)
```

Computer response:

```
ans =

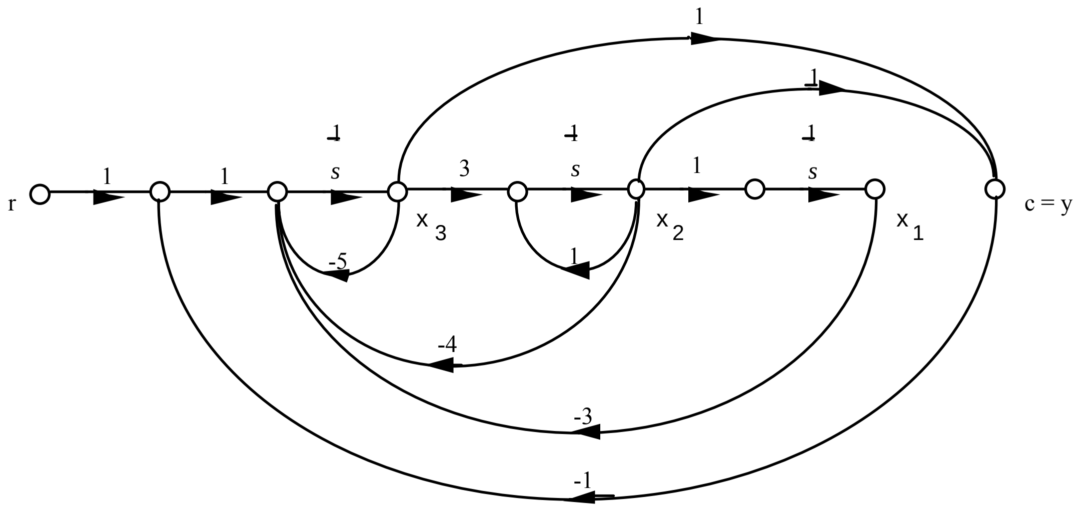
   1.0000
   1.5000 + 1.3229i
   1.5000 - 1.3229I
```

46.

Writing the open-loop state and output equations we get,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_2 + 3x_3$$
$$\dot{x}_3 = -3x_1 - 4x_2 - 5x_3 + u$$
$$y = x_2 + x_3$$

Drawing the signal-flow diagram and including the unity feedback path yields,



Writing the closed-loop state and output equations from the signal-flow diagram,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_2 + 3x_3$$
$$\begin{aligned}\dot{x}_3 &= -3x_1 - 4x_2 - 5x_3 + r - c \\ &= -3x_1 - 4x_2 - 5x_3 + r - (x_2 + x_3) \\ &= -3x_1 - 5x_2 - 6x_3 + r\end{aligned}$$
$$y = x_2 + x_3$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 3 \\ -3 & -5 & -6 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$
$$y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\mathbf{x}$$

Now, find the characteristic equation.

$$|s\mathbf{I}-A| = \left|\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 3 \\ -3 & -5 & -6 \end{bmatrix}\right| = \begin{bmatrix} s & -1 & 0 \\ 0 & (s-1) & -3 \\ 3 & 5` & (s+6) \end{bmatrix}$$

$$= s^3 + 5s^2 + 9s + 9$$

Forming a Routh table to determine stability

s^3	1	9
s^2	5	9
s^1	$\frac{36}{5}$	0
s^0	9	0

Since there are no sign changes, the closed-loop system is stable.

47.

Program:

```
A=[0,1,0;0,1,3;-3,-4,-5];
B=[0;0;1];
C=[0,1,1];
D=0;
'G'
G=ss(A,B,C,D)
'T'
T=feedback(G,1)
'Eigenvalues of T'
ssdata(T);
eig(T)
```

Computer response:

```
ans =

G


a =
        x1  x2  x3
   x1    0   1   0
   x2    0   1   3
   x3   -3  -4  -5


b =
        u1
   x1    0
   x2    0
   x3    1


c =
        x1  x2  x3
   y1    0   1   1


d =
        u1
```

```
   y1   0

Continuous-time model.

ans =

T


a =
       x1  x2  x3
   x1   0   1   0
   x2   0   1   3
   x3  -3  -5  -6


b =
       u1
   x1   0
   x2   0
   x3   1


c =
       x1  x2  x3
   y1   0   1   1


d =
       u1
   y1   0

Continuous-time model.

ans =

Eigenvalues of T


ans =

  -1.0000 + 1.4142i
  -1.0000 - 1.4142i
  -3.0000
```

SOLUTIONS TO DESIGN PROBLEMS

48.

$$T(s) = \frac{K(s+1)(s+10)}{s^3 + (5.45+K)s^2 + (11.91+11K)s + (43.65+10K)}$$

s^3	1	11.91+11K
s^2	5.45+K	43.65+10K
s^1	$\frac{11K^2 + 61.86K + 21.26}{5.45+K}$	0
s^0	43.65+10K	0

For stability, - 0.36772 < K < ∞. Stable for all positive K.

49.

$$T(s) = \frac{0.7K(s+0.1)}{s^4 + 2.2s^3 + 1.14s^2 + 0.193s + (0.07K+0.01)}$$

s^4	1	1.14	0.07K+0.01
s^3	2.2	0.193	0
s^2	1.0523	0.07K+0.01	0
s^1	0.17209 - 0.14635K	0	0
s^0	0.07K+0.01	0	0

For stability, - 0.1429 < K < 1.1759

50.

$$T(s) = \frac{0.6K + 10Ks^2 + 60.1Ks}{s^5 + 130s^4 + 3229s^3 + 10(K + 2348)s^2 + (60.1K + 58000)s + 0.6K}$$

s^5	1	3229	60.1K+58000
s^4	130	10K+23480	0.6K
s^3	-10K+396290	7812.4K+7540000	0
s2	$\frac{-100K^2+2712488K+8.3247E9}{-10K+396290}$	0.6K	0
s^1	$\frac{7813E3K^4-5.1401E11K^3+7.2469E15K^2+3.3213E19K+2.4874E22}{1000K^3-66753880K^2+9.9168E11K+3.299E15}$	0	0
s^0	0.6K	0	0

Note: s^3 row was multiplied by 130

From s^1 row after canceling common roots:

$$\frac{7813000(K-39629)(K+967.31586571671)(K+2776.9294183336)(K-29908.070615165)}{1000(K-39629)(K+2783.405672635)(K-29908.285672635)}$$

From s^0 row: K > 0

From s^3 row: K < 39629

From s^2 row: K < 29908.29; 39629 < K

From s^1 row: 29908.29 < K, or K < 29908.07;

Therefore, for stability, 0 < K < 29908.07

51.

s^5	1	1311.2	1000(100K+1)
s^4	112.1	10130	60000K
s^3	1220.8	99465K+1000	0
s^2	10038-9133.4K	60000K	0
s^1	$99465\dfrac{(K+0.010841)(K-1.0192)}{(K-1.0991)}$	0	0
s^0	60000K	0	0

From s^2 row: K < 1.099

From s^1 row: -0.010841 < K < 1.0192; K > 1.0991

From s^0 row: 0 < K

Therefore, 0 < K < 1.0192

52.

Find the closed-loop transfer function.

$$G(s) = \frac{63\text{x}10^6 K}{(s+30)(s+140)(s+2.5)}$$

$$T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{63\text{x}10^6 K}{s^3 + 172.5s^2 + 4625s + (10500 + 63\text{x}10^6 K)}$$

Make a Routh table.

s^3	1	4625
s^2	172.5	$10500+63\text{x}10^6K$
s^1	$4564.13-365217.39K$	0
s^0	$10500+63\text{x}10^6K$	0

The s^1 line says $K < 1.25\text{x}10^{-2}$ for stability. The s^0 line says $K > -1.67\text{x}10^{-4}$ for stability.

Hence, $-1.67\text{x}10^{-4} < K < 1.25\text{x}10^{-2}$ for stability.

53.

Find the closed-loop transfer function.

$$G(s) = \frac{7570Kp(s+103)(s+0.8)}{s(s+62.61)(s-62.61)}$$

$$T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{7570Kp(s+103)(s+0.8)}{s^3 + 7570K_p s^2 + (785766K_p - 3918.76)s + 623768K_p}$$

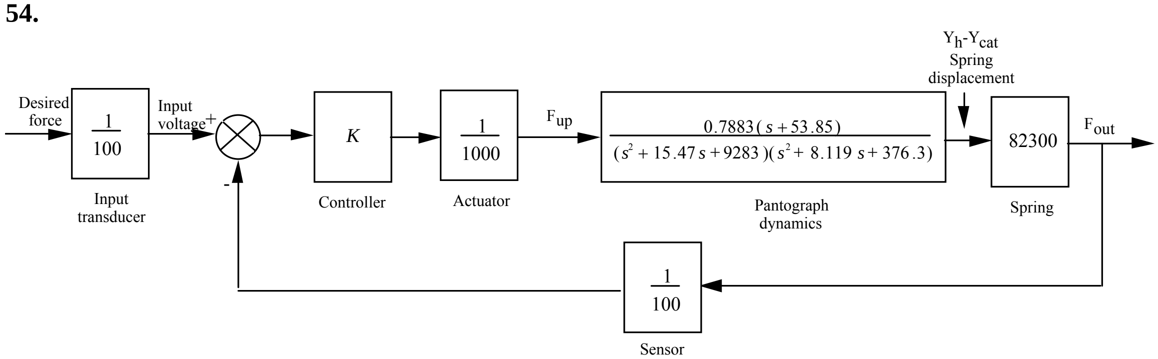
Make a Routh table.

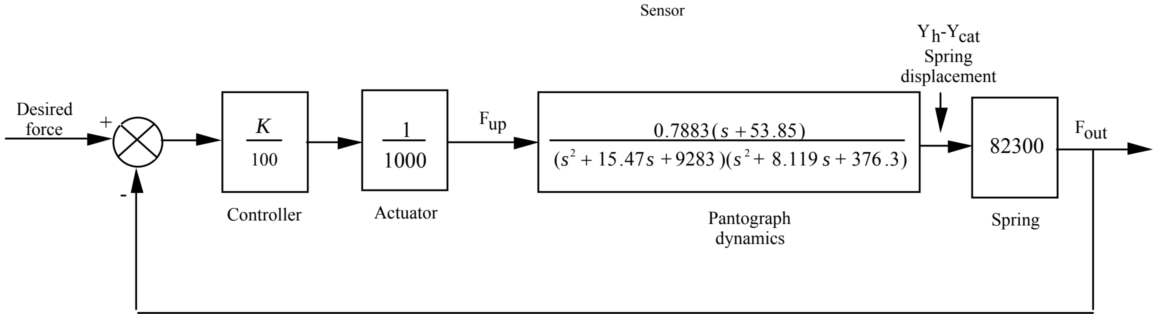
s^3	1	$785766K_p - 3918.76$
s^2	7570	$623768K_p$
s^1	$785766K_p - 4001.16$	0
s^0	$623768K_p$	0

The s^1 line says $K_p > 5.09\text{x}10^{-3}$ for stability. The s^0 line says $K_p > 0$ for stability.

Hence, $K_p > 5.09\text{x}10^{-3}$ for stability.

54.





$$G(s) = \frac{Y_h(s) - Y_{cat}(s)}{F_{up}(s)} = \frac{0.7883(s+53.85)}{(s^2+15.47s+9283)(s^2+8.119s+376.3)}$$

Ge(s)=(K/100)*(1/1000)*G(s)*82.3e3

$$G_e(s) = \frac{0.6488K\,(s+53.85)}{(s^2 + 8.119s + 376.3)\,(s^2 + 15.47s + 9283)}$$

$$T(s) = \frac{0.6488K\,(s+53.85)}{s^4 + 23.589\,s^3 + 9784.90093\,s^2 + (0.6488\,K + 81190.038)s + (34.94\,K + 0.34931929\,10^7)}$$

s^4	1	9785	(0.3493e7+34.94K)	+
s^3	23.59	(0.6488K+81190)	0	+
s^2	(-0.0275K+6343)	(0.3493e7+34.94K)	0	K<230654
s^1	$\frac{-0.0178K^2 + 10587K + 432.59e6}{-.0275K + 6343}$		0	-128966<K<188444
s^0	(0.3493e7+34.94K)		0	-99971<K

The last column evaluates the range of *K* for stability for each row. Therefore $-99971 < K < 188444$.

S E V E N

Steady-State Errors

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Steady-State Error Design via Gain

a. $G(s) = \frac{76.39K}{s(s+150)(s+1.32)}$. System is Type 1. Step input: $e(\infty) = 0$; Ramp input:

$e(\infty) = \frac{1}{K_v} = \frac{1}{\frac{76.39K}{150 \times 1.32}} = \frac{2.59}{K}$; Parabolic input: $e(\infty) = \infty$.

b. $\frac{1}{K_v} = \frac{2.59}{K} = 0.2$. Therefore, K = 12.95. Now test the closed-loop transfer function,

$T(s) = \frac{989.25}{s^3+151.32s^2+198s+989.25}$, for stability. Using Routh-Hurwitz, the system is stable.

s^3	1	198
s^2	151.32	989.25
s^1	191.46253	0
s^0	989.25	0

Video Laser Disc Recorder: Steady-State Error Design via Gain

a. The input, $15t^2$, transforms into $30/s^3$. $e(\infty) = 30/K_a = 0.005$.

$K_a = \frac{0.2*600}{20000} * K_1K_2K_3 = 6x10^{-3} K_1K_2K_3$. Therefore: $e(\infty) = 30/K_a = \frac{30}{6x10^{-3}K_1K_2K_3}$

$= 5x10^{-3}$. Therefore $K_1K_2K_3 = 10^6$.

b. Using $K_1K_2K_3 = 10^6$, $G(s) = \frac{2x10^5(s+600)}{s^2(s+2x10^4)}$. Therefore, T(s) =

$\frac{2x10^5(s+600)}{s^3+2x10^4s^2+2x10^5s+1.2x10^8}$.

Making a Routh table,

s^3	1	$2x10^5$
s^2	$2x10^4$	$1.2x10^8$
s^1	194000	0
s^0	120000000	0

we see that the system is stable.

c.
Program:
```
numg=200000*[1 600];
deng=poly([0 0 -20000]);
G=tf(numg,deng);
'T(s)'
T=feedback(G,1)
poles=pole(T)
```

Computer response:
```
ans =

T(s)

Transfer function:
         200000 s + 1.2e008
------------------------------------
s^3 + 20000 s^2 + 200000 s + 1.2e008

poles =

  1.0e+004 *

  -1.9990
  -0.0005 + 0.0077i
  -0.0005 - 0.0077I
```

ANSWERS TO REVIEW QUESTIONS

1. Nonlinear, system configuration

2. Infinite

3. Step(position), ramp(velocity), parabola(acceleration)

4. Step(position)-1, ramp(velocity)-2, parabola(acceleration)-3

5. Decreases the steady-state error

6. Static error coefficient is much greater than unity.

7. They are exact reciprocals.

8. A test input of a step is used; the system has no integrations in the forward path; the error for a step input is 1/10001.

9. The number of pure integrations in the forward path

10. Type 0 since there are no poles at the origin

11. Minimizes their effect

12. If each transfer function has no pure integrations, then the disturbance is minimized by decreasing the plant gain and increasing the controller gain. If any function has an integration then there is no control over its effect through gain adjustment.

13. No

14. A unity feedback is created by subtracting one from H(s). G(s) with H(s)-1 as feedback form an equivalent forward path transfer function with unity feedback.

15. The fractional change in a function caused by a fractional change in a parameter

16. Final value theorem and input substitution methods

SOLUTIONS TO PROBLEMS

1.

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

where

$$G(s) = \frac{450(s+12)(s+8)(s+15)}{s(s+38)(s^2+2s+28)}.$$

For step, $e(\infty) = 0$. For 37tu(t) , $R(s) = \frac{37}{s^2}$. Thus, $e(\infty) = 6.075\text{x}10^{-2}$. For parabolic input, $e(\infty) = \infty$

2.

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{s(60/s^3)}{1+\frac{20(s+3)(s+4)(s+8)}{s^2(s+2)(s+15)}} = 0.9375$$

3.

Reduce the system to an equivalent unity feedback system by first moving 1/s to the left past the summing junction. This move creates a forward path consisting of a parallel pair, $\left(\frac{1}{s}+1\right)$ in cascade with a feedback loop consisting of $G(s) = \frac{2}{s+3}$ and $H(s) = 7$. Thus,

$$G_e(s) = \left(\frac{(s+1)}{s}\right)\left(\frac{2/(s+3)}{1+14/(s+3)}\right) = \frac{2(s+1)}{s(s+17)}$$

Hence, the system is Type 1 and the steady-state errors are as follows:

Steady-state error for $15u(t) = 0$.

Steady-state error for $15tu(t) = \frac{15}{K_v} = \frac{15}{2/17} = 127.5$.

Steady-state error for $15t^2u(t) = \infty$

4.

System is type 0. $K_p = \frac{5}{2}$.

For 40u(t), $e(\infty) = \frac{40}{1+K_p} = \frac{80}{7} = 11.43$

For 70tu(t), $e(\infty) = \infty$

For $80t^2u(t)$, $e(\infty) = \infty$

5.

$$E(s) = \frac{R(s)}{1+G(s)} = \frac{72/S^4}{1+\frac{200(S+2)(S+5)(S+7)(S+9)}{S^3(S+3)(S+10)(S+15)}}$$

Thus,

$$e(\infty) = \lim_{s\to 0} sE(s) = \frac{72}{\frac{(200)(2)(5)(7)(9)}{(3)(10)(15)}} = 0.2571$$

6.

$$\mathcal{L}\left[\frac{de}{dt}\right] = sE(s)$$

Therefore, $\dot{e}(\infty) = \lim_{s\to 0} s^2E(s) = \lim_{s\to 0} s^2\frac{R(s)}{1+G(s)} = \lim_{s\to 0}\frac{s^2\frac{6}{s^4}}{1+\frac{100(s+1)(s+2)}{s^2(s+10)(s+3)}} = \frac{9}{10}$.

7.

$e(\infty) = \frac{15}{1+K_p}$; $K_p = \frac{1000(12)(25)(32)}{(61)(73)(87)} = 24.78$. Therefore, $e(\infty) = 0.582$.

8.

For 8u(t), $e_{ss} = \frac{8}{1+K_p} = 2$; For 8tu(t), $e_{ss} = \infty$, since the system is Type 0.

9.

a. The closed-loop transfer function is,

$$T(s) = \frac{5000}{s^2+75s+5000}$$

from which, $\omega_n = \sqrt{5000}$ and $2\zeta\omega_n = 75$. Thus, $\zeta = 0.53$ and

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 14.01\%$.

b. $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{75/2} = 0.107$ second.

c. Since system is Type 1, e_{ss} for $5u(t)$ is zero.

d. Since K_v is $\frac{5000}{75} = 66.67$, $e_{ss} = \frac{5}{K_v} = 0.075$.

e. $e_{ss} = \infty$, since system is Type 1.

10.

$$K_v = \lim_{s\to 0} sG(s) = \frac{10^5(3)(10)(20)}{(25)(\alpha)(30)} = 10^4$$

Thus, $\alpha = 8$.

11.

$K_a = \lim_{s\to 0} s^2 G(s) = \frac{Kx2x4x6}{5x7} = 10{,}000$. Therefore, K = 7291.667.

12.

a. $G_e(s) = \dfrac{\dfrac{5}{s(s+1)(s+2)}}{1+\dfrac{5(s+3)}{s(s+1)(s+2)}} = \dfrac{5}{s^3+3s^{2+}+7s+15}$

Therefore, $K_p = 1/3$; $K_v = 0$; and $K_a = 0$.

b. For 50u(t), $e(\infty) = \frac{50}{1+K_p} = 37.5$; For 50tu(t), $e(\infty) = \infty$; For $50t^2u(t)$, $e(\infty) = \infty$

c. Type 0

13.

$$E(s) = \frac{R(s)}{1+G(s)}. \text{ Thus, } e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s\frac{6}{s^4}}{1+\frac{1000(s^2+4s+20)(s^2+20s+15)}{s^3(s+2)(s+10)}}$$

$= 4x10^{-4}$.

14.

Collapsing the inner loop and multiplying by 1000/*s* yields the equivalent forward-path transfer function as,

$$G_e(s) = \frac{10^5(s+2)}{s(s^2+1005s+2000)}$$

Hence, the system is Type 1.

15.

$$\dot{e}(\infty) = \lim_{s\to 0} s^2 E(s) = \lim_{s\to 0} s^2 \frac{R(s)}{1+G(s)}.$$

For Type 0, step input: $R(s) = \frac{1}{s}$, and $\dot{e}(\infty) = \lim_{s\to 0} \frac{s}{1+G(s)} = 0$

For Type 0, ramp input: $R(s) = \frac{1}{s^2}$, and

$$\dot{e}(\infty) = \lim_{s\to 0} \frac{1}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+K_p}$$

For Type 0, parabolic input: $R(s) = \frac{1}{s^3}$, and $\dot{e}(\infty) = \lim_{s\to 0} \frac{1}{s+sG(s)} = \infty$

For Type 1, step input: $R(s) = \frac{1}{s}$, and $\dot{e}(\infty) = \lim_{s\to 0} \frac{s}{1+G(s)} = 0$

For Type 1, ramp input: $R(s) = \frac{1}{s^2}$, and $\dot{e}(\infty) = \lim_{s\to 0} \frac{1}{1+G(s)} = 0$

For Type 1, parabolic input: $R(s) = \frac{1}{s^3}$, and $\dot{e}(\infty) = \lim_{s\to 0} \frac{1}{s+sG(s)} = \frac{1}{K_v}$

For Type 2, step input: $R(s) = \frac{1}{s}$, and $\dot{e}(\infty) = \lim_{s\to 0} \frac{s}{1+G(s)} = 0$

For Type 2, ramp input: $R(s) = \frac{1}{s^2}$, and $\dot{e}(\infty) = \lim_{s\to 0} \frac{1}{1+G(s)} = 0$

For Type 2, parabolic input: $R(s) = \frac{1}{s^3}$, and $\dot{e}(\infty) = \lim_{s\to 0} \frac{1}{s+sG(s)} = 0$

		TYPE		
		0	**1**	**2**
INPUT	**Step**	0	0	0
	Ramp	$\frac{1}{1+K_p}$	0	0
	Parabola	∞	$\frac{1}{K_v}$	0

16.

a. $e(\infty) = \frac{1/10}{K_v} = 0.01$; where $K_v = \frac{7K}{5x8x12} = 10$. Thus, $K = 685.71$.

b. $K_v = 10$.

c. The minimum error will occur for the maximum gain before instability. Using the Routh-Hurwitz Criterion along with $T(s) = \dfrac{K(s+7)}{s^4 + 25s^3 + 196s^2 + (480+K)s + 7K}$:

s^4	1	196	7*K*	For Stability
s^3	25	480+*K*		
s^2	4420-*K*	175*K*		*K* < 4420
s^1	$-K^2 - 435K + 2121600$			-1690.2 < *K* < 1255.2
s^0	175*K*			*K* > 0

Thus, for stability and minimum error *K* = 1255.2. Thus, $K_v = \dfrac{7K}{5x8x12} = 18.3$ and

$$e(\infty) = \frac{1/10}{K_v} = \frac{1/10}{18.3} = 0.0055.$$

17.

$e(\infty) = \dfrac{15}{K_v} = \dfrac{15}{Ka/10} = \dfrac{150}{Ka} = 0.003$. Hence, Ka = 50,000.

18.

Find the equivalent G(s) for a unity feedback system. $G(s) = \dfrac{\frac{K}{s(s+1)}}{1 + \frac{10}{s+1}} = \dfrac{K}{s(s+11)}$. Thus, $e(\infty) = \dfrac{100}{K_v} = \dfrac{100}{K/11} = 0.01$; from which K = 110,000.

19.

$K_a = \dfrac{2K}{4}$. $e(\infty) = \dfrac{20}{K_a} = 0.01$. Hence, K = 4000.

20.

a. $e(\infty) = \dfrac{10}{K_v} = \dfrac{1}{6000}$. But, $K_v = \dfrac{30K}{5} = 60{,}000$. Hence, K = 10,000. For finite error for a ramp input, n = 1.

b. $K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \dfrac{10000(s^2+3s+30)}{s(s+5)} = \infty$

$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s\dfrac{10000(s^2+3s+30)}{s(s+5)} = 60{,}000$

$K_a = \lim_{s \to 0} s^2G(s) = \lim_{s \to 0} s^2\dfrac{10000(s^2+3s+30)}{s(s+5)} = 0$

21.

a. Type 0

b. $E(s) = \frac{R(s)}{1 + G(s)}$. Thus, $e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{10/s}{1 + \frac{K(s^2+2s+5)}{(s+2)^2(s+3)}} = \frac{120}{12+5K}$.

c. $e(\infty) = \infty$, since the system is Type 0.

22.

$$e(\infty) = \frac{25}{K_v} = \frac{25}{150K/420} = 0.1. \text{ Thus, } K = 700.$$

23.

$$e(\infty) = \frac{50}{1 + K_p} = \frac{50}{1 + \frac{4K}{26}} = 0.05. \text{ Thus, } K = 6493.5.$$

24.

The system is stable for $0 < K < 2000$. Since the maximum K_v is $K_v = \frac{K}{320} = \frac{2000}{320} = 6.25$, the minimum steady-state error is $\frac{1}{K_v} = \frac{1}{6.25} = 0.16$.

25.

To meet steady-state error characteristics:

$$E(s) = \frac{R(s)}{1 + G(s)} = \frac{1}{s\left(1 + \frac{K(s+\alpha)}{(s+\beta)^2}\right)}$$

$$e(t)\Big|_{t \to \infty} = sE(s)\Big|_{s \to 0} = \frac{1}{1 + \frac{K\alpha}{\beta^2}} = \frac{\beta^2}{\beta^2 + K\alpha} = 0.1$$

Therefore, $K\alpha = 9\beta^2$.

To meet the transient requirement: Since $T(s) = \frac{K(s+\alpha)}{s^2 + (K+2\beta)s + (\beta^2 + K\alpha)}$,

$\omega_n^2 = 10 = \beta^2 + K\alpha$; $2\zeta\omega_n = \sqrt{10} = K+2\beta$. Solving for β, $\beta = \pm 1$. For $\beta = +1$, $K = 1.16$ and $\alpha = 7.76$.

An alternate solution is $\beta = -1$, $K = 5.16$, and $\alpha = 1.74$.

26.

a. System Type = 1

b. Assume $G(s) = \frac{K}{s(s+\alpha)}$. Therefore, $e(\infty) = \frac{1}{K_v} = \frac{1}{K/\alpha} = 0.01$, or $\frac{K}{\alpha} = 100$.

But, $T(s) = \frac{G(s)}{1 + G(s)} = \frac{K}{s^2+\alpha s+K}$.

Since $\omega_n = 10$, $K = 100$, and $\alpha = 1$. Hence, $G(s) = \frac{100}{s(s+1)}$.

c. $2\zeta\omega_n = \alpha = 1$. Thus, $\zeta = \frac{1}{20}$.

27.

$T(s) = \frac{G(s)}{1+G(s)} = \frac{K(s+\alpha)}{s^2+(K+\beta)s+\alpha K}$. Hence, $K+\beta = 2$, $K\alpha = \omega_n^2 = (1^2+1^2) = 2$.

Also, $e(\infty) = \frac{1}{K_v} = \frac{\beta}{K\alpha} = 0.1$. Therefore, $\beta = 0.1K\alpha = 0.2$, $K = 1.8$, and $\alpha = 1.111$.

28.

System Type = 1. $T(s) = \frac{G(s)}{1+G(s)} = \frac{K}{s^2+as+K}$. From G(s), $K_v = \frac{K}{a} = 100$. For 10% overshoot, $\zeta = 0.6$. Therefore, $2\zeta\omega_n = a$, and $\omega_n^2 = K$. Hence, $a = 1.2\sqrt{K}$.

Also, $a = \frac{K}{100}$ Solving simultaneously,

$K = 1.44 \times 10^4$, and $a = 1.44 \times 10^2$.

29.

a. For 20% overshoot, $\zeta = 0.456$. Also, $K_v = 1000 = \frac{K}{a}$. Since $T(s) = \frac{K}{s^2+as+K}$, $2\zeta\omega_n = a$, and $\omega_n = \sqrt{K}$. Hence, $a = 0.912\sqrt{K}$. Solving for a and K, K = 831,744, and a = 831.744.

b. For 10% overshoot, $\zeta = 0.591$. Also, $\frac{1}{K_v} = 0.01$. Thus, $K_v = 100 = \frac{K}{a}$. Since $T(s) = \frac{K}{s^2+as+K}$, $2\zeta\omega_n = a$, and $\omega_n = \sqrt{K}$. Hence, $a = 1.182\sqrt{K}$. Solving for a and K, K = 13971 and a = 139.71.

30.

a. For the inner loop:

$$G_1(s) = \frac{\frac{1}{s^2(s+1)}}{1+\frac{1}{s^3(s+1)}} = \frac{s}{s^4+s^3+1}$$

$$G_e(s) = \frac{1}{s^2(s+3)}\ G_1(s) = \frac{1}{s(s^5+4s^4+3s^3+s+3)}$$

$$T(s) = \frac{G_e(s)}{1+G_e(s)} = \frac{1}{s^6+4s^5+3s^4+s^2+3s+1}$$

b. From $G_e(s)$, system is Type 1.

c. Since system is Type 1, $e_{ss} = 0$

d. ; From $G_e(s)$, $K_v = \lim_{s\to 0} sG_e(s) = \frac{1}{3}$. Therefore, $e_{ss} = \frac{5}{K_v} = 15$.

e. Poles of T(s) = -3.0190, -1.3166, 0.3426 ± j0.7762, -0.3495. Therefore, system is unstable and results of (c) and (d) are meaningless

31.

a. For the inner loop:

$$G_1(s) = \frac{\frac{10}{s(s+1)(s+3)(s+4)}}{1+\frac{20}{(s+1)(s+3)(s+4)}} = \frac{10}{s(s^3+8s^2+19s+32)}$$

$$G_e(s) = \frac{20}{s(s^3+8s^2+19s+32)}$$

$$T(s) = \frac{G_e(s)}{1+G_e(s)} = \frac{20}{s^4+8s^3+19s^2+32s+20}$$

b. From $G_e(s)$, system is Type 1.

c. Since system is Type 1, $e_{ss} = 0$

d. From $G_e(s)$, $K_v = \lim_{s\to 0} sG_e(s) = \frac{20}{32} = \frac{5}{8}$. Therefore, $e_{ss} = \frac{5}{K_v} = 8$.

e. Poles of T(s) = -5.4755, -0.7622 ± j1.7526, -1. Therefore, system is stable and results of parts c and d are valid.

32.

Program:

```
numg1=[1 7];deng1=poly([0 -4 -8 -12]);
'G1(s)='
G1=tf(numg1,deng1)
numg2=5*poly([-9 -13]);deng2=poly([-10 -32 -64]);
'G2(s)='
G2=tf(numg2,deng2)
numh1=10;denh1=1;
'H1(s)='
H1=tf(numh1,denh1)
numh2=1;denh2=[1 3];
'H2(s)='
H2=tf(numh2,denh2)
%Close loop with H1 and form G3
'G3(s)=G2(s)/(1+G2(s)H1(s)'
G3=feedback(G2,H1)
%Form G4=G1G3
'G4(s)=G1(s)G3(s)'
G4=series(G1,G3)
%Form Ge=G4/1+G4H2
'Ge(s)=G4(s)/(1+G4(s)H2(s))'
Ge=feedback(G4,H2)
%Form T(s)=Ge(s)/(1+Ge(s)) to test stability
'T(s)=Ge(s)/(1+Ge(s))'
T=feedback(Ge,1)
'Poles of T(s)'
pole(T)
%Computer response shows that system is stable. Now find error specs.
Kp=dcgain(Ge)
'sGe(s)='
sGe=tf([1 0],1)*Ge;
'sGe(s)'
sGe=minreal(sGe)
Kv=dcgain(sGe)
's^2Ge(s)='
s2Ge=tf([1 0],1)*sGe;
's^2Ge(s)'
s2Ge=minreal(s2Ge)
Ka=dcgain(s2Ge)
essstep=30/(1+Kp)
essramp=30/Kv
essparabola=60/Ka
```

Computer response:

```
ans =
```

```
G1(s)=



Transfer function:
            s + 7
------------------------------
s^4 + 24 s^3 + 176 s^2 + 384 s
ans =
G2(s)=
Transfer function:
     5 s^2 + 110 s + 585
------------------------------
s^3 + 106 s^2 + 3008 s + 20480
ans =
H1(s)=
Transfer function:
10
ans =
H2(s)=
Transfer function:
  1
-----
s + 3
ans =
G3(s)=G2(s)/(1+G2(s)H1(s)
Transfer function:
     5 s^2 + 110 s + 585
```

```
------------------------------
s^3 + 156 s^2 + 4108 s + 26330


ans =
G4(s)=G1(s)G3(s)
Transfer function:
            5 s^3 + 145 s^2 + 1355 s + 4095
-------------------------------------------------------
s^7 + 180 s^6 + 8028 s^5 + 152762 s^4 + 1.415e006 s^3
                          + 6.212e006 s^2 + 1.011e007 s
ans =
Ge(s)=G4(s)/(1+G4(s)H2(s))
Transfer function:
      5 s^4 + 160 s^3 + 1790 s^2 + 8160 s + 12285
-------------------------------------------------------
s^8 + 183 s^7 + 8568 s^6 + 176846 s^5 + 1.873e006 s^4
        + 1.046e007 s^3 + 2.875e007 s^2 + 3.033e007 s
                                                 + 4095
ans =
T(s)=Ge(s)/(1+Ge(s))
Transfer function:
      5 s^4 + 160 s^3 + 1790 s^2 + 8160 s + 12285
-------------------------------------------------------
s^8 + 183 s^7 + 8568 s^6 + 176846 s^5 + 1.873e006 s^4
        + 1.046e007 s^3 + 2.875e007 s^2 + 3.034e007 s
                                                + 16380
```

```
ans =

Poles of T(s)


ans =

-124.7657

  -21.3495

  -12.0001

   -9.8847

   -7.9999

   -4.0000

   -2.9994

   -0.0005

Kp =

     3

ans =

sGe(s)=

ans =

sGe(s)

Transfer function:

   5 s^5 + 160 s^4 + 1790 s^3 + 8160 s^2 + 1.229e004 s

---------------------------------------------------------

s^8 + 183 s^7 + 8568 s^6 + 1.768e005 s^5 + 1.873e006 s^4

        + 1.046e007 s^3 + 2.875e007 s^2 + 3.033e007 s

                                                + 4095

Kv =

     0

ans =
```

```
s^2Ge(s)=



ans =

s^2Ge(s)

Transfer function:

  5 s^6 + 160 s^5 + 1790 s^4 + 8160 s^3 + 1.229e004 s^2

---------------------------------------------------------

s^8 + 183 s^7 + 8568 s^6 + 1.768e005 s^5 + 1.873e006 s^4

        + 1.046e007 s^3 + 2.875e007 s^2 + 3.033e007 s

                                                 + 4095

Ka =

     0

essstep =

    7.5000

Warning: Divide by zero.

(Type "warning off MATLAB:divideByZero" to suppress this warning.)

> In D:\My Documents\Control Systems Engineering Book\CSE 4th ed\Solutions Manual\Chap

7 References\p7_32.m at line 40

essramp =

   Inf

Warning: Divide by zero.

(Type "warning off MATLAB:divideByZero" to suppress this warning.)

> In D:\My Documents\Control Systems Engineering Book\CSE 4th ed\Solutions Manual\Chap

7 References\p7_32.m at line 41

essparabola =

   Inf
```

33.

The equivalent forward transfer function is, $G(s) = \frac{10K_1}{s(s+1+10K_f)}$.

Also, $T(s) = \frac{G(s)}{1 + G(s)} = \frac{10K_1}{s^2+(10K_f+1)s+10K_1}$. From the problem statement, $K_v = \frac{10K_1}{1+10K_f} = 10$.

Also, $2\zeta\omega_n = 10K_f+1 = 2(0.5)\sqrt{10K_1} = \sqrt{10K_1}$. Solving for K_1 and K_f simultaneously, $K_1 = 10$ and $K_f = 0.9$.

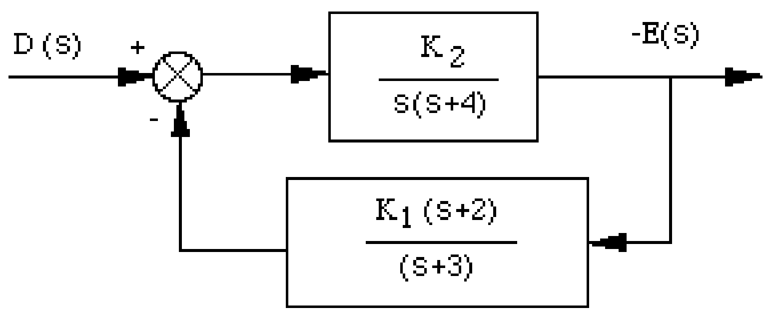
34.

$e(\infty) = \lim_{s\to 0} \frac{sR(s) - sD(s)G_2(s)}{1 + G_1(s)G_2(s)}$, where $G_1(s) = \frac{1}{s+5}$ and $G_2 = \frac{100}{s+2}$. From the problem statement,

$R(s) = D(s) = \frac{1}{s}$. Hence, $e(\infty) = \lim_{s\to 0} \frac{1 - \frac{100}{s+2}}{1 + \frac{1}{s+5}\frac{100}{s+2}} = -\frac{49}{11}$.

35.

Error due only to disturbance: Rearranging the block diagram to show D(s) as the input,



Therefore,

$$-E(s) = D(s)\frac{\frac{K_2}{s(s+4)}}{1 + \frac{K_1K_2(s+2)}{s(s+3)(s+4)}} = D(s)\frac{K_2(s+3)}{s(s+3)(s+4) + K_1K_2(s+2)}$$

For $D(s) = \frac{1}{s}$, $e_D(\infty) = \lim_{s\to 0} sE(s) = -\frac{3}{2K_1}$.

Error due only to input: $e_R(\infty) = \frac{1}{K_v} = \frac{1}{\frac{K_1K_2}{6}} = \frac{6}{K_1K_2}$.

Design:

$e_D(\infty) = -0.000012 = -\frac{3}{2K_1}$, or $K_1 = 125{,}000$.

$e_R(\infty) = 0.003 = \frac{6}{K_1K_2}$, or $K_2 = 0.016$

36.

$$\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_2(s)H_1(s)}; \quad \therefore \frac{E_{a1}(s)}{R(s)} = \frac{G_1(s)}{1+G_2(s)H_1(s)}$$

$$e_{a1}(\infty) = \lim_{s\to 0}\frac{sR(s)G_1(s)}{1+G_2(s)H_1(s)}$$

37.

System 1:

Forming a unity-feedback path, the equivalent unity feedback system has a forward transfer function of

$$G_e(s) = \frac{\frac{10(s+10)}{s(s+2)}}{1+\frac{10(s+10)(s+3)}{s(s+2)}} = \frac{10(s+10)}{11s^2+132s+300}$$

a. Type 0 System; **b.** $K_p = \lim_{s\to 0} G_e(s) = 1/3$; **c.** step input; **d.** $e(\infty) = \frac{1}{1+K_p} = 3/4$;

e. $e_{a-step}(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0}\frac{s\left(\frac{1}{s}\right)}{1+\frac{10(s+10)(s+4)}{s(s+2)}} = 0$.

System 2:

Forming a unity-feedback path, the equivalent unity feedback system has a forward transfer function of

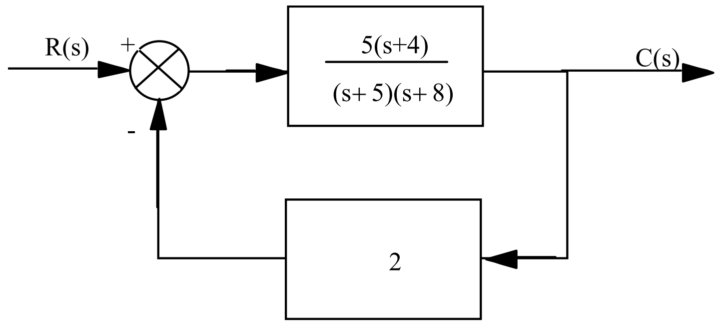
$$G_e(s) = \frac{\frac{10(s+10)}{s(s+2)}}{1+\frac{10(s+10)s}{s(s+2)}} = \frac{10(s+10)}{s(11s+102)}$$

a. Type 1 System; **b.** $K_v = \lim_{s\to 0} sG_e(s) = 0.98$; **c.** ramp input; **d.** $e(\infty) = \frac{1}{K_v} = 1.02$;

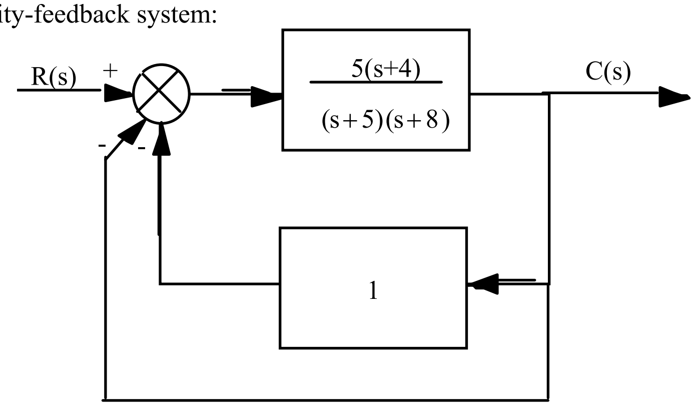
e. $e_{a-ramp}(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0}\frac{s\left(\frac{1}{s^2}\right)}{1+\frac{10(s+10)(s+1)}{s(s+2)}} = \frac{1}{50}$.

38.

System 1. Push 5 to the right past the summing junction:



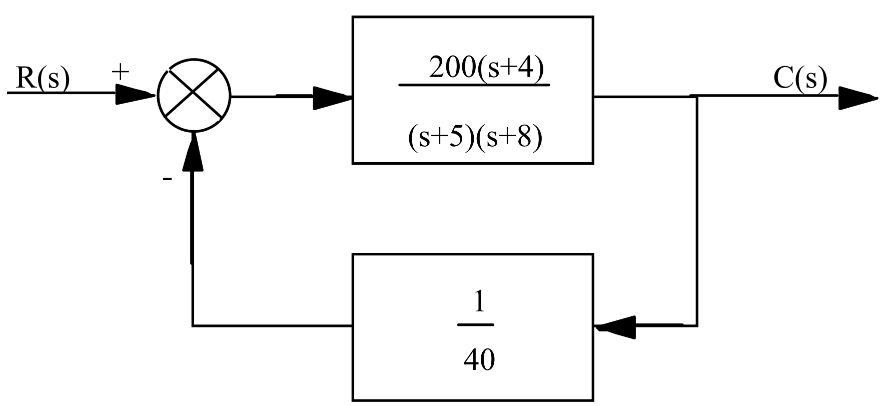
Produce a unity-feedback system:



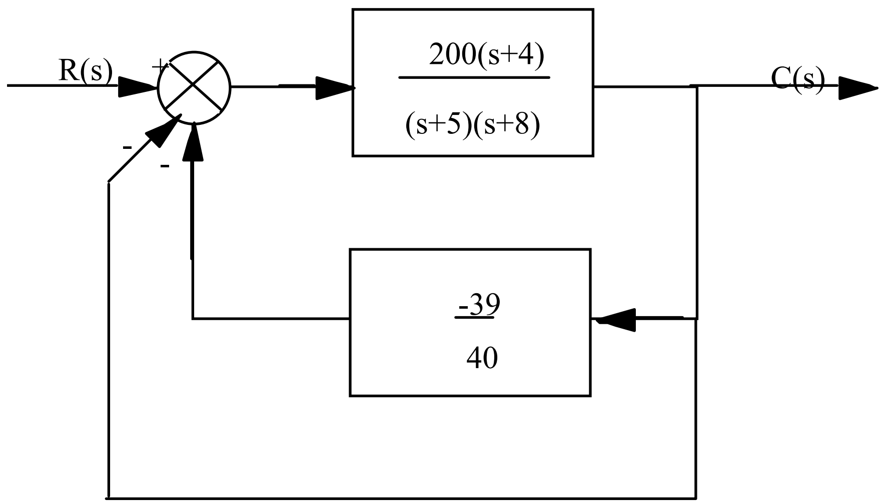
Thus, $G_e(s) = \dfrac{\dfrac{5(s+4)}{(s+5)(s+8)}}{1+\dfrac{5(s+4)}{(s+5)(s+8)}} = \dfrac{5(s+4)}{s^2+18s+60}$. $K_p = \dfrac{1}{3}$. $e_{step} = \dfrac{1}{1+K_p}$ = 0.75, $e_{ramp} = \infty$, $e_{parabola} = \infty$.

Checking for stability, from first block diagram above, T(s) = $\dfrac{5(s+4)}{s^2+23s+80}$. The system is stable.

System 2. Push 20 to the right past the summing junction and push 10 to the left past the pickoff point:



Produce a unity-feedback system:



Thus, $G_e(s) = \dfrac{\dfrac{200(s+4)}{(s+5)(s+8)}}{1 - \dfrac{200(s+4)}{(s+5)(s+8)}\left(\dfrac{39}{40}\right)} = \dfrac{200(s+4)}{s^2 - 182s - 740}$. $K_p = \dfrac{200(4)}{-740} = -1.08$.

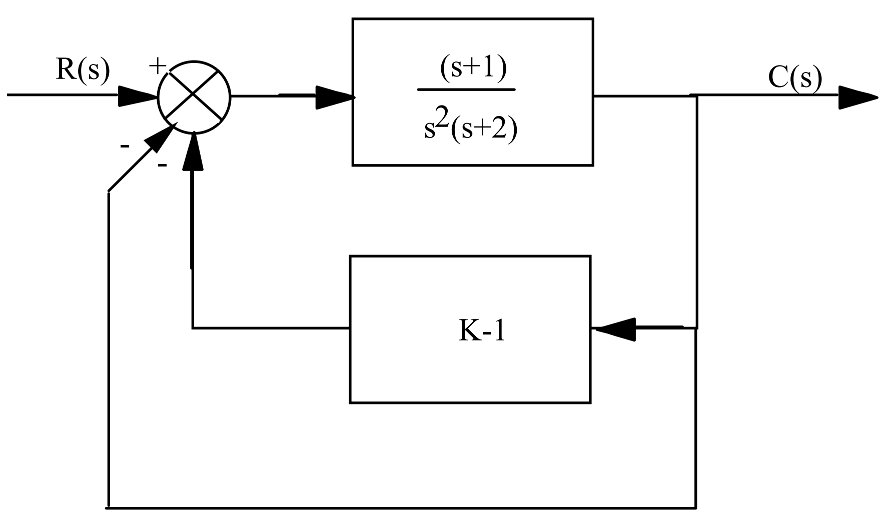
$e_{step} = \dfrac{1}{1+K_p}$ = -12.5, $e_{ramp} = \infty$, $e_{parabola} = \infty$.

Checking for stability, from first block diagram above, $T(s) = \dfrac{G_e(s)}{1+G_e(s)} = \dfrac{200(s+4)}{s^2+18s+60}$.

Therefore, system is stable and steady-state error calculations are valid.

39.

Produce a unity-feedback system:



Thus, $G_e(s) = \dfrac{\dfrac{(s+1)}{s^2(s+2)}}{1+\dfrac{(s+1)(K-1)}{s^2(s+2)}} = \dfrac{s+1}{s^3+2s^2+(K-1)s+(K-1)}$. Error = 0.001 = $\dfrac{1}{1+K_p}$.

Therefore, $K_p = 999 = \dfrac{1}{K-1}$. Hence, K = 1.001001.

Check stability: Using original block diagram, $T(s) = \dfrac{\dfrac{(s+1)}{s^2(s+2)}}{1+\dfrac{K(s+1)}{s^2(s+2)}} = \dfrac{s+1}{s^3+2s^2+Ks+K}$.

Making a Routh table:

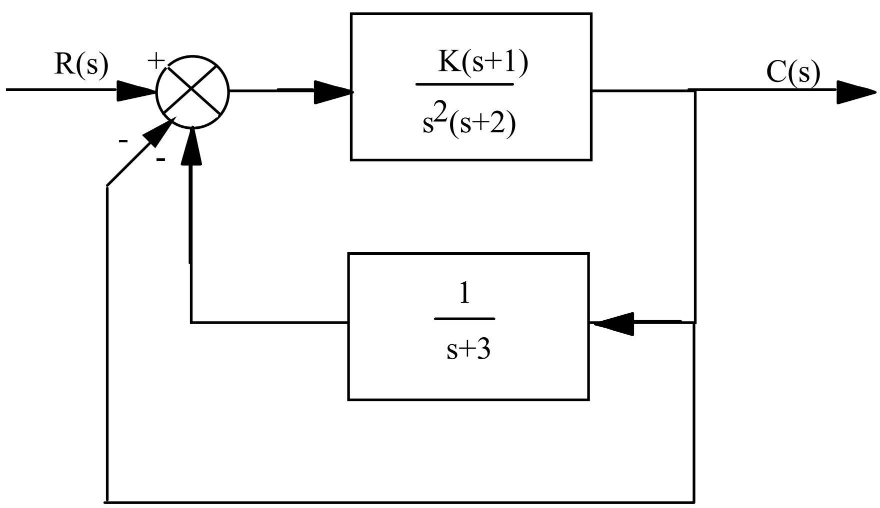
s^3	1	K
s^2	2	K
s^1	$\frac{K}{2}$	0
s^0	K	0

Therefore, system is stable and steady-state error calculations are valid.

40.

a. Produce a unity-feedback system:

$$H_1(s) = \frac{s+4}{s+3} - 1 = \frac{1}{s+3}$$



Thus, $G_e(s) = \dfrac{\dfrac{K(s+1)}{s^2(s+2)}}{1+\dfrac{\frac{1}{s+3}K(s+1)}{s^2(s+2)}} = \dfrac{K(s+1)(s+3)}{s^4+5s^3+6s^2+Ks+K}$. System is Type 0.

b. Since Type 0, appropriate static error constant is K_p.

c. $K_p = \frac{3K}{K} = 3$

d. $e_{step} = \frac{1}{1+K_p} = \frac{1}{4}$

Check stability: Using original block diagram, $T(s) = \dfrac{\dfrac{K(s+1)}{s^2(s+2)}}{1+\dfrac{(s+4)K(s+1)}{(s+3)s^2(s+2)}} = \dfrac{K(s+1)(s+3)}{s^4+5s^3+(K+6)s^2+5Ks+4K}$.

Making a Routh table:

s^4	1	K+6	4K
s^3	5	5K	0
s^2	6	4K	0
s^1	$\frac{5}{3}K$	0	0
s^0	4K	0	0

Therefore, system is stable for $0 < K$ and steady-state error calculations are valid.

41.

Program:

```
K=10
numg1=K*poly([-1 -2]);deng1=poly([0 0 -3 -4 -5]);
'G1(s)='
G1=tf(numg1,deng1)
numh1=[1 6];denh1=poly([-7 -8]);
'H1(s)='
H1=tf(numh1,denh1)
'H2(s)=H1-1'
H2=H1-1
%Form Ge(s)=G1(s)/(1+G1(s)H2(s)
'Ge(s)=G1(s)/(1+G1(s)H2(s))'
Ge=feedback(G1,H2)
%Test system stability
'T(s)=Ge(s)/(1+Ge(s))'
T=feedback(Ge,1)
pole(T)
Kp=dcgain(Ge)
'sGe(s)'
sGe=tf([1 0],1)*Ge;
sGe=minreal(sGe)
Kv=dcgain(sGe)
's^2Ge(s)'
s2Ge=tf([1 0],1)*sGe;
s2Ge=minreal(s2Ge)
Ka=dcgain(s2Ge)
essstep=30/(1+Kp)
essramp=30/Kv
essparabola=60/Ka

K=1E6
numg1=K*poly([-1 -2]);deng1=poly([0 0 -3 -4 -5]);
'G1(s)='
G1=tf(numg1,deng1)
numh1=[1 6];denh1=poly([-7 -8]);
```

```
'H1(s)='
H1=tf(numh1,denh1)
'H2(s)=H1-1'
H2=H1-1
%Form Ge(s)=G1(s)/(1+G1(s)H2(s)
'Ge(s)=G1(s)/(1+G1(s)H2(s))'
Ge=feedback(G1,H2)
%Test system stability
'T(s)=Ge(s)/(1+Ge(s))'
T=feedback(Ge,1)
pole(T)
Kp=dcgain(Ge)
'sGe(s)'
sGe=tf([1 0],1)*Ge;
sGe=minreal(sGe)
Kv=dcgain(sGe)
's^2Ge(s)'
s2Ge=tf([1 0],1)*sGe;
s2Ge=minreal(s2Ge)
Ka=dcgain(s2Ge)
essstep=30/(1+Kp)
essramp=30/Kv
essparabola=60/Ka
```

Computer response:

```
K =

    10


ans =

G1(s)=


Transfer function:
      10 s^2 + 30 s + 20
------------------------------
s^5 + 12 s^4 + 47 s^3 + 60 s^2


ans =

H1(s)=


Transfer function:
     s + 6
---------------
s^2 + 15 s + 56


ans =

H2(s)=H1-1


Transfer function:
-s^2 - 14 s - 50
----------------
s^2 + 15 s + 56


ans =

Ge(s)=G1(s)/(1+G1(s)H2(s))
```

```
Transfer function:

             10 s^4 + 180 s^3 + 1030 s^2 + 1980 s + 1120
---------------------------------------------------------------------
s^7 + 27 s^6 + 283 s^5 + 1427 s^4 + 3362 s^3 + 2420 s^2 - 1780 s

                                                               - 1000

ans =

T(s)=Ge(s)/(1+Ge(s))


Transfer function:
             10 s^4 + 180 s^3 + 1030 s^2 + 1980 s + 1120
---------------------------------------------------------------------
s^7 + 27 s^6 + 283 s^5 + 1437 s^4 + 3542 s^3 + 3450 s^2 + 200 s + 120


ans =

  -7.6131
  -7.4291
  -5.2697
  -3.3330 + 0.1827i
  -3.3330 - 0.1827i
  -0.0111 + 0.1898i
  -0.0111 - 0.1898i


Kp =

   -1.1200


ans =

sGe(s)


Transfer function:

           10 s^5 + 180 s^4 + 1030 s^3 + 1980 s^2 + 1120 s
---------------------------------------------------------------------
s^7 + 27 s^6 + 283 s^5 + 1427 s^4 + 3362 s^3 + 2420 s^2 - 1780 s

                                                               - 1000


Kv =

     0


ans =

s^2Ge(s)


Transfer function:

          10 s^6 + 180 s^5 + 1030 s^4 + 1980 s^3 + 1120 s^2
---------------------------------------------------------------------
s^7 + 27 s^6 + 283 s^5 + 1427 s^4 + 3362 s^3 + 2420 s^2 - 1780 s

                                                               - 1000
```

```
Ka =

     0


essstep =

 -250.0000

Warning: Divide by zero.
(Type "warning off MATLAB:divideByZero" to suppress this warning.)
> In D:\My Documents\Control Systems Engineering Book\CSE 4th ed\Solutions Manual\Chap 7
References\p7_41.m at line 27

essramp =

   Inf

Warning: Divide by zero.
(Type "warning off MATLAB:divideByZero" to suppress this warning.)
> In D:\My Documents\Control Systems Engineering Book\CSE 4th ed\Solutions Manual\Chap 7
References\p7_41.m at line 28

essparabola =

   Inf


K =

     1000000


ans =

G1(s)=


Transfer function:
 1e006 s^2 + 3e006 s + 2e006
------------------------------
s^5 + 12 s^4 + 47 s^3 + 60 s^2


ans =

H1(s)=


Transfer function:
     s + 6
---------------
s^2 + 15 s + 56


ans =

H2(s)=H1-1


Transfer function:
-s^2 - 14 s - 50
----------------
s^2 + 15 s + 56
```

```
ans =

Ge(s)=G1(s)/(1+G1(s)H2(s))


Transfer function:

   1e006 s^4 + 1.8e007 s^3 + 1.03e008 s^2 + 1.98e008 s + 1.12e008
---------------------------------------------------------------------
s^7 + 27 s^6 + 283 s^5 - 998563 s^4 - 1.7e007 s^3 - 9.4e007 s^2

                                                 - 1.78e008 s - 1e008

ans =

T(s)=Ge(s)/(1+Ge(s))


Transfer function:

   1e006 s^4 + 1.8e007 s^3 + 1.03e008 s^2 + 1.98e008 s + 1.12e008
---------------------------------------------------------------------
s^7 + 27 s^6 + 283 s^5 + 1437 s^4 + 1.004e006 s^3 + 9.003e006 s^2

                                                  + 2e007 s + 1.2e007


ans =

 -26.9750 +22.2518i
 -26.9750 -22.2518i
  17.9750 +22.2398i
  17.9750 -22.2398i
  -6.0000
  -1.9998
  -1.0002


Kp =

   -1.1200


ans =

sGe(s)


Transfer function:

 1e006 s^5 + 1.8e007 s^4 + 1.03e008 s^3 + 1.98e008 s^2 + 1.12e008 s
---------------------------------------------------------------------
s^7 + 27 s^6 + 283 s^5 - 9.986e005 s^4 - 1.7e007 s^3 - 9.4e007 s^2

                                                 - 1.78e008 s - 1e008

Kv =

     0


ans =

s^2Ge(s)
```

```
Transfer function:

1e006 s^6 + 1.8e007 s^5 + 1.03e008 s^4 + 1.98e008 s^3 + 1.12e008 s^2
---------------------------------------------------------------------
s^7 + 27 s^6 + 283 s^5 - 9.986e005 s^4 - 1.7e007 s^3 - 9.4e007 s^2

                                                 - 1.78e008 s - 1e008

Ka =

     0


essstep =

 -250.0000

Warning: Divide by zero.
(Type "warning off MATLAB:divideByZero" to suppress this warning.)
> In D:\My Documents\Control Systems Engineering Book\CSE 4th ed\Solutions Manual\Chap 7
References\p7_41.m at line 56

essramp =

   Inf

Warning: Divide by zero.
(Type "warning off MATLAB:divideByZero" to suppress this warning.)
> In D:\My Documents\Control Systems Engineering Book\CSE 4th ed\Solutions Manual\Chap 7
References\p7_41.m at line 57

essparabola =

   Inf
```

42.

$$Y(s) = R(s)\frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} + \frac{D(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

$$E(s) = R(s) - Y(s) = R(s) - \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}\ R(s) - \frac{D(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

$$= \left[1 - \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}\right] R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}\ D(s)$$

Thus,

$$e(\infty) = \lim_{s\varnothing 0} sE(s) = \lim_{s\varnothing 0} \left\{ \left[1 - \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}\right] R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} D(s)\right\}$$

43.

a. $E(s) = R(s) - C(s)$. But, $C(s) = [R(s) - C(s)H(s)]G_1(s)G_2(s) + D(s)$. Solving for $C(s)$,

$$C(s) = \frac{R(s)G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} + \frac{D(s)}{1 + G_1(s)G_2(s)H(s)}$$

Substituting into $E(s)$,

$$E(s) = \left[1 - \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}\right] R(s) - \frac{1}{1 + G_1(s)G_2(s)H(s)}\ D(s)$$

b. For $R(s) = D(s) = \frac{1}{s}$,

$$e(\infty) = \lim_{s\to 0} sE(s) = 1 - \frac{\lim\limits_{s\to 0} G_1(s)G_2(s)}{1 + \lim\limits_{s\to 0} G_1(s)G_2(s)H(s)} - \frac{1}{1 + \lim\limits_{s\to 0} G_1(s)G_2(s)H(s)}$$

c. Zero error if $G_1(s)$ and/or $G_2(s)$ is Type 1. Also, $H(s)$ is Type 0 with unity dc gain.

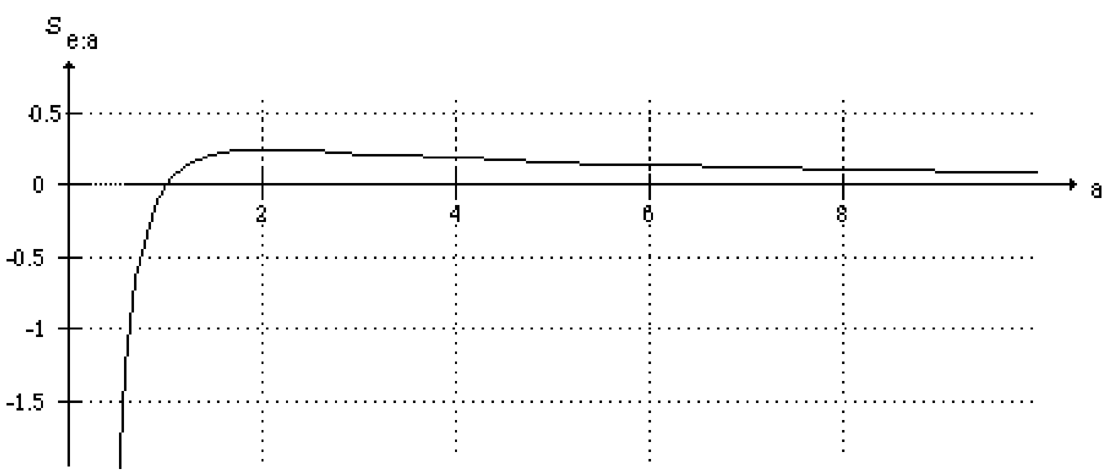
44.

First find the forward transfer function of an equivalent unity feedback system.

$$G_e(s) = \frac{\dfrac{K}{s(s+1)(s+4)}}{1 + \dfrac{K(s+a-1)}{s(s+1)(s+4)}} = \frac{K}{s^3 + 5s^2 + (K+4)s + K(a-1)}$$

Thus, $e(\infty) = e(\infty) = \dfrac{1}{1+K_p} = \dfrac{1}{1+\dfrac{K}{K(a-1)}} = \dfrac{a-1}{a}$

Finding the sensitivity of $e(\infty)$, $S_{e:a} = \dfrac{a}{e}\dfrac{\delta e}{\delta a} = \dfrac{a}{\dfrac{a}{a-1}}\left(\dfrac{a-(a-1)}{a^2}\right) = \dfrac{a-1}{a^2}$.



45.

From Eq. (7.70),

$$e(\infty) = 1 - \lim_{s\to 0}\left(\frac{\dfrac{K_1K_2}{(s+2)}}{1 + \dfrac{K_1K_2(s+1)}{(s+2)}}\right) - \lim_{s\to 0}\left(\frac{\dfrac{K_2}{(s+2)}}{1 + \dfrac{K_1K_2(s+1)}{(s+2)}}\right) = \frac{2-K_2}{2+K_1K_2}$$

Sensitivity to K_1:

$$S_{e:K_1} = \frac{K_1}{e}\frac{\delta e}{\delta K_1} = -\frac{K_1K_2}{2+K_1K_2} = -\frac{(100)(0.1)}{2+(100)(0.1)} = -0.833$$

Sensitivity to K_2:

$$S_{e:K_2} = \frac{K_2}{e}\frac{\delta e}{\delta K_2} = \frac{2K_2(1+K_1)}{(K_2-2)(2+K_1K_2)} = \frac{2(0.1)(1+100)}{(0.1-2)(2+(100)(0.1))} = -0.89$$

46.

a. Using Eq. (7.89) with

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^3+20s^2+111s+164}\begin{bmatrix} s^2+15s+50 & -(4s+22) & -(2s+20) \\ -(3s+15) & s^2+10s+23 & 6 \\ -(s+13) & s+9 & s^2+15s+38 \end{bmatrix}$$

yields e(∞) = 1.09756 for a step input and e(∞) = ∞ for a ramp input. The same results are obtained using

$$\mathbf{A}^{-1} = -\frac{1}{164}\begin{pmatrix} 50 & -22 & -20 \\ -15 & 23 & 6 \\ -13 & 9 & 38 \end{pmatrix}$$

and Eq. (7.96) for a step input and Eq. (7.103) for a ramp input.

b. Using Eq. (7.89) with

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^3+9s^2+5s+7}\begin{bmatrix} s^2+9s & s & 7 \\ -(5s+7) & s^2 & 7s \\ -(s+9) & -1 & s^2+9s+5 \end{bmatrix}$$

yields e(∞) = 0 for a step input and $e(\infty) = \frac{5}{7}$ for a ramp input. The same results are obtained using

$$\mathbf{A}^{-1} = -\frac{1}{7}\begin{pmatrix} 0 & 0 & 7 \\ -7 & 0 & 0 \\ -9 & -1 & 5 \end{pmatrix}$$

and Eq. (12.123) for a step input and Eq. (12.130) for a ramp input.

c. Using Eq. (7.89) with

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^3+14s^2+43s+17}\begin{pmatrix} s^2+5s-4 & -5s-23 & -s+10 \\ s+11 & s^2+14s+42 & -2s-19 \\ -3s-2 & -2s-3 & s^2+9s+5 \end{pmatrix}$$

yields e(∞) = 6 for a step input and e(∞) = ∞ for a ramp input. The same results are obtained using

$$\mathbf{A}^{-1} = -\frac{1}{17}\begin{pmatrix} -4 & -23 & 10 \\ 11 & 42 & -19 \\ -2 & -3 & 5 \end{pmatrix}$$

and Eq. (7.96) for a step input and Eq. (7.103) for a ramp input.

47.

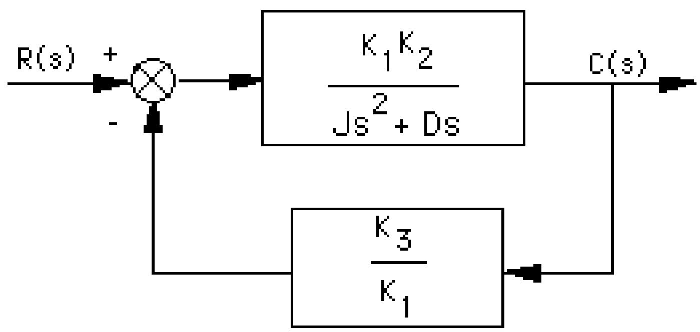
Find $G_4(s)$: Since 100 mi/hr = 146.67 ft/sec, the velocity response of $G_4(s)$ to a step displacement of the accelerator is $v(t) = 146.67(1 - e^{-\alpha t})$. Since 60 mi/hr = 88 ft/sec, the velocity equation at 10 seconds becomes $88 = 146.67\,(1 - e^{-\alpha 10})$. Solving for α yields $\alpha = 0.092$. Thus, $G_4(s) = \frac{K_1}{s+0.092}$. But, from the velocity equation, the dc value of $G_4(s)$ is $\frac{K_1}{0.092} = 146.67$. Solving for K_1, $G_4(s) = \frac{13.49}{s+0.092}$.

Find error: The forward transfer function of the velocity control loop is

$G_3(s)G_4(s) = \dfrac{13.49K}{s(s+1)(s+0.092)}$. Therefore, $K_v = \frac{13.49K}{0.092}$. $e(\infty) = \frac{1}{K_v} = 6.82 \times 10^{-3}K$.

48.

First, reduce the system to an equivalent unity feedback system. Push K_1 to the right past the summing junction.



Convert to a unity feedback system by adding a unity feedback path and subtracting unity from $\frac{K_1}{K_3}$. The equivalent forward transfer function is,

$$G_e(s) = \frac{\dfrac{K_1K_2}{Js^2+Ds}}{1 + \dfrac{K_1K_2}{Js^2+Ds}\left(\dfrac{K_3}{K_1} - 1\right)} = \frac{K_1K_2}{Js^2 + Ds + K_2(K_3 - K_1)}$$

The system is Type 0 with $K_p = \dfrac{K_1}{K_3 - K_1}$. Assuming the input concentration is R_o,

$e(\infty) = \dfrac{R_o}{1+K_p} = \dfrac{R_o(K_3 - K_1)}{K_3}$. The error can be reduced if $K_3 = K_1$.

49.

a. For the inner loop, $G_{1e}(s) = \dfrac{K\dfrac{(s+0.01)}{s^2}}{1+K\dfrac{(s+0.01)}{s^2}} = \dfrac{K(s+0.01)}{s^2+Ks+0.01K}$, where $K = \dfrac{K_c}{J}$.

Form $G_e(s) = G_{1e}(s)\dfrac{(s+0.01)}{s^2} = K\dfrac{(s+0.01)^2}{s^2(s^2+Ks+0.01K)}$.

System is Type 2. Therefore, $e_{step} = 0$,

b. $e_{ramp} = 0$,

c. $e_{parabola} = \dfrac{1}{K_a} = \dfrac{1}{0.01} = 100$

d. $T(s) = \dfrac{G_e(s)}{1+G_e(s)} = \dfrac{K(s+0.01)^2}{s^4+Ks^3+1.01Ks^2+0.02Ks+10^{-4}K}$

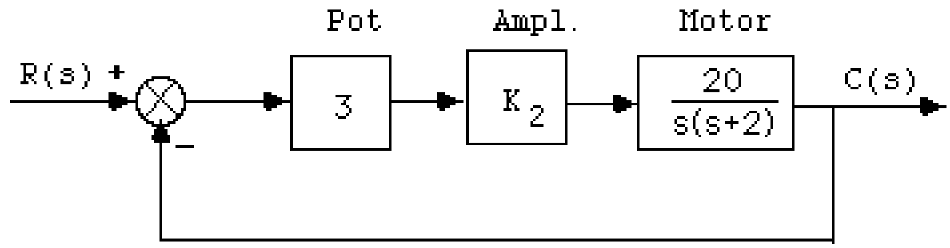
s^4	1	1.01K	10^{-4}K	
s^3	K	0.02K	0	0 < K
s^2	1.01K-0.02	10^{-4}K	0	0.0198 < K
s^1	$\dfrac{0.0201K^2-0.0004K}{1.01K-0.02}$	0	0	0.0199 < K
s^0	10^{-4}K			0 < K

Thus, for stability $K = \dfrac{K_c}{J} > 0.0199$

SOLUTIONS TO DESIGN PROBLEMS

50.

Pot gains: $K_1 = \dfrac{3\pi}{\pi} = 3$; Amplifier gain: K_2 ; Motor transfer function: Since time constant = 0.5, α = 2. Also, $\dfrac{K}{\alpha} = \dfrac{100}{10} = 10$. Hence, K = 20. The motor transfer function is now computed as $\dfrac{C(s)}{E_a(s)} = \dfrac{20}{s(s+2)}$. The following block diagram results after pushing the potentiometers to the right past the summing junction:



Finally, since $K_v = 10 = \frac{60K_2}{2}$, from which $K_2 = \frac{1}{3}$.

51.

First find K_V: Circumference = 2π nautical miles. Therefore, boat makes 1 revolution

in $\frac{2\pi}{20} = 0.314$ hr.

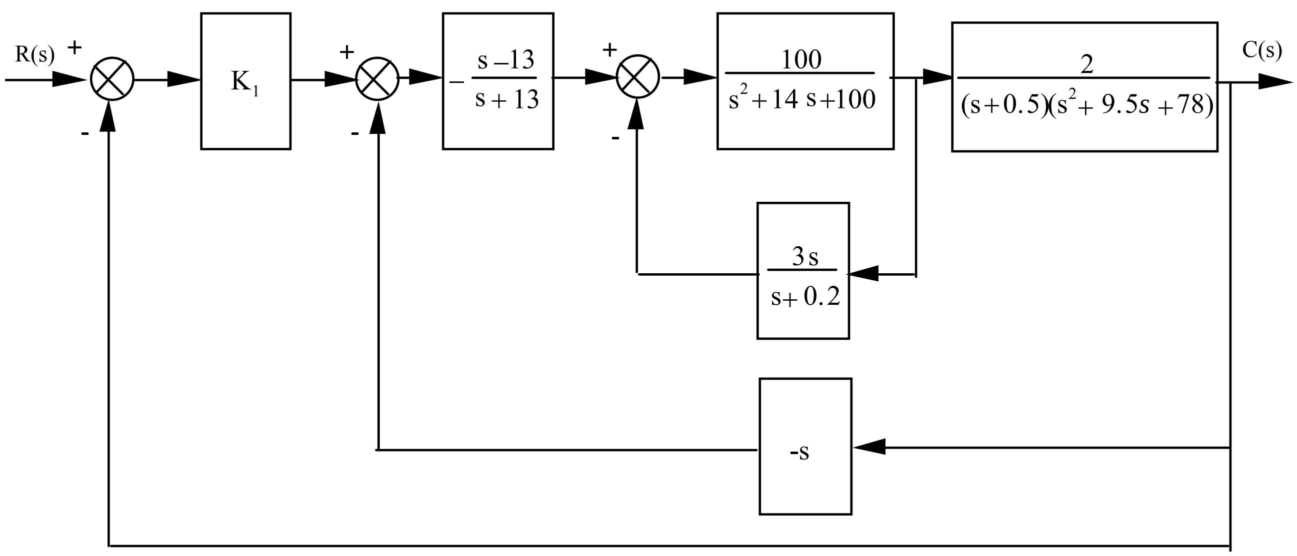
Angular velocity is thus, $\frac{1}{0.314}\frac{\text{rev}}{\text{hr}} = \frac{2\pi}{3600 \text{ x } 0.314}\frac{\text{rad}}{\text{sec}} = 5.56 \text{ x } 10^{-3}\frac{\text{rad}}{\text{sec}}$.

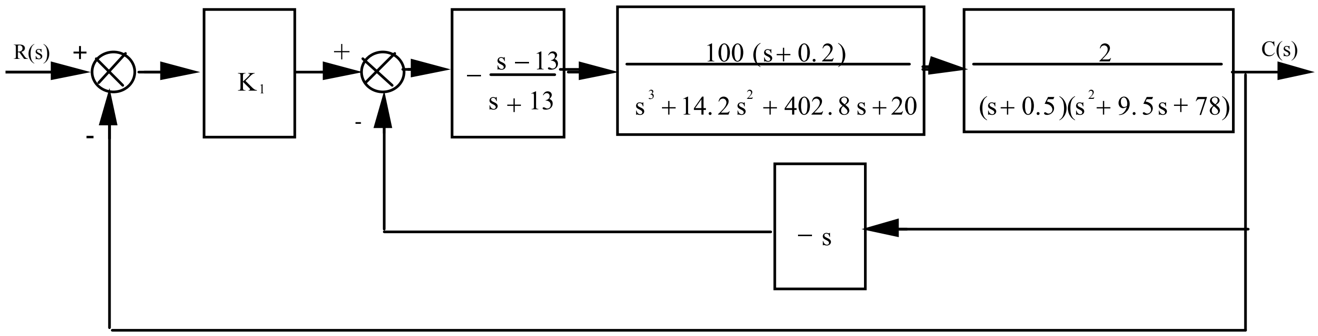
For 0.1^o error, $e(\infty) = \frac{1/10^o}{360^o} \text{ x } 2\pi \text{ rad} = \frac{5.56 \text{ x } 10^{-3}}{K_V}$. Thus $K_V = 3.19 = \frac{K}{4}$; from which,

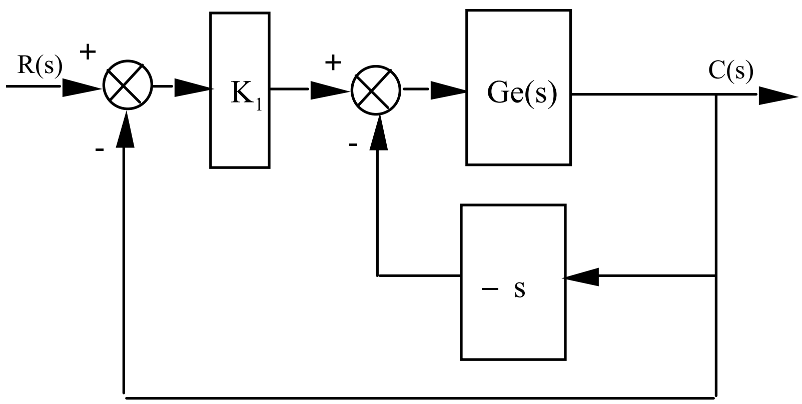
K = 12.76.

52.

a. Performing block diagram reduction:







$$G_e(s) = (-200)\frac{s^2 - 12.8s - 2.6}{s^7 + 37.2s^6 + 942.15s^5 + 13420s^4 + 1.0249\times10^5 s^3 + 4.6048\times10^5 s^2 + 2.2651\times10^5 s + 10140}$$

System is unity feedback with a forward transfer function, $G_t(s)$, where

$$G_t(s) = -200K_1 \frac{s^2 - 12.8s - 2.6}{s^7 + 37.2s^6 + 942.15s^5 + 13420s^4 + 1.0269\times10^5 s^3 + 4.5792\times10^5 s^2 + 2.2599\times10^5 s + 10140}$$

Thus, system is Type 0.

b. From $G_t(s)$, $K_p = \frac{520K_1}{10140} = 700$. Thus, $K_1 = 13650$.

c. $T(s) = \frac{G_t}{1+G_t}$

For $K_1 = 13650$,

$$T(s) = -2730000 \frac{s^2 - 12.8s - 2.6}{s^7 + 37.2s^6 + 942.15s^5 + 13420s^4 + 1.0269\times10^5 s^3 - 2.2721\times10^6 s^2 + 3.517\times10^7 s + 7108140}$$

Because of the negative coefficient in the denominator the system is unstable and the pilot would not be hired.

53.

The force error is the actuating signal. The equivalent forward-path transfer function is

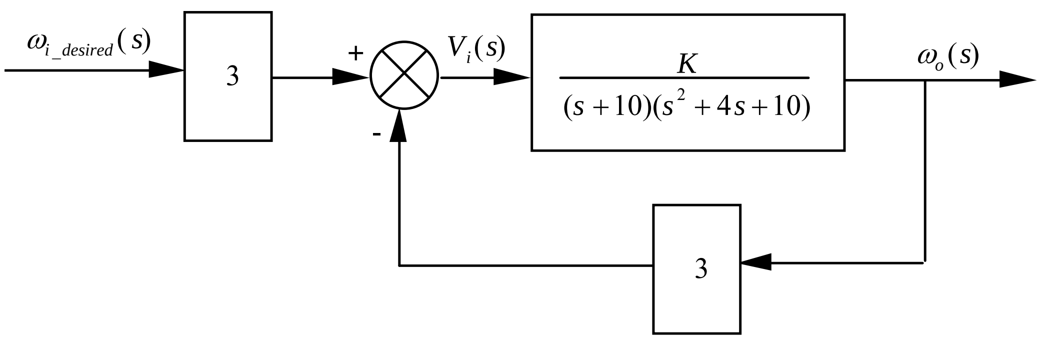
$G_e(s) = \dfrac{K_1}{s(s+K_1K_2)}$. The feedback is $H(s) = D_e s + K_e$. Using Eq. (7.72)

$E_a(s) = \dfrac{R(s)}{1+G_e(s)H(s)}$. Applying the final value theorem,

$e_{a_ramp}(\infty) = \lim\limits_{s\to 0} \dfrac{s\left(\dfrac{1}{s^2}\right)}{1+\dfrac{K_1(D_e s + K_e)}{s(s+K_1K_2)}} = \dfrac{K_2}{K_e} < 0.1$. Thus, $K_2 < 0.1K_e$. Since the closed-loop system is second-order with positive coefficients, the system is always stable.

54.

a. The minimum steady-state error occurs for a maximum setting of gain, *K*. The maximum *K* possible is determined by the maximum gain for stability. The block diagram for the system is shown below.



Pushing the input transducer to the right past the summing junction and finding the closed-loop transfer function, we get

$$T(s) = \frac{\dfrac{3K}{(s+10)(s^2+4s+10)}}{1+\dfrac{3K}{(s+10)(s^2+4s+10)}} = \frac{3K}{s^3+14s^2+50s+(3K+100)}$$

Forming a Routh table,

s^3	1	50
s^2	14	3*K*+100
s^1	$\dfrac{-3K+600}{14}$	0
s^0	3*K*+100	0

The s^1 row says $-\infty < K < 200$. The s^0 row says $-\frac{100}{3} < K$. Thus for stability,

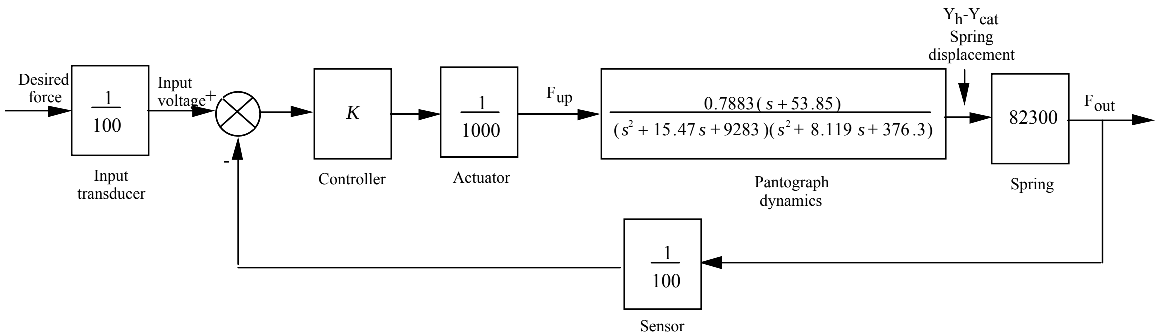
$-\frac{100}{3} < K < 200$. Hence, the maximum value of K is 200.

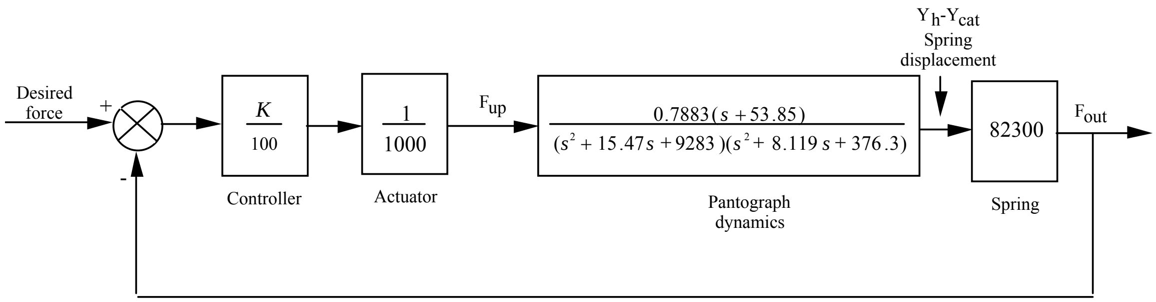
b. $K_p = \frac{3K}{100} = 6$. Hence, $e_{step}(\infty) = \frac{1}{1+K_p} = \frac{1}{7}$.

c. Step input

55.

a.





b.

$$G(s) = \frac{Y_h(s) - Y_{cat}(s)}{F_{up}(s)} = \frac{0.7883(s+53.85)}{(s^2+15.47s+9283)(s^2+8.119s+376.3)}$$

$$Ge(s) = (K/100)*(1/1000)*G(s)*82.3e3$$

$$G_e(s) = \frac{0.6488K\,(s+53.85)}{(s^2 + 8.119s + 376.3)\,(s^2 + 15.47s + 9283)}$$

$$K_p = 0.6488K*53.85/[(376.3)(9283)] = K*1.0002E\text{-}5$$

Maximum K minimizes the steady-state error. Maximum K possible is that which yields stability. From Chapter 6 maximum K for stability is $K = 1.88444 \times 10^5$. Therefore, $K_p = 1.8848$.

c. $e_{ss} = 1/(1+K_p) = 0.348$.

E I G H T

Root Locus Techniques

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Transient Design via Gain

a. From the Chapter 5 Case Study Challenge:

$$G(s) = \frac{76.39K}{s(s+150)(s+1.32)}$$

Since $T_s = 8$ seconds, we search along $-\frac{1}{2}$, the real part of poles with this settling time, for 180^o.

We find the point to be - 0.5+j6.9 with 76.39K = 7194.23, or K = 94.18. Second-order approximation is OK since third pole is much more than 5 times further from the imaginary axis than the dominant second-order pair.

b.

Program:

```
numg= 1;
deng=poly([0 -150 -1.32]);
'G(s)'
G=tf(numg,deng)
rlocus(G)
axis([-2,0,-10,10]);
title(['Root Locus'])
grid on
[K1,p]=rlocfind(G)
K=K1/76.39
```

Computer response:

```
ans =

G(s)


Transfer function:
           1
-----------------------
s^3 + 151.3 s^2 + 198 s

Select a point in the graphics window

selected_point =

  -0.5034 + 6.3325i
```

```
K1 =

  6.0690e+003


p =

  1.0e+002 *

  -1.5027
  -0.0052 + 0.0633i
  -0.0052 - 0.0633i


K =

   79.4469

>>
ans =

G(s)


Transfer function:
           1
-----------------------
s^3 + 151.3 s^2 + 198 s

Select a point in the graphics window

selected_point =

  -0.5000 + 6.2269i


K1 =

  5.8707e+003

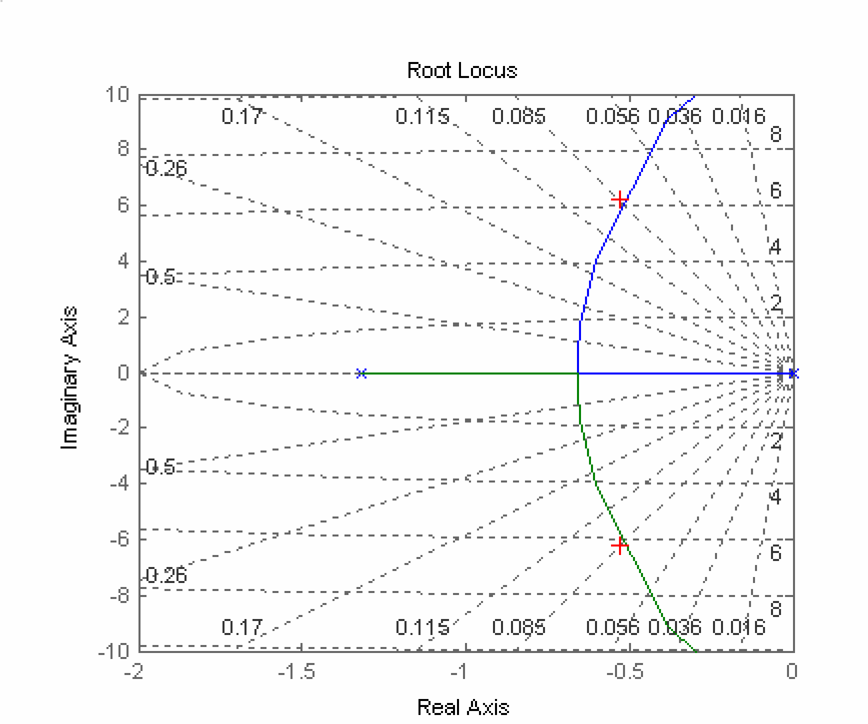

p =

  1.0e+002 *

  -1.5026
  -0.0053 + 0.0623i
  -0.0053 - 0.0623i


K =

   76.8521
```



UFSS Vehicle: Transient Design via Gain

a. Push $-K_1$ to the right past the summing junction yielding $G(s) = \frac{K(s+0.437)}{s(s+2)(s+1.29)(s+0.193)}$, where $K = 0.25K_1$. Combine the parallel feedback paths and obtain $H(s) = (s+1)$. Hence, $G(s)H(s) = \frac{K(s+0.437)(s+1)}{s(s+2)(s+1.29)(s+0.193)}$. The root locus is shown below in (b). Searching the 10% overshoot line ($\zeta = 0.591$; $\theta = 126.24^o$), we find the operating point to be $-1.07 \pm j1.46$ where $K = 3.389$, or $K_1 = 13.556$.

b.
Program:
```
numg= [1 0.437];
deng=poly([0 -2 -1.29 -0.193]);
G=tf(numg,deng);
numh=[1 1];
denh=1;
H=tf(numh,denh);
GH=G*H;
rlocus(GH)
pos=(10);
z=-log(pos/100)/sqrt(pi^2+[log(pos/100)]^2);
sgrid(z,0)
title(['Root Locus with ' , num2str(pos), ' Percent Overshoot Line'])
[K,p]=rlocfind(GH);
pause
K1=K/0.25
T=feedback(K*G,H)
T=minreal(T)
step(T)
```

```
title(['Step Response for Design of ' , num2str(pos), ' Percent'])
```

Computer response:

```
Select a point in the graphics window

selected_point =

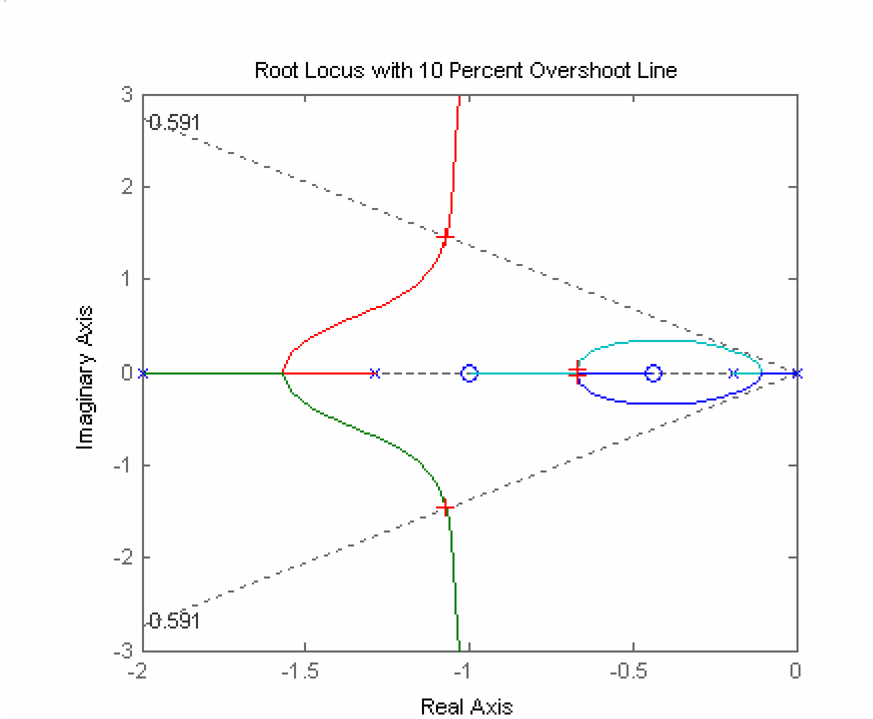
  -1.0704 + 1.4565i

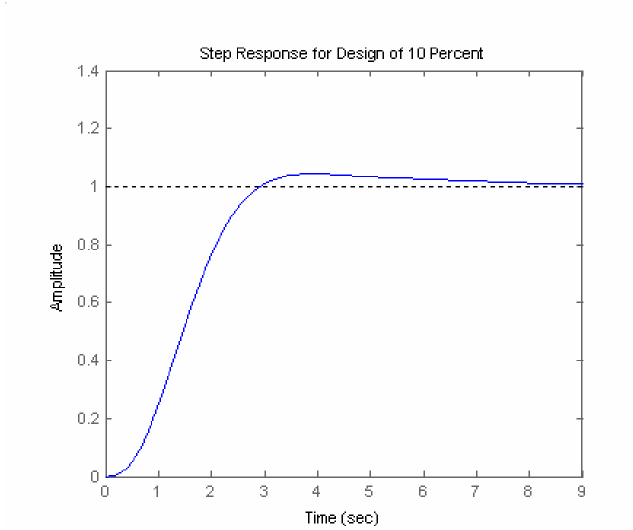

K1 =

   13.5093


Transfer function:
               3.377 s + 1.476
---------------------------------------------
s^4 + 3.483 s^3 + 6.592 s^2 + 5.351 s + 1.476


Transfer function:
               3.377 s + 1.476
---------------------------------------------
s^4 + 3.483 s^3 + 6.592 s^2 + 5.351 s + 1.476
```





ANSWERS TO REVIEW QUESTIONS

1. The plot of a system's closed-loop poles as a function of gain

2. (1) Finding the closed-loop transfer function, substituting a range of gains into the denominator, and factoring the denominator for each value of gain. (2) Search on the s-plane for points that yield 180 degrees when using the open-loop poles and zeros.

3. K = 1/5

4. No

5. At the zeros of G(s) and the poles of H(s)

6. (1) Apply Routh-Hurwitz to the closed-loop transfer function's denominator. (2) Search along the imaginary axis for 180 degrees.

7. If any branch of the root locus is in the rhp, the system is unstable.

8.If the branch of the root locus is vertical, the settling time remains constant for that range of gain on the vertical section.

9. If the root locus is circular with origin at the center

10. Determine if there are any break-in or breakaway points

11. (1) Poles must be at least five times further from the imaginary axis than the dominant second order pair, (2) Zeros must be nearly canceled by higher order poles.

12. Number of branches, symmetry, starting and ending points

13. The zeros of the open loop system help determine the root locus. The root locus ends at the zeros. Thus, the zeros are the closed-loop poles for high gain.

SOLUTIONS TO PROBLEMS

1.

a. No: Not symmetric; On real axis to left of an even number of poles and zeros

b. No: On real axis to left of an even number of poles and zeros

c. No: On real axis to left of an even number of poles and zeros

d. Yes

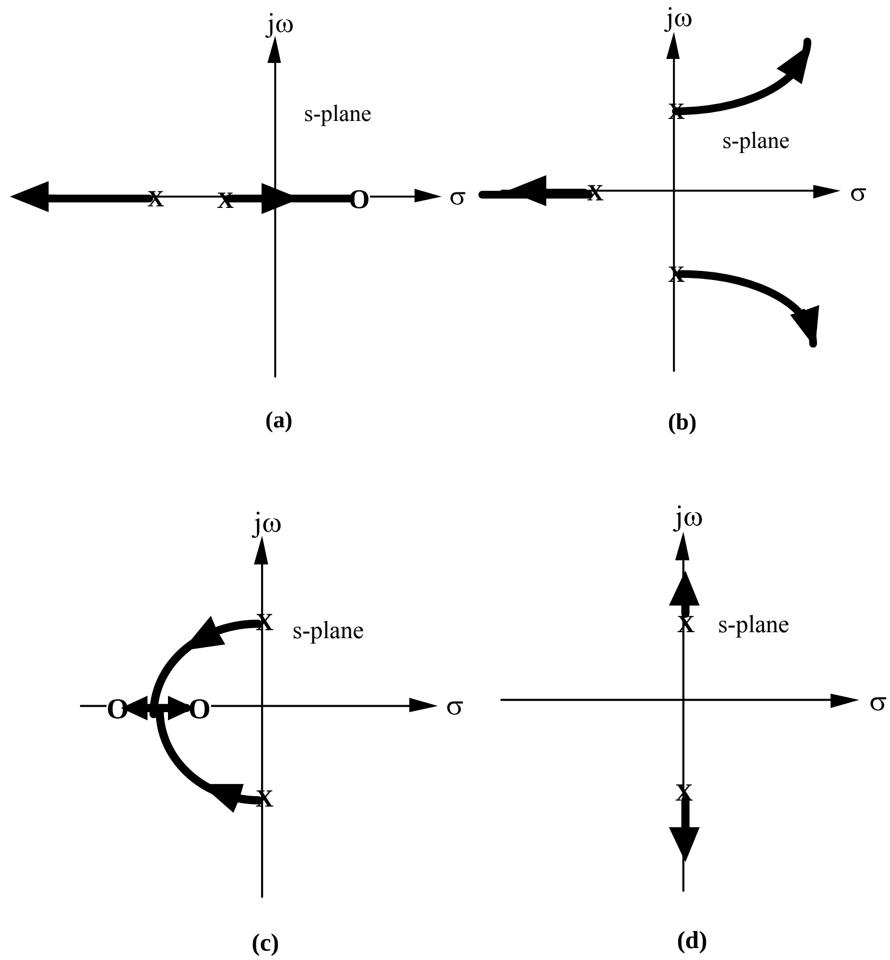
e. No: Not symmetric; Not on real axis to left of odd number of poles and/or zeros

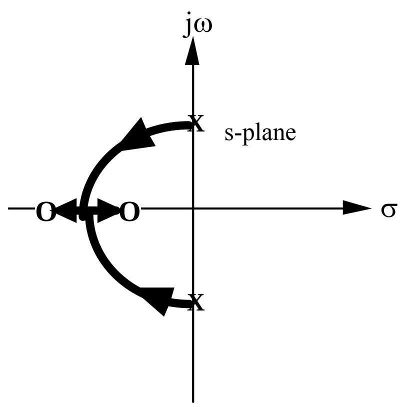
f. Yes

g. No: Not symmetric; real axis segment is not to the left of an odd number of poles

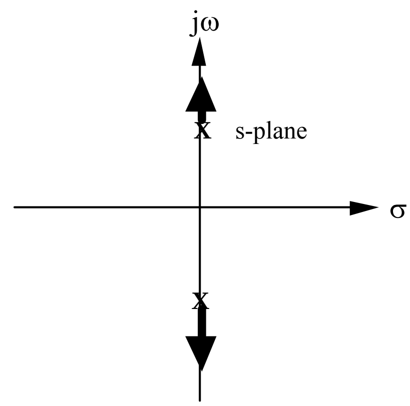
h. Yes

2.

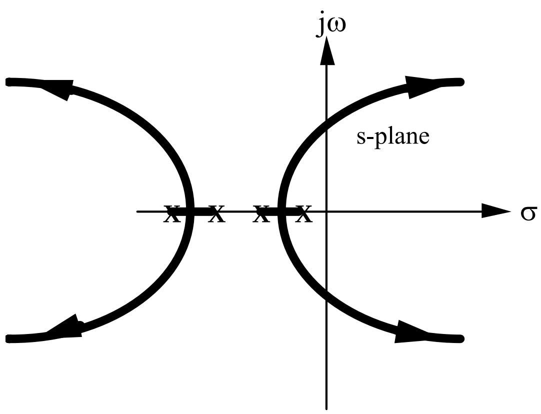



jω
s-plane
σ

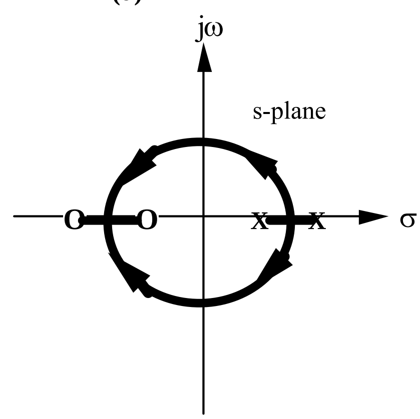
(c)


jω
s-plane
σ

(d)

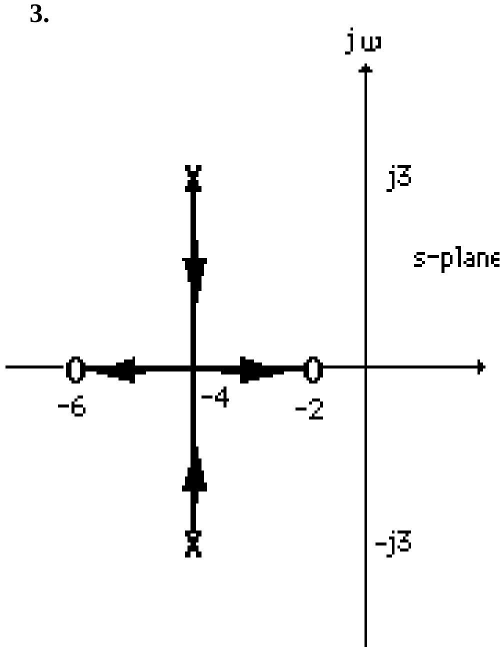

jω
s-plane
σ

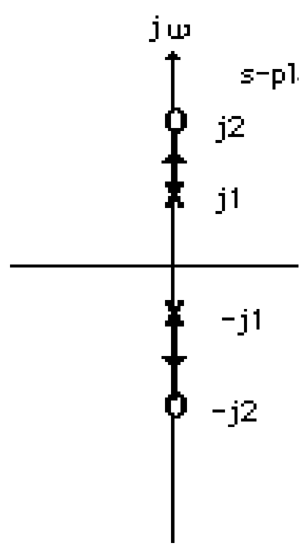
(e)

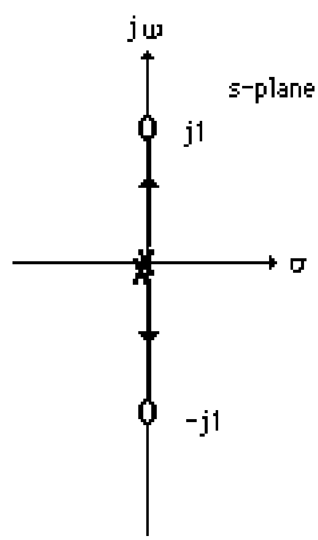

jω
s-plane
σ

(f)

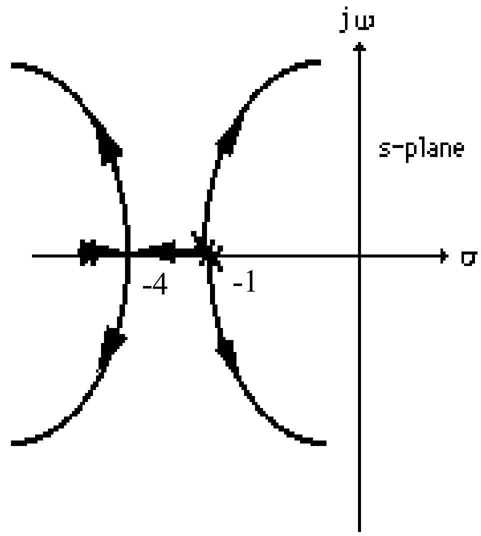
3.


jω
j3
s-plane
-6
-4
-2
-j3


jω
j2
j1
-j1
-j2

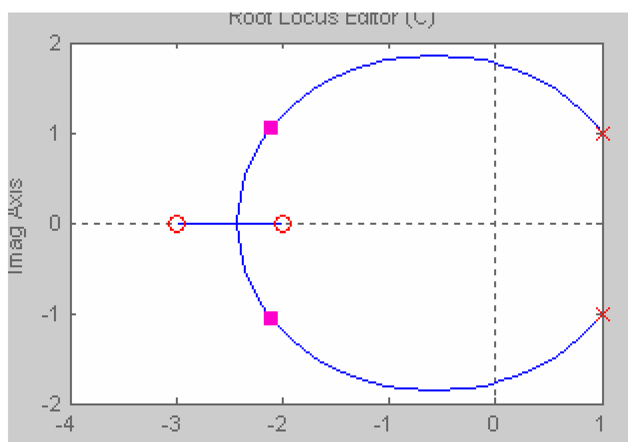

jω
s-plane
j1
σ
-j1

a. **b.** **c.**



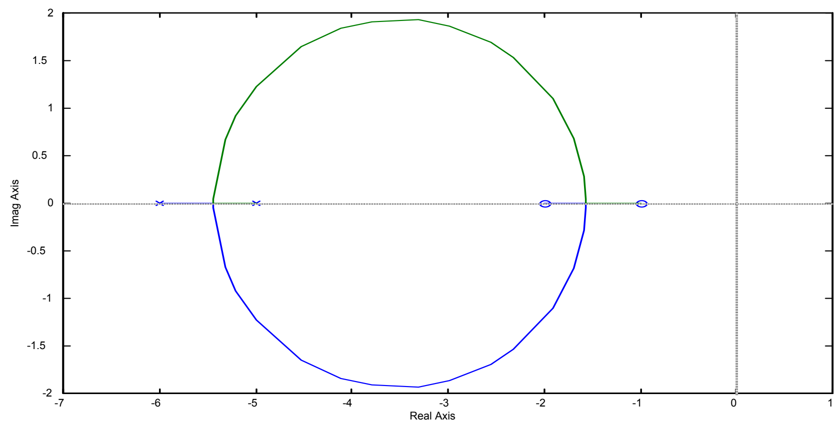
d.

4.



Breakaway: σ = -2.43 for K = 52.1

5.



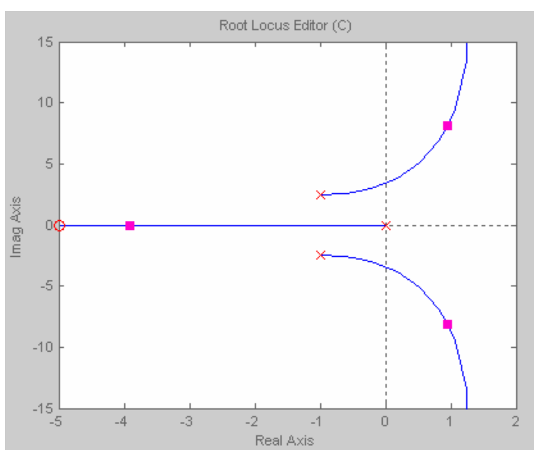
Break-in: σ = -1.5608 for K = 61.986; Breakaway: σ = -5.437 for K = 1.613.

6.

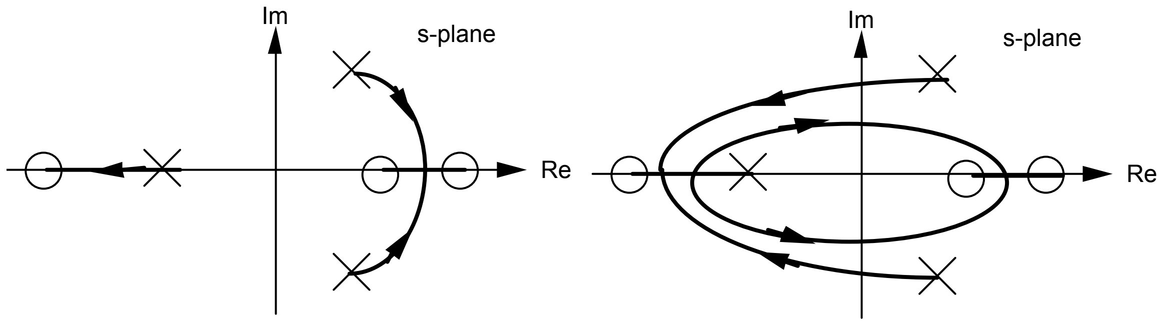
Convert the denominator to the following form: $D(s) = 1 + \frac{20K(s+5)}{s^3 + 2s^2 + 7s}$ and thus identify

$$G(s) = \frac{20K(s+5)}{s^3 + 2s^2 + 7s} = \frac{20K(s+5)}{s(s^2 + 2s + 7)}.$$

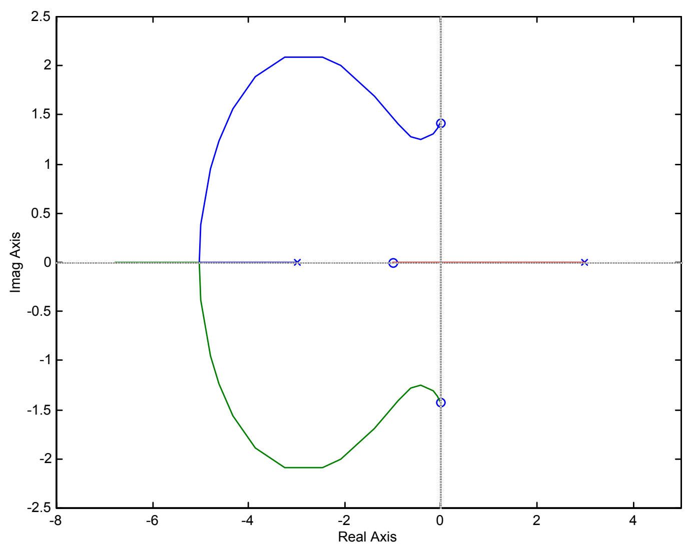
Plotting the root locus yields



7.



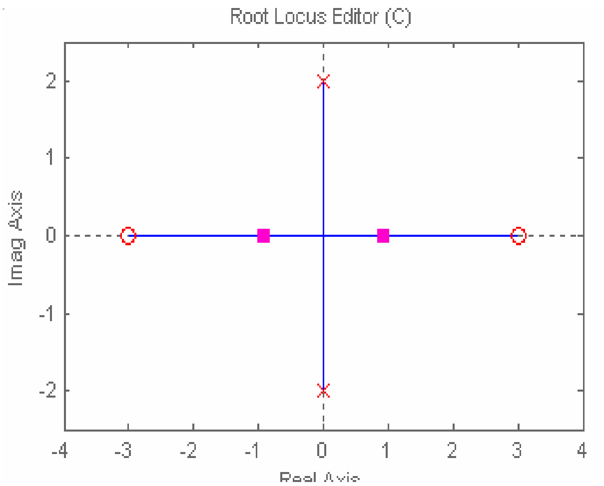
8.



Closed-loop poles will be in the left-half-plane when rhp pole reaches the origin,

or $K > \dfrac{(3)(3)}{\left(\sqrt{2}\right)\left(\sqrt{2}\right)(1)} = \dfrac{9}{2}$.

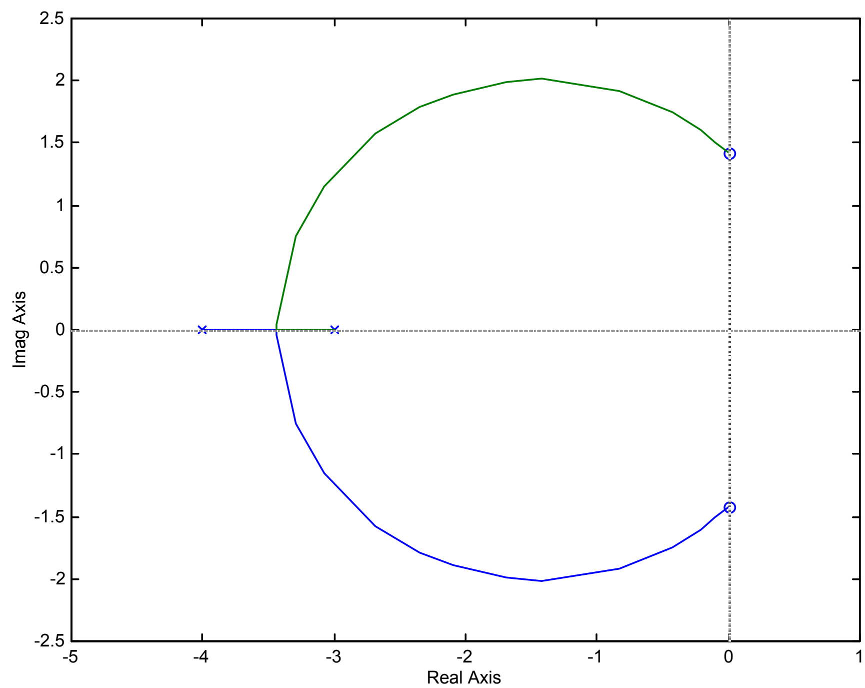
9.



Closed-loop poles will be in the right-half-plane for $K > \frac{(2)(2)}{(3)(3)} = \frac{4}{9}$ (gain at the origin).

Therefore, stable for $K < 4/9$; unstable for $K > 4/9$.

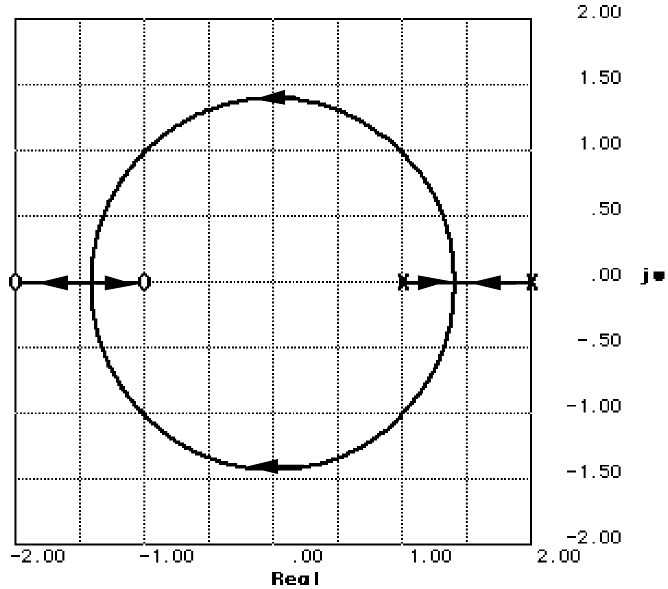
10.



Breakaway: σ = -3.436 for K = 1.781. System is never unstable. System is marginally stable for K = ∞.

11.

System 1:



(a)

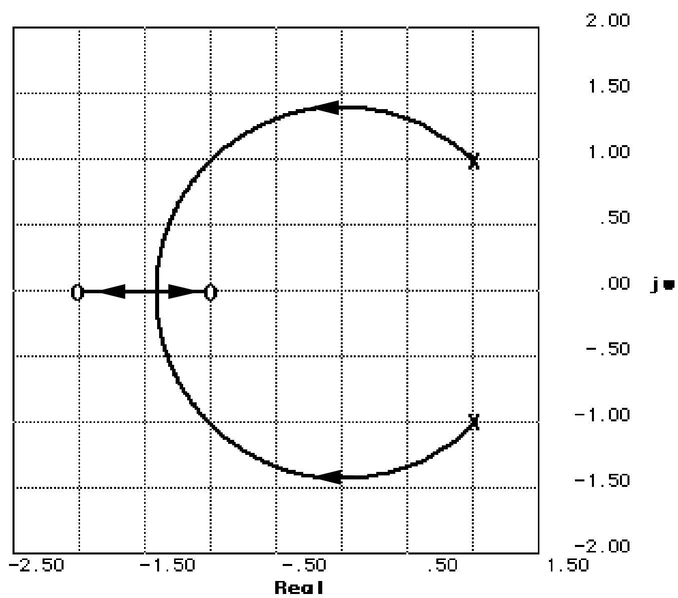
a. Breakaway: σ = 1.41 for K = 0.03; Break-in: σ = -1.41 for K = 33.97.

b. Imaginary axis crossing at j1.41 for K = 1. Thus stable for K > 1.

c. At break-in point, poles are multiple. Thus, K = 33.97.

d. Searching along 135^0 line for 180^0, K = 5 at $1.414 \angle 135^0$.

System 2:



(b)

a. Break-in: σ = -1.41 for K = 28.14.

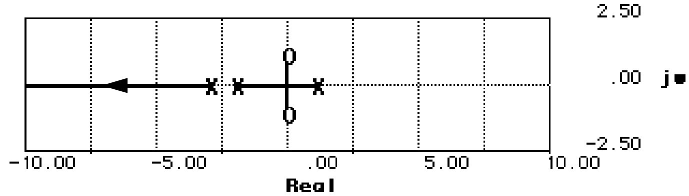
b. Imaginary axis crossing at j1.41 for K = 0.67. Thus stable for K > 0.67.

c. At break-in point, poles are multiple. Thus, K = 28.14.

d. Searching along 135^0 line for 180^0, K = 4 at 1.414 ∠ 135^0.

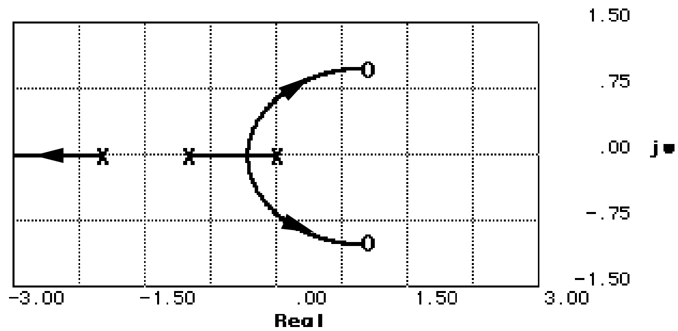
12.

a.



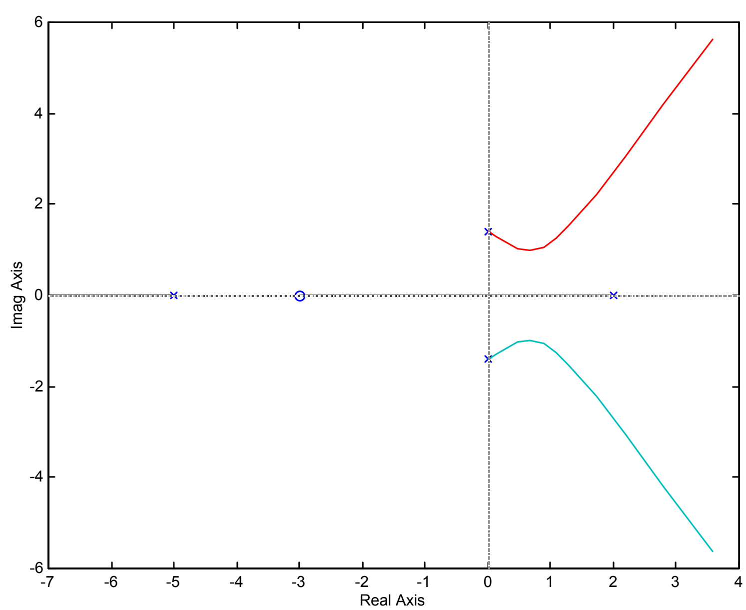
Root locus crosses the imaginary axis at the origin for K = 6. Thus the system is stable for K > 6.

b.



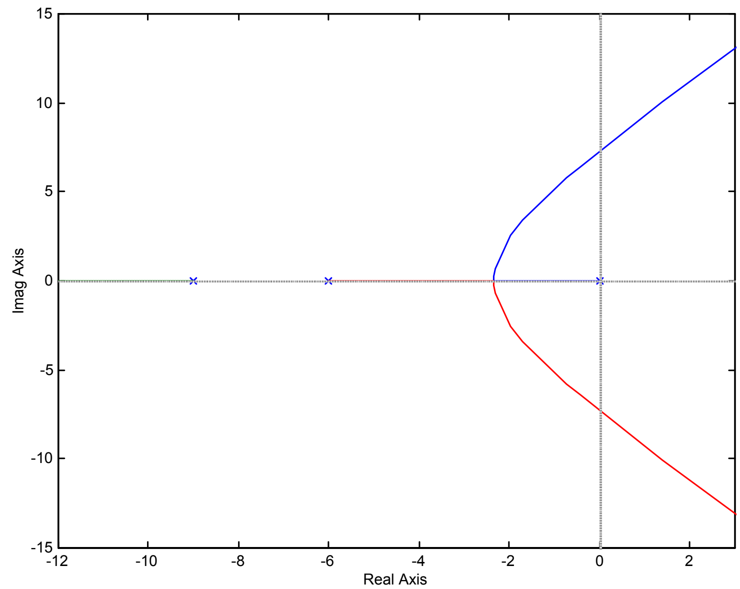
Root locus crosses the imaginary axis at j0.65 for K = 0.79. Thus, the system is stable for K < 0.79.

13.



There will be only two right-half-plane poles when pole at +2 moves into the left-half-plane at the origin. Thus $K = \frac{(5)(\sqrt{2})(\sqrt{2})(2)}{3} = 6.67$.

14.



Root locus crosses the imaginary axis at j7.348 with a gain of 810. Real axis breakaway is at –2.333 at a gain of 57.04. Real axis intercept for the asymptotes is $\frac{-15}{3} = -5$. The angle of the asymptotes is $= \frac{\pi}{3}$, π, $\frac{5\pi}{3}$. Some other points on the root locus are:

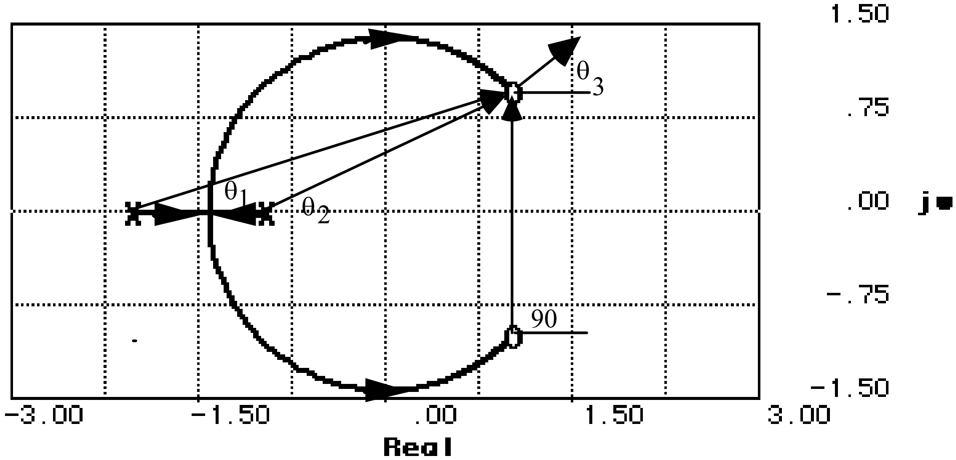
$\zeta = 0.4$: -1.606 + j3.68, K = 190.1

$\zeta = 0.6$: -1.956 + j2.6075, K = 117.8

$\zeta = 0.8$: -2.189 + j1.642, K = 79.55

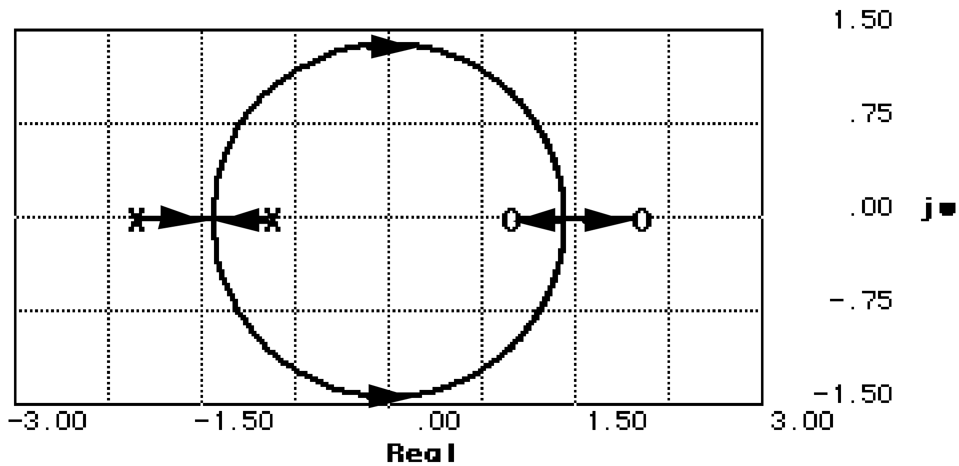
15.

a.



Imaginary axis crossing: j1.41 at K = 1.5. Stability: K < 1.5. Breakaway: -1.41 at K = 0.04. Points on root locus: -1.5 ± j0, K = 0.0345; -0.75 ± j1.199, K = 0.429; 0 ± j1.4142, K = 1.5; 0.75 ± j1.1989, K = 9. Finding angle of arrival: $90 - \theta_1 - \theta_2 + \theta_3 = 90^o - \tan^{-1}(1/3) - \tan^{-1}(1/2) + \theta_3 = 180^o$. Thus, $\theta_3 = 135^o$.

b.



Imaginary axis crossing: j1.41 at K = 1. Stability: K < 1. Breakaway: -1.41 at K = 0.03. Break-in: 1.41 at K = 33.97. Points on root locus: -1.5 ± j0, K = 0.02857; -0.75 ± j1.199, K = 0.33; 0 ± j1.4142, K = 1; 0.75 ± j1.1989, K = 3.

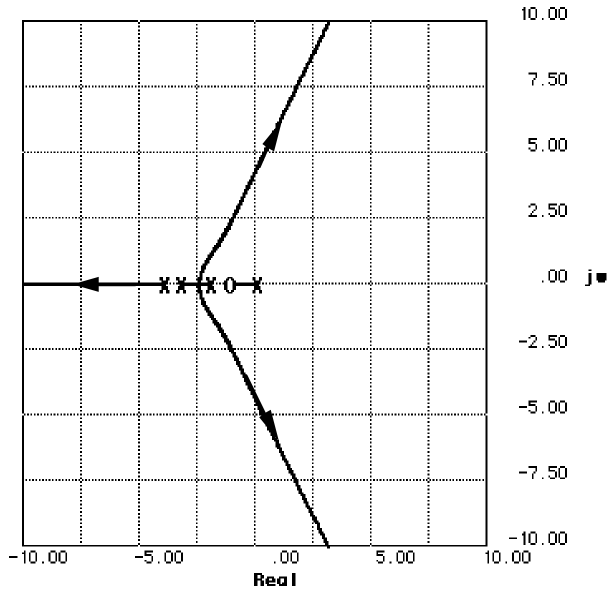
16.

a. Root locus crosses the imaginary axis at $\pm j3.162$ at $K = 52$.

b. Since the gain is the product of pole lengths to -5, $K = (1)\left(\sqrt{4^2+1^2}\right)\left(\sqrt{4^2+1^2}\right) = 17$.

17.

a.



b. $\sigma_a = \frac{(0 - 2 - 3 - 4) - (-1)}{3} = -\frac{8}{3}$; Angle $= \frac{(2k+1)\pi}{3} = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

c. Root locus crosses imaginary axis at j4.28 with K = 140.8.

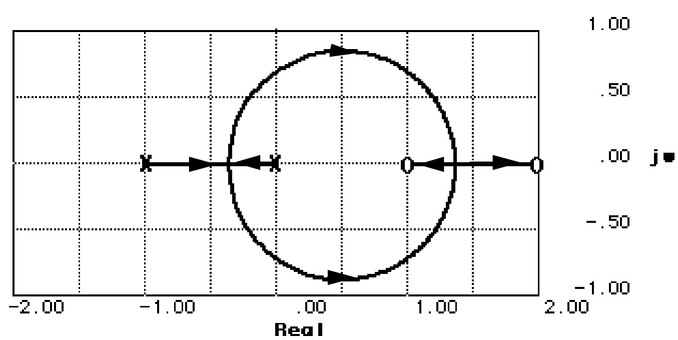
d. K = 13.125

18.

Assume that root locus is epsilon away from the asymptotes. Thus, $\sigma_a = \frac{(0 - 3 - 6) - (-\alpha)}{2} \approx -1$;

Angle $= \frac{(2k+1)\pi}{2} = \frac{\pi}{2}, \frac{3\pi}{2}$. Hence $\alpha = 7$. Checking assumption at $-1 \pm j100$ yields -180^o with K = 9997.02.

19.



a. Breakaway: -0.37 for K = 0.07. Break-in: 1.37 for K = 13.93

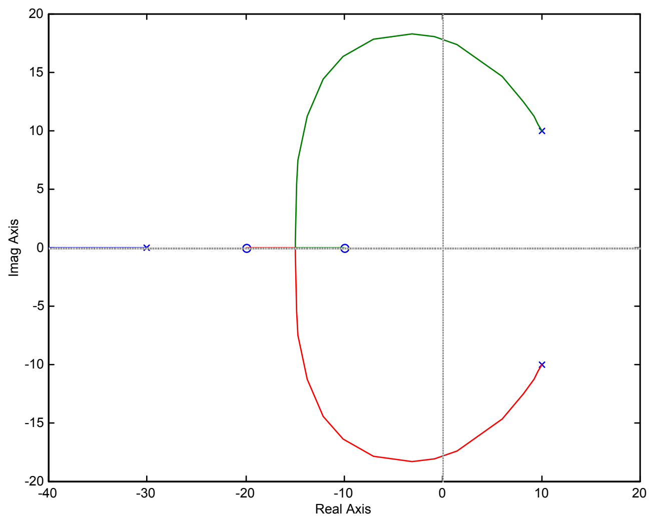
b. Imaginary axis crossing: ±j0.71 for K = 0.33

c. System stable for K < 0.33

d. Searching 120° find point on root locus at $0.5\angle 120^{\circ}$ = - 0.25 ± j0.433 for K = 0.1429

20.

a.

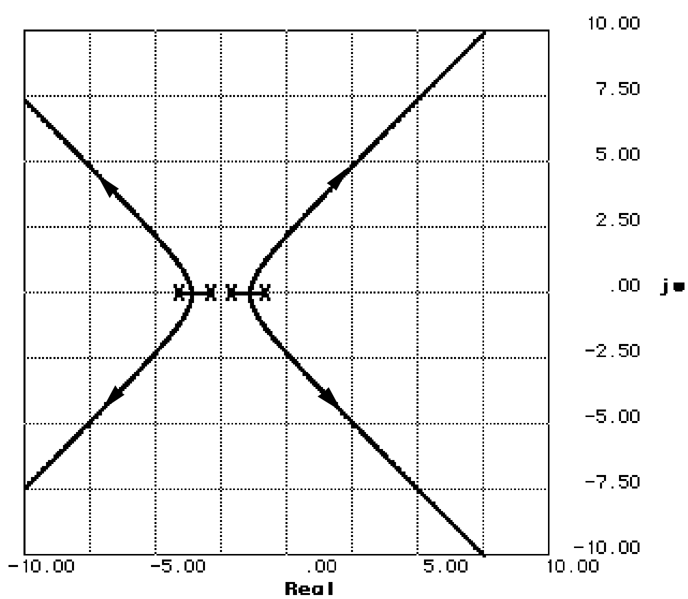


b. $0 < K < 23.93$

c. $K = 81.83$ @ -13.04 ± j13.04

d. At the break-in point, $s = -14.965$, $K = 434.98$.

21.



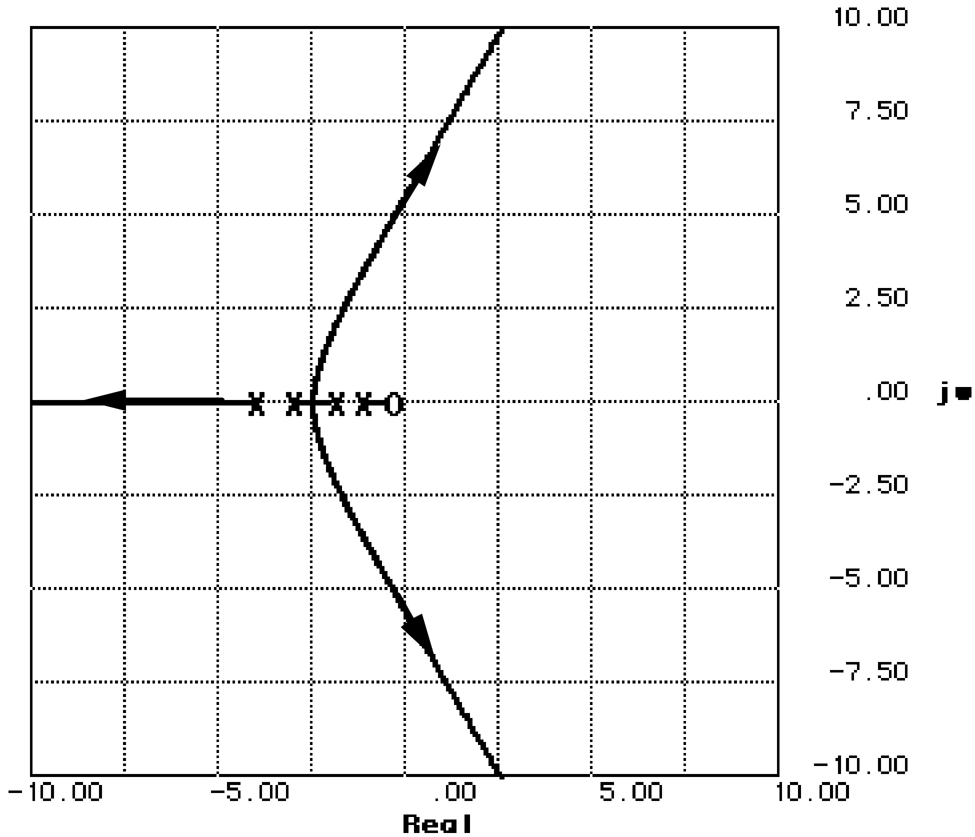
a. Asymptotes: $\sigma_{int} = \frac{(-1\ -2\ -3\ -4) - (0)}{4} = -\frac{5}{2}$; Angle $= \frac{(2k+1)\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

b. Breakaway: -1.38 for K = 1 and -3.62 for K = 1

c. Root locus crosses the imaginary axis at ±j2.24 for K = 126. Thus, stability for K < 126.

d. Search 0.7 damping ratio line (134.427 degrees) for 180^0. Point is $1.4171\angle 134.427^o$ =
- 0.992 ± j1.012 for K = 10.32.

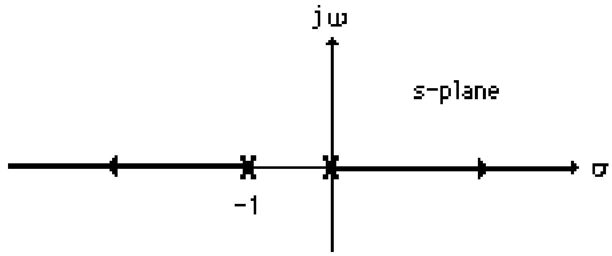
e. Without the zero, the angles to the point ±j5.5 add up to -265.074^o. Therefore the contribution of the zero must be 265.074 - 180 = 85.074^o. Hence, $\tan 85.074^o = \frac{5.5}{z_c}$, where $-z_c$ is the location of the zero. Thus, $z_c = 0.474$.



f. After adding the zero, the root locus crosses the imaginary axis at ±j5.5 for K = 252.5. Thus, the system is stable for K < 252.5.

g. The new root locus crosses the 0.7 damping ratio line at $2.7318\angle 134.427^o$ for K = 11.075 compared to $1.4171\angle 134.427^o$ for K = 10.32 for the old root locus. Thus, the new system's settling time is shorter, but with the same percent overshoot.

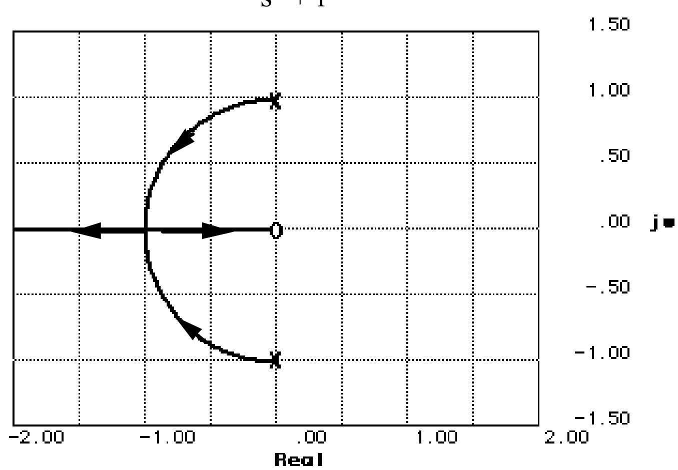
22.



23.

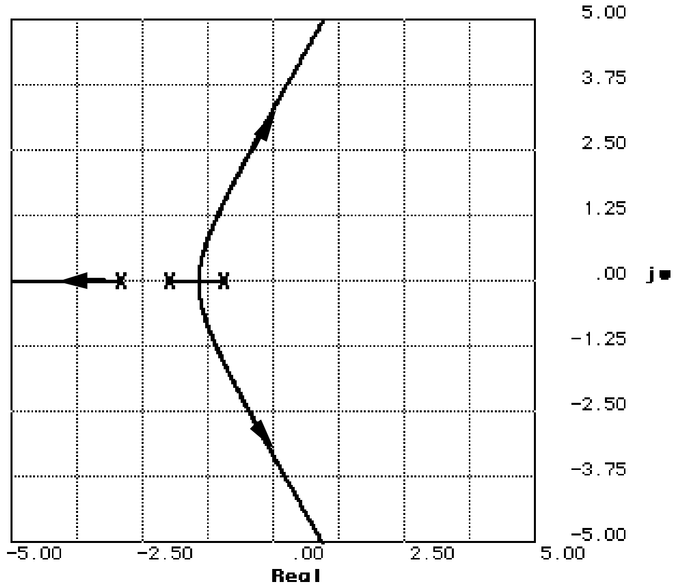
$$T(s) = \frac{1}{s^2 + \alpha s + 1} = \frac{\frac{1}{s^2+1}}{1 + \frac{\alpha s}{s^2+1}}$$. Thus an equivalent system has $G(s) = \frac{1}{s^2+1}$ and $H(s) = \alpha s$.

Plotting a root locus for $G(s)H(s) = \frac{\alpha s}{s^2+1}$, we obtain,



24.

a.



b. Root locus crosses 20% overshoot line at $1.8994 \angle 117.126^{o} = -0.866 \pm j1.69$ for $K = 9.398$.

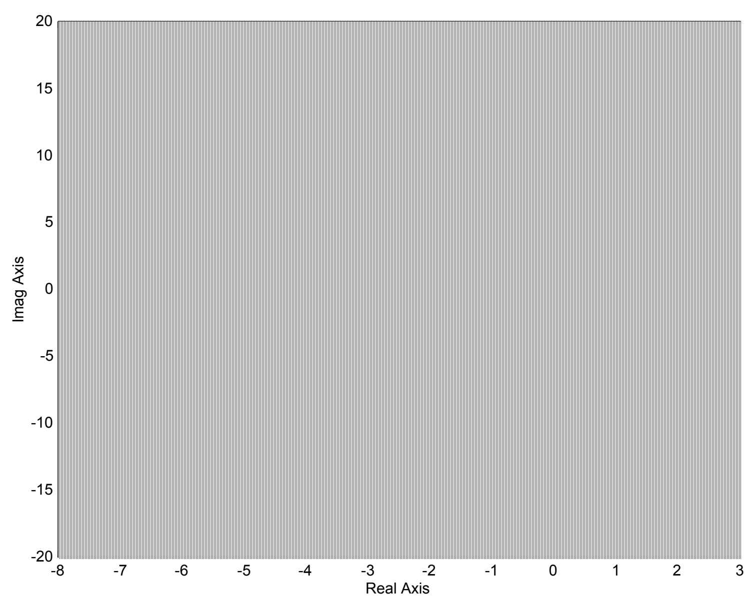
c. $T_s = \dfrac{4}{0.866} = 4.62$ seconds; $T_p = \dfrac{\pi}{1.69} = 1.859$ seconds

d. Root locus crosses imaginary axis at $\pm j3.32$ for $K = 60$. Therefore stability for $K < 60$.

e. Other poles with same gain as dominant poles: $\sigma = -4.27$

25.

a.



b.

$$\sigma_a = \frac{(-6-5-4-2)-(2)}{4-2} = -9.5$$

$$\theta_a = \frac{(2k+1)\pi}{4-2} = \frac{\pi}{2}, \frac{3\pi}{2}$$

c. At the $j\omega$ axis crossing, $K = 115.6$. Thus for stability, $0 < K < 115.6$.

d. Breakaway points at $\sigma = -2.524$ @ $K = 0.496$ and $\sigma = -5.576$ @ $K = 0.031$.

e. For 25% overshoot, Eq. (4.39) yields $\zeta = 0.404$. Searching along this damping ratio line, we find the 180^0 point at $-0.6608 + j1.496$ where $K = 35.98$.

f. $-7.839 \pm j7.425$

g. Second-order approximation not valid because of the existence of closed-loop zeros in the rhp.

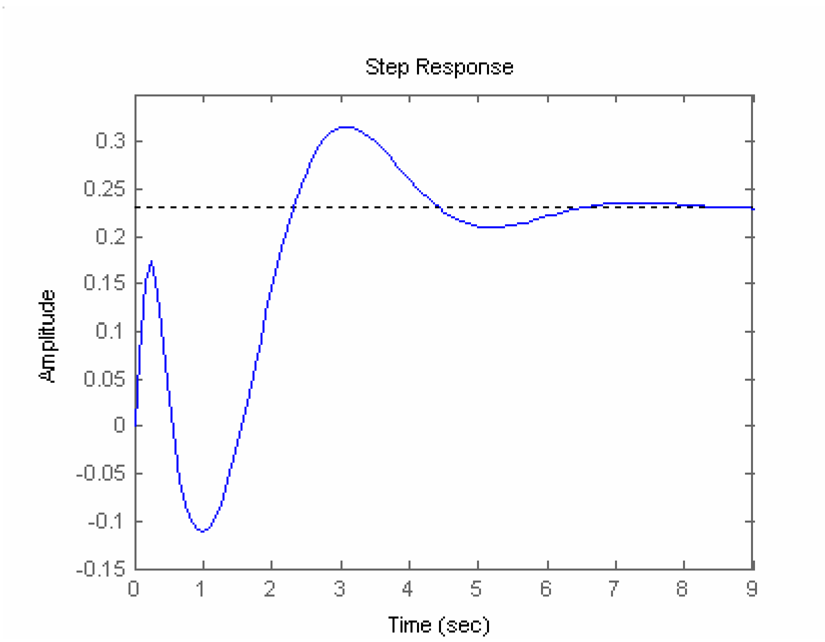
h.

Program:

```
numg=35.98*[1 -2 2];
deng=poly([-2 -4 -5 -6]);
G=tf(numg,deng);
T=feedback(G,1)
step(T)
```

Computer response:

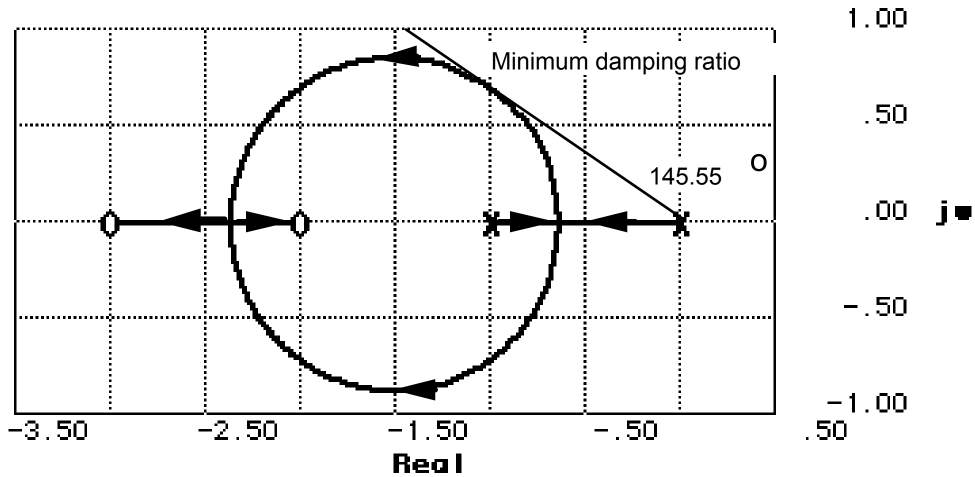
```
Transfer function:
    35.98 s^2 - 71.96 s + 71.96
------------------------------------
s^4 + 17 s^3 + 140 s^2 + 196 s + 312
```



Simulation shows over 30% overshoot and nonminimum-phase behavior. Second-order approximation not valid.

26.

a. Draw root locus and minimum damping ratio line.



Minimum damping ratio is ζ = cos (180 - 145.55) = cos 34.45^o = 0.825. Coordinates at tangent point of ζ = 0.825 line with the root locus is approximately –1 + j0.686. The gain at this point is 0.32.

b. Percent overshoot for ζ = 0.825 is 1.019%.

c. $T_s = \frac{4}{1}$ = 4 seconds; $T_p = \frac{\pi}{0.6875}$ = 4.57 seconds

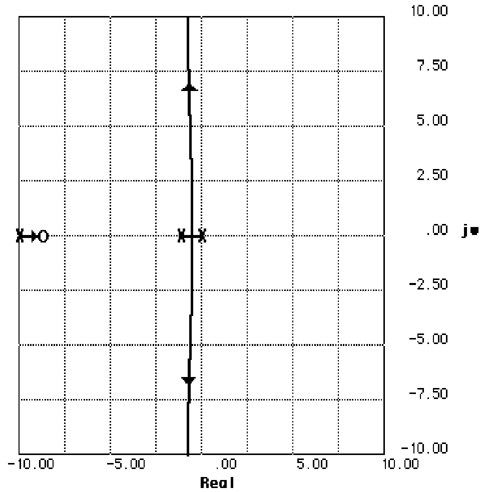
d. Second-order approximation is not valid because of the two zeros and no pole-zero cancellation.

27.

The root locus intersects the 0.55 damping ratio line at –7.217 + j10.959 with K = 134.8. A justification of a second-order approximation is not required. The problem stated the requirements in terms of damping ratio and not percent overshoot, settling time, or peak time. A second-order approximation is required to draw the equivalency between percent overshoot, settling time, and peak time and damping ratio and natural frequency.

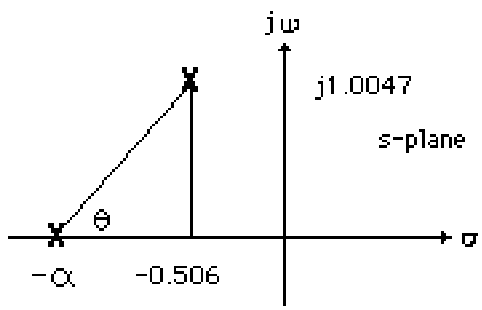
28.

Since the problem stated the settling time at large values of K, assume that the root locus is approximately close to the vertical asymptotes. Hence, $\sigma_{int} = \frac{-11+\alpha}{2} = -\frac{4}{T_s}$. Since T_s is given as 4 seconds, σ_{int} = -1 and $\alpha = 9$. The root locus is shown below.



29.

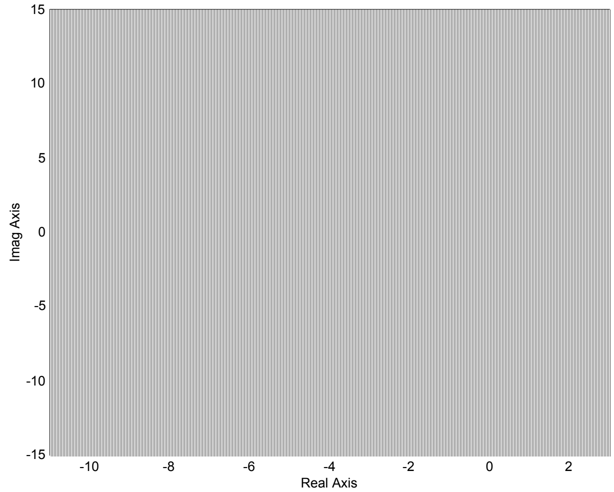
The design point is $-0.506 \pm j1.0047$. Excluding the pole at $-\alpha$, the sum of angles to the design point is -141.37^o. Thus, the contribution of the pole at $-\alpha$ is $141.37 - 180 = -38.63^o$. The following geometry applies:



Hence, $\tan\theta = \dfrac{1.0047}{\alpha - 0.506} = \tan 38.63 = 0.799$. Thus $\alpha = 1.763$. Adding this pole at -1.763 yields 180^o at $-0.506 \pm j1.0047$ with $K = 7.987$.

30.

a.



b. Searching along the 10% overshoot line (angle = 126.239°), the point - 0.7989 + *j*1.0898 yields 180° for K = 81.74.

c. Higher-order poles are located at approximately –6.318 and –7.084. Since these poles are more than 5 times further from the imaginary axis than the dominant pole found in (b), the second-order approximation is valid.

d. Searching along the imaginary axis yields 180° at *j*2.53, with K = 394.2.

Hence, for stability, $0 < K < 394.2$.

31.

Program:

```
pos=10;
z=-log(pos/100)/sqrt(pi^2+[log(pos/100)]^2)
numg=1;
deng=poly([0 -3 -4 -8]);
G=tf(numg,deng)
Gzpk=zpk(G)
rlocus(G,0:1:100)
pause
axis([-2,0,-2,2])
sgrid(z,0)
pause
[K,P]=rlocfind(G)
T=feedback(K*G,1)
pause
step(T)
```

Computer response:

```
z =

    0.5912


Transfer function:
             1
```

```
-----------------------------
s^4 + 15 s^3 + 68 s^2 + 96 s


Zero/pole/gain:
         1
-------------------
s (s+8) (s+4) (s+3)

Select a point in the graphics window

selected_point =

  -0.7994 + 1.0802i

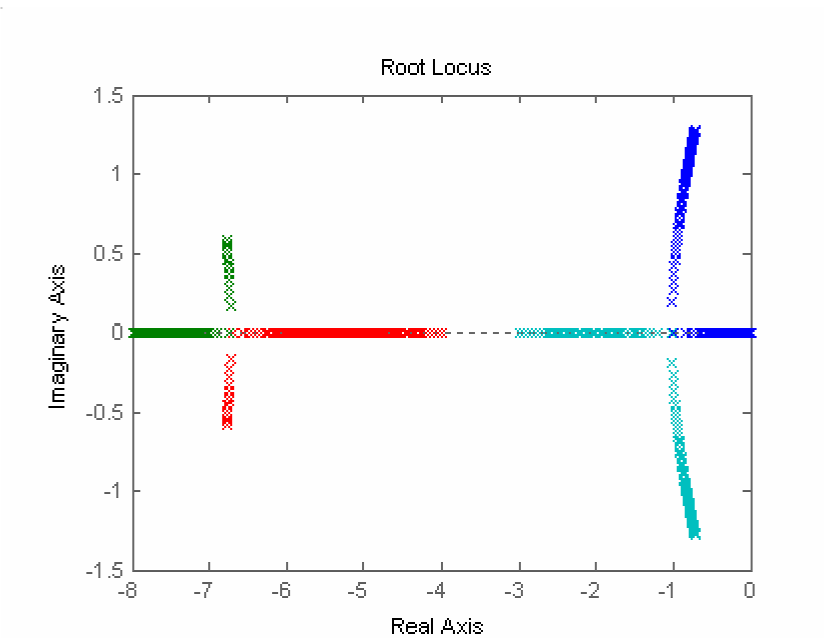

K =

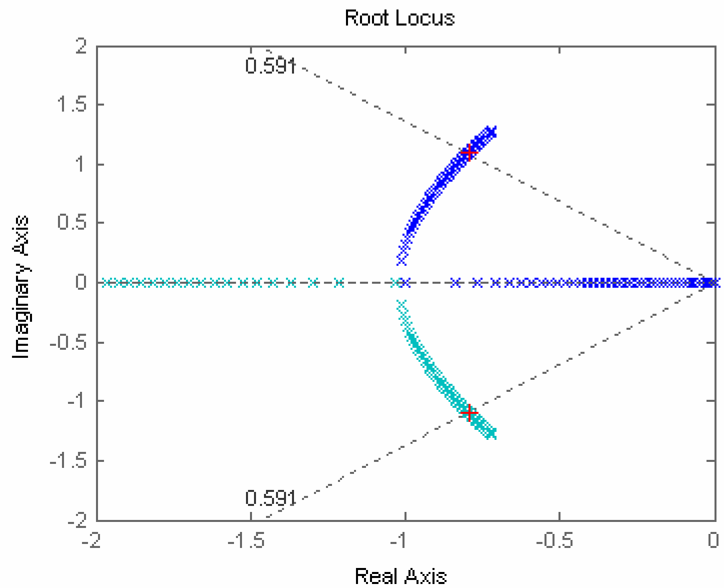
   81.0240

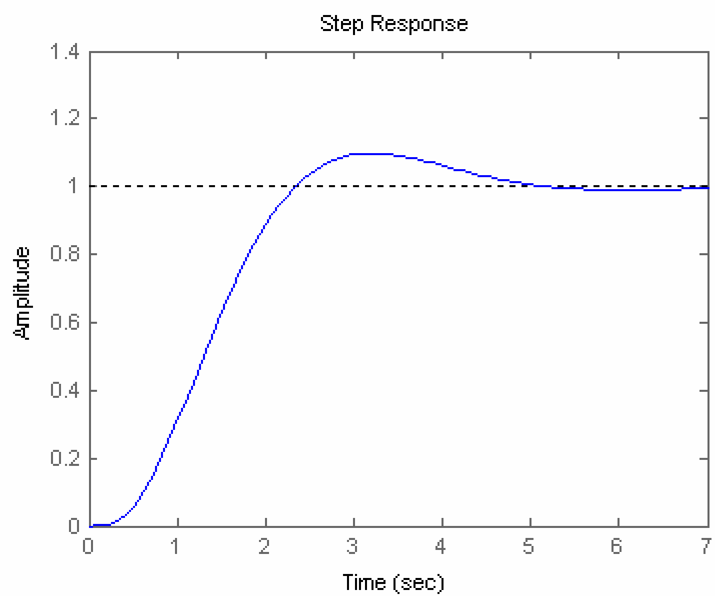

P =

  -7.1058
  -6.2895
  -0.8023 + 1.0813i
  -0.8023 - 1.0813i


Transfer function:
               81.02
------------------------------------
s^4 + 15 s^3 + 68 s^2 + 96 s + 81.02
```

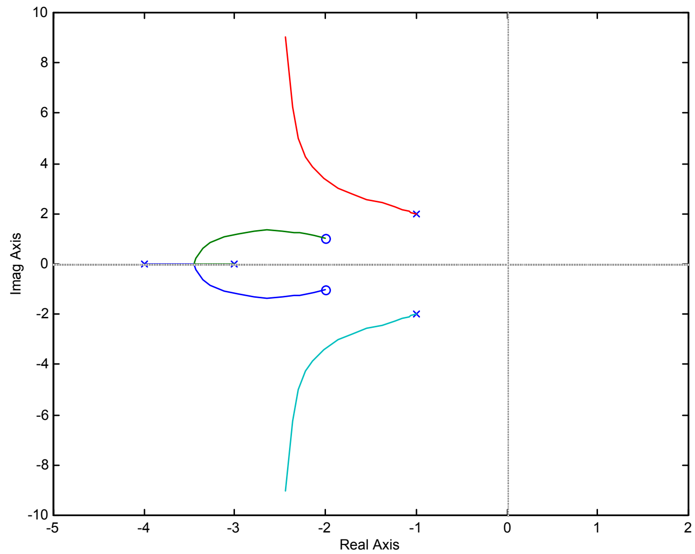






32.

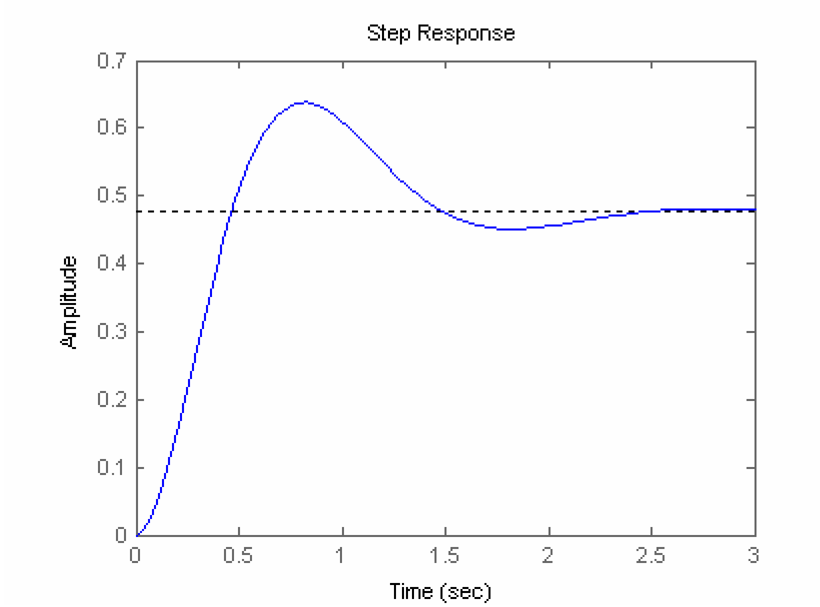
a. For a peak time of 1s, search along the horizontal line, Im = $\pi/ T_p = \pi$, to find the point of intersection with the root locus. The intersection occurs at $-2 \pm j\pi$ at a gain of 11.



b.

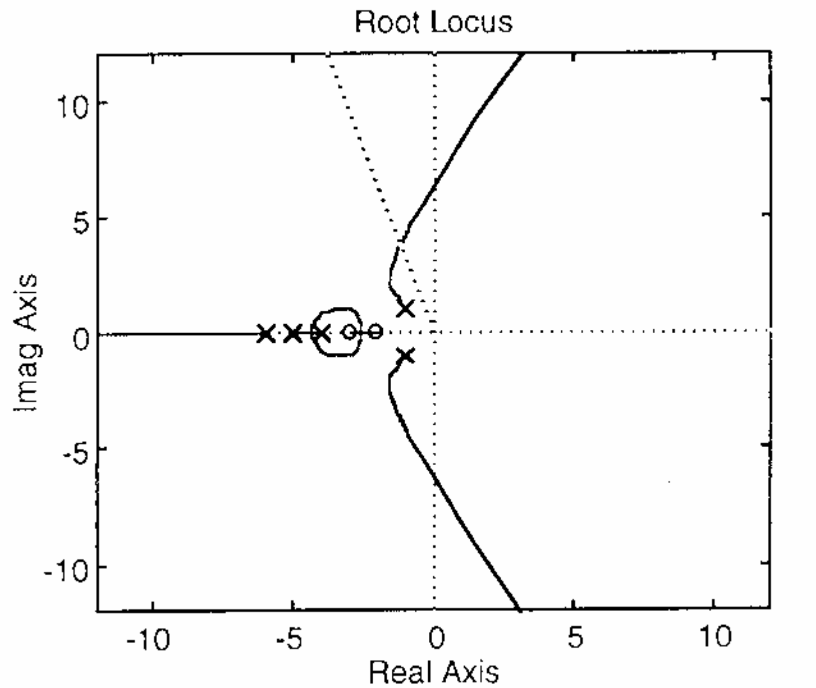
Program:

```
numg=11*[1 4 5];
deng=conv([1 2 5],poly([-3 -4]));
G=tf(numg,deng);
T=feedback(G,1);
step(T)
```



33.

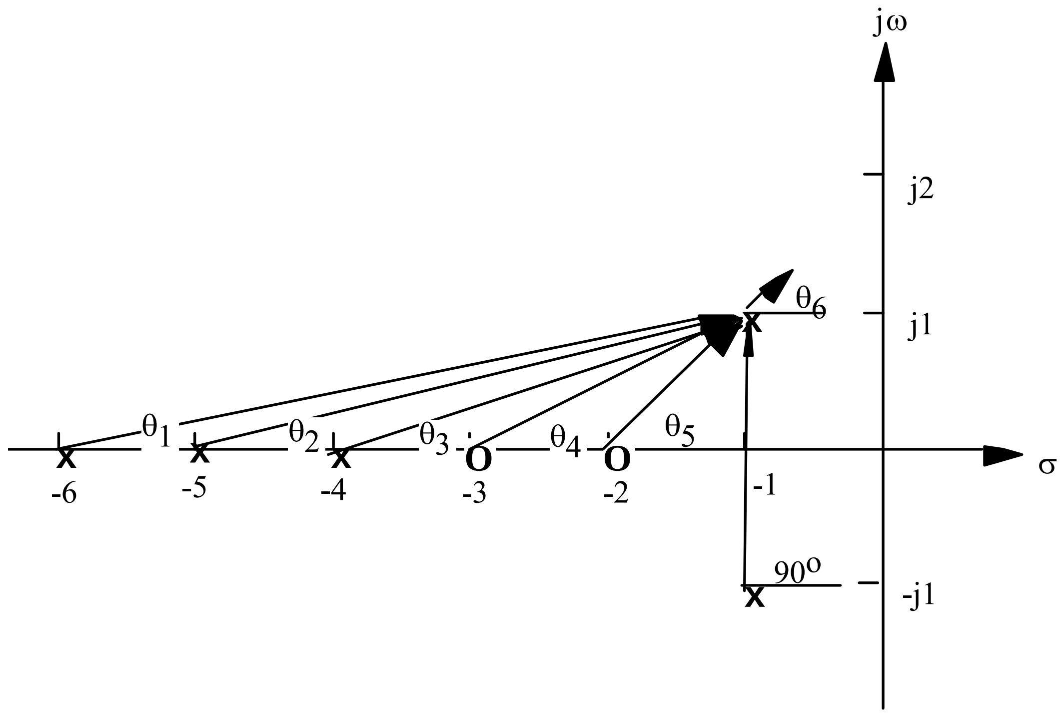
a.



b. Searching the jω axis for 180°, we locate the point j6.29 at a gain of 447.83.

c. Searching for maximum gain between -4 and -5 yields the breakaway point, -4.36. Searching for minimum gain between -2 and -3 yields the break-in point, -2.56.

d.



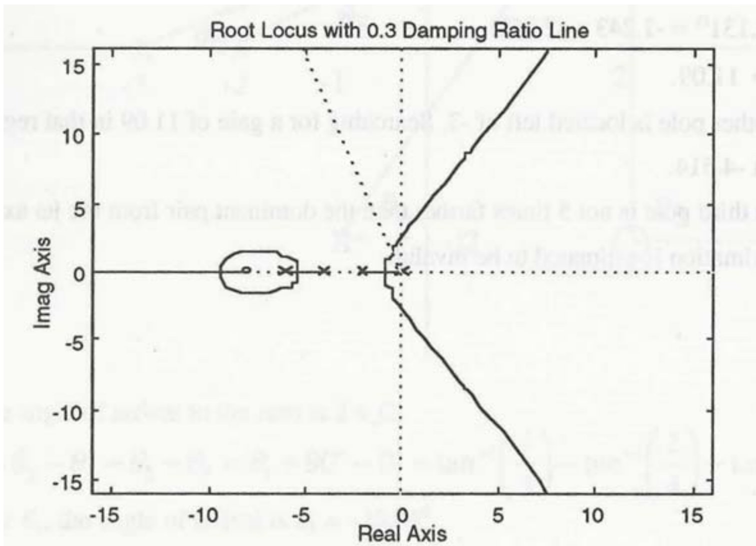
To find the angle of departure from the poles at -1±j1: $-\theta_1 - \theta_2 - \theta_3 + \theta_4 + \theta_5 - \theta_6 - 90^0$

$= -\tan^{-1}(1/5) - \tan^{-1}(1/4) - \tan^{-1}(1/3) + \tan^{-1}(1/2) + \tan^{-1}(1/1) - \theta_6 - 90^o = 180^0$. Thus, $\theta_6 = -242.22^o$

e. Searching along the $\zeta = 0.3$ line ($\theta = 180 - \cos^{-1}(\zeta) = 107.458^o$) for 180^o we locate the point $3.96 \angle 107.458^o = -1.188 \pm j3.777$. The gain is 127.133.

34.

a.

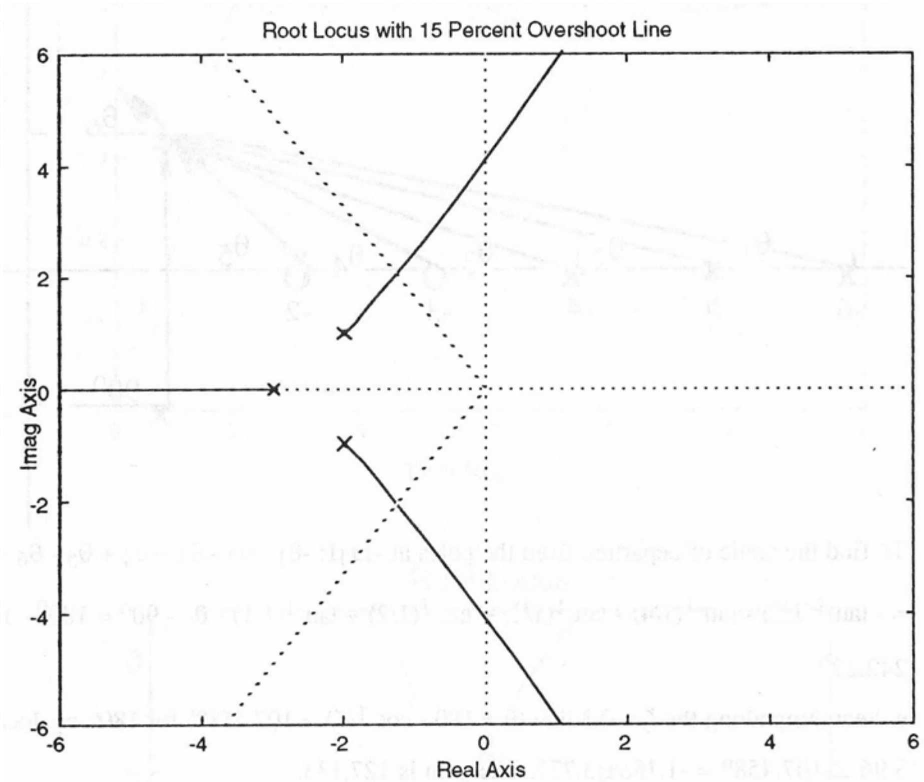


b. Searching the jω axis for 180^o, we locate the point j2.56 at a gain of 30.686.

c. Searching for maximum gain between 0 and -2 yields the breakaway point, -0.823. Searching for maximum gain between -4 and -6 yields the breakaway point, -5.37. Searching for minimum gain beyond -8 yields the break-in point, -9.39.

e. Searching along the $\zeta = 0.3$ line ($\theta = 180 - \cos^{-1}(\zeta) = 107.458^o$) for 180^o we locate the point $1.6 \angle 107.458^o = -0.48 \pm j1.53$. The gain is 9.866.

35.



a. Searching the 15% overshoot line ($\zeta = 0.517$; $\theta = 121.131^{o}$) for 180^{o}, we find the point 2.404 $\angle\, 121.131^{o} = -1.243 + j2.058$.

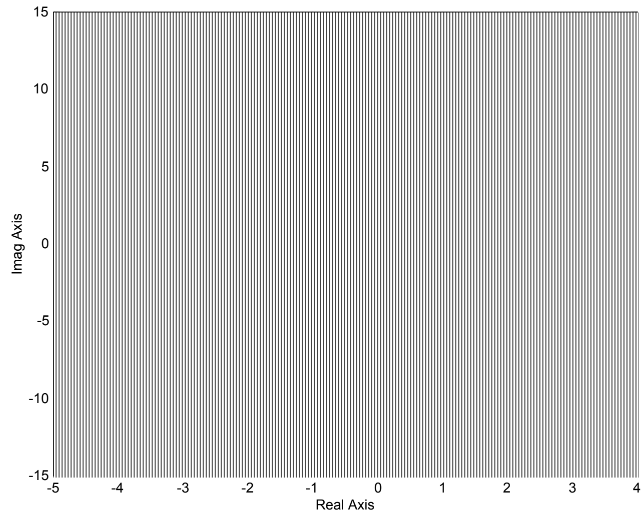
b. K = 11.09.

c. Another pole is located left of -3. Searching for a gain of 11.09 in that region, we find the third pole at -4.514.

d. The third pole is not 5 times farther than the dominant pair from the jω axis. the second-order approximation is estimated to be invalid.

36.

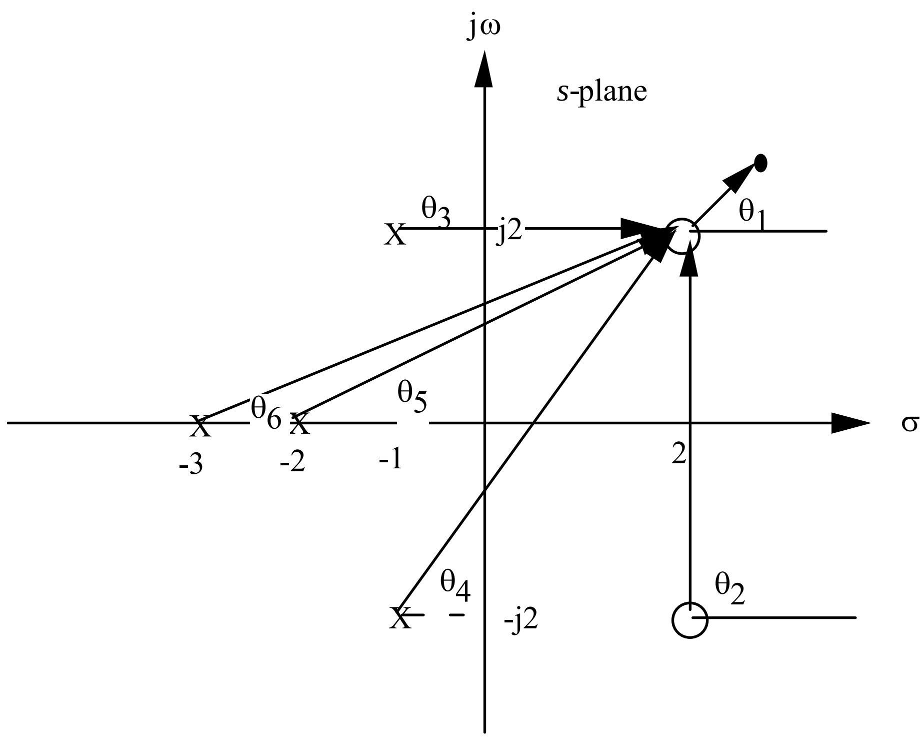
a.



b. Searching the $j\omega$ axis for 180°, we locate the point $j1.69$ at a gain of 4.249.

c. Searching between -2 and -3 for maximum gain, the breakaway is found at -2.512.

d.



To find the angle of arrival to the zero at $2 + j2$:

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 - \theta_5 - \theta_6 = \theta_1 + 90^\circ - 0^\circ - \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{2}{4}\right) - \tan^{-1}\left(\frac{2}{5}\right) = 180^\circ$$

Solving for θ_1, the angle of arrival is $\theta_1 = -191.5^0$.

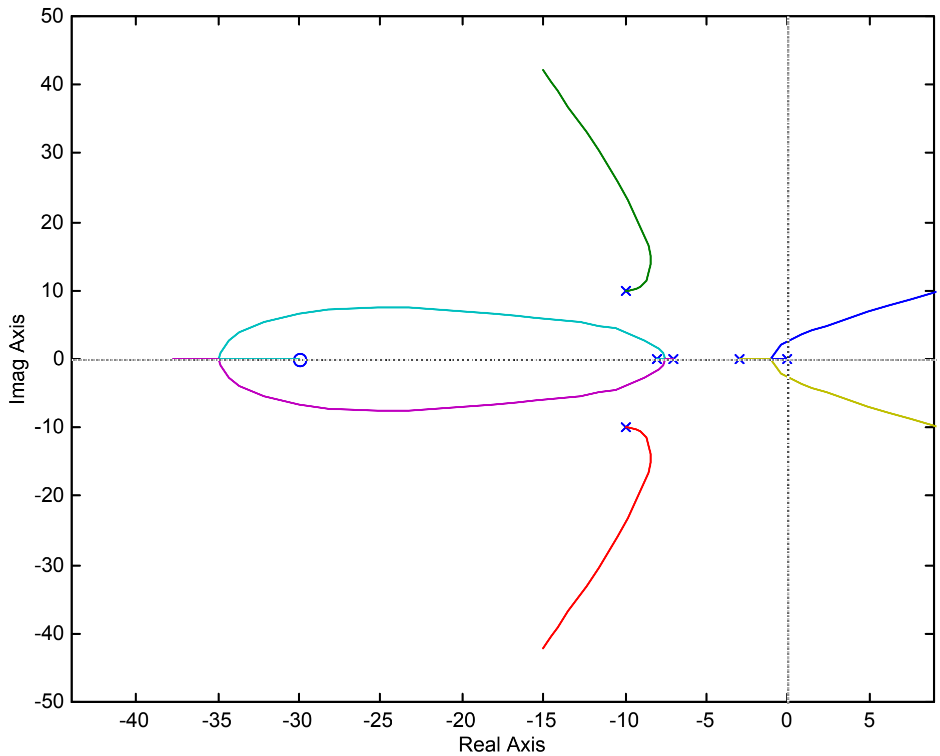
e. The closed-loop zeros are the poles of *H(s)*, or $-1 \pm j2$.

f. Searching the $\zeta = 0.358$; ($\theta = 110.97^{\circ}$) for 180°, we find the point

= $-0.6537+j1.705$. The gain, $K = 0.8764$.

g. Higher-order poles are at $-2.846 \pm j1.731$. These are not 5 times further than the dominant poles. Further, there are closed-loop zeros at $-1 \pm j2$ that are not cancelled any higher-order poles. Thus, the second-order approximation is not valid.

37.



a. The root locus crosses the imaginary axis at $j2.621$ with $K = 4365$. Therefore, the system is stable for $0 < K < 4365$.

b. Search the 0.707 damping ratio line for 180° and find $-0.949 + j0.949$ with $K = 827.2$.

c. Assume critical damping where root locus breaks away from the real axis. Locus breaks away at –1.104 with K = 527.6.

38.

Program:

```
numg=1;
deng=poly([0 -3 -7 -8]);
numh=[1 30];
denh=[1 20 200];
G=tf(numg,deng)
Gzpk=zpk(G)
H=tf(numh,denh)
```

```
rlocus(G*H)
pause
K=0:10:1e4;
rlocus(G*H,K)
sgrid(0.707,0)
axis([-2,2,-5,5]);
pause
for i=1:1:3;
[K,P]=rlocfind(G*H)
end
T=feedback(K*G,H)
step(T)
```

Computer response:

```
Transfer function:
              1
------------------------------
s^4 + 18 s^3 + 101 s^2 + 168 s


Zero/pole/gain:
         1
-------------------
s (s+8) (s+7) (s+3)


Transfer function:
     s + 30
----------------
s^2 + 20 s + 200

Select a point in the graphics window

selected_point =

  -0.9450 + 0.9499i


K =

  828.1474


P =

  -9.9500 +10.0085i
  -9.9500 -10.0085i
  -8.1007 + 1.8579i
  -8.1007 - 1.8579i
  -0.9492 + 0.9512i
  -0.9492 - 0.9512i

Select a point in the graphics window

selected_point =

   0.0103 + 2.6385i


K =

  4.4369e+003


P =

  -9.7320 +10.0691i
```

```
  -9.7320 -10.0691i
  -9.2805 + 3.3915i
  -9.2805 - 3.3915i
   0.0126 + 2.6367i
   0.0126 - 2.6367i

Select a point in the graphics window

selected_point =

  -1.0962 - 0.0000i


K =

  527.5969

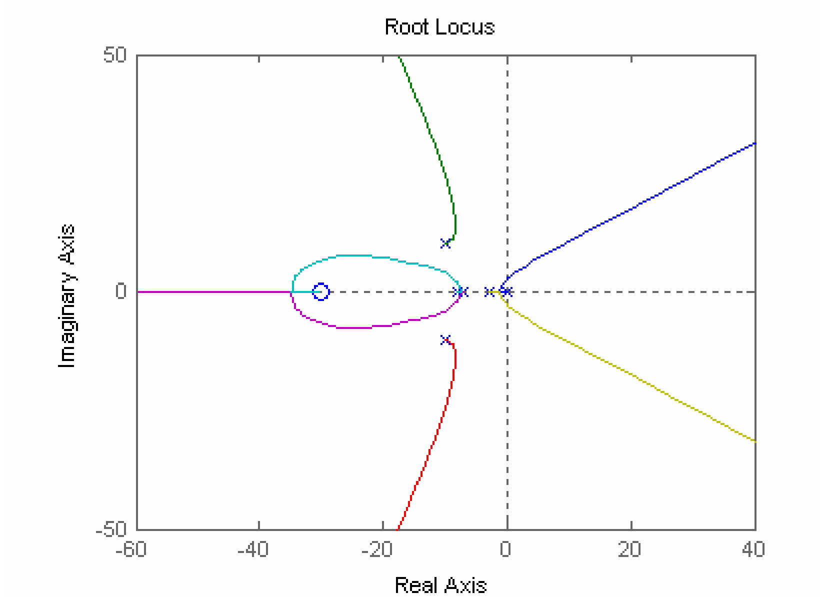

P =

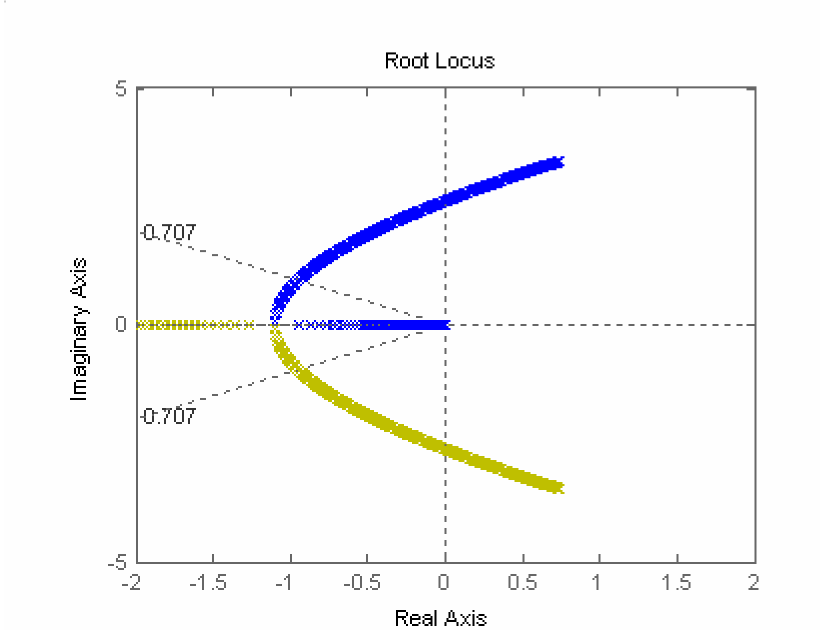
  -9.9682 +10.0052i
  -9.9682 -10.0052i
  -7.9286 + 1.5303i
  -7.9286 - 1.5303i
  -1.1101
  -1.0962

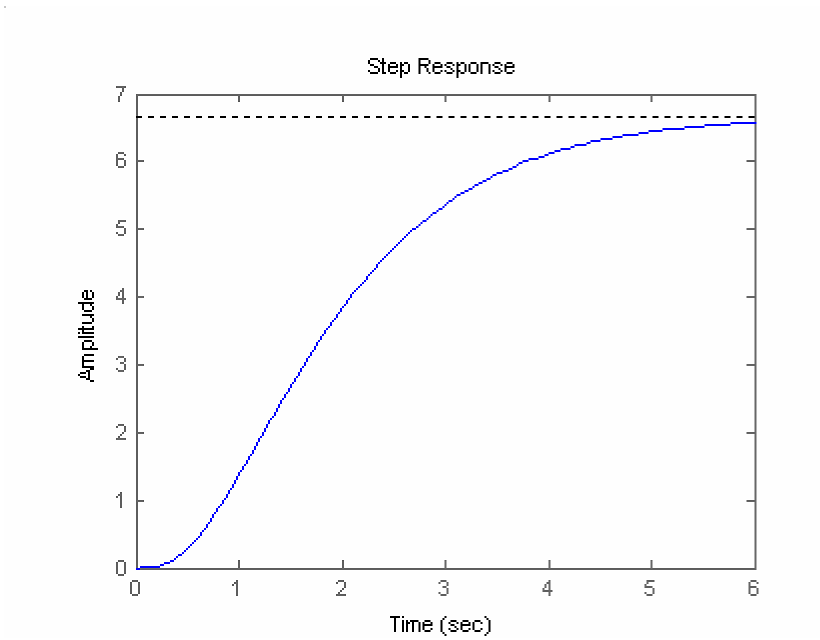

Transfer function:

       527.6 s^2 + 1.055e004 s + 1.055e005
-------------------------------------------------
s^6 + 38 s^5 + 661 s^4 + 5788 s^3 + 23560 s^2

                        + 3.413e004 s + 1.583e004
```







39.

a. Search $j\omega = j10$ line for 180^o and find $-4.533 + j10$ with $K = 219.676$.

b. $K_a = \dfrac{219.676 \text{ x } 6}{20}$

c. A settling time of 0.4 seconds yields a real part of -10. Thus if the zero is at the origin, G(s)

$\dfrac{K}{s(s+20)}$, which yields complex poles with -10 as the real part. At the design point, $-10 + j10$, K = 200.

40.

a. Searching along $\zeta\omega_n$ = -1 for 180°, find –1 + j2.04 with K = 170.13.

b. Assume critical damping when root locus breaks away form the real axis. Searching for maximum gain, the breakaway point is at -1.78 with K = 16.946.

41.

$T(s) = \frac{K}{s^3 + 6s^2 + 5s + K}$. Differentiating the characteristic equation, $s^3 + 6s^2 + 5s + K = 0$, yields,

$$3s^2 \frac{\delta s}{\delta K} + 12s \frac{\delta s}{\delta K} + 5 \frac{\delta s}{\delta K} + 1 = 0.$$

Solving for $\frac{\delta s}{\delta K}$,

$$\frac{\delta s}{\delta K} = \frac{-1}{3s^2 + 12s + 5}$$

The sensitivity of s to K is

$$S_{s:K} = \frac{K}{s}\frac{\delta s}{\delta K} = \frac{K}{s}\frac{-1}{3s^2 + 12s + 5}$$

a. Search along the ζ = 0.591 line and find the root locus intersects at s = 0.7353∠126.228° = - 0.435 + j0.593 with K = 2.7741. Substituting s and K into $S_{s:K}$ yields

$$S_{s:K} = 0.487 - j0.463 = 0.672\angle -43.553^\circ$$

b. Search along the ζ = 0.456 line and find the root locus intersects at s = 0.8894∠117.129° = - 0.406 + j0.792 with K = 4.105. Substituting s and K into $S_{s:K}$ yields

$$S_{s:K} = 0.482 - j0.358 = 0.6\angle -36.603^\circ$$

c. Least sensitive: ζ = 0.456.

42.

The sum of the feedback paths is $H_e(s) = 1 + 0.02s + \frac{0.00076s^3}{s+0.06}$. Thus,

$$H_e(s) = \frac{0.00076\left(s^3 + 26.316s^2 + 1317.4s + 78.947\right)}{s + 0.06}$$

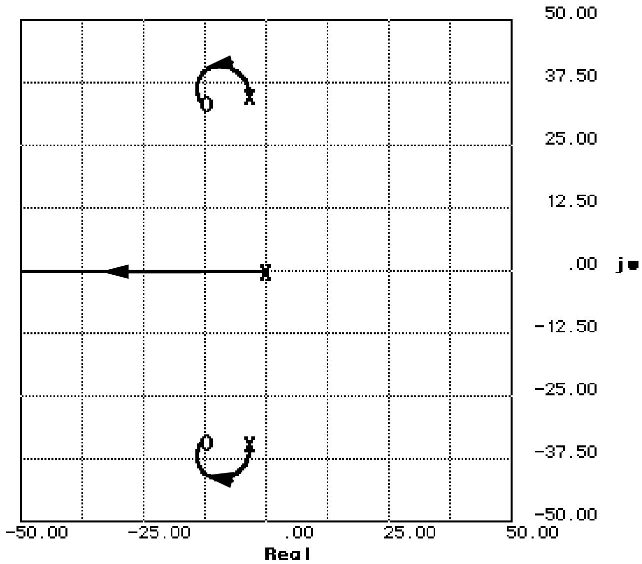
and

$$G(s)H_e(s) = 0.00076\frac{K\left(s^3 + 26.316s^2 + 1317.4s + 78.947\right)}{s(s + 0.06)\left(s^2 + 7s + 1220\right)}$$

$$G(s)H_e(s) = 0.00076\frac{K([s + 0.06][(s + 13.128 + 33.815i)(s + 13.128 - 33.815i)])}{s(s + 0.06)\left(s^2 + 7s + 1220\right)}$$

$$G(s)H_e(s) = 0.00076\frac{K([s + 13.128 + 33.815i][s + 13.128 - 33.815i])}{s([s + 3.5 + 34.753i][s + 3.5 - 34.753i])}$$

Plotting the root locus,



Searching vertical lines to calibrate the root locus, we find that 0.00076K is approximately 49.03 at $-10 \pm j41.085$. Searching the real axis for 0.00076K = 49.03, we find the third pole at -36.09.

a. $\zeta = \cos\left(\tan^{-1}\left(\frac{41.085}{10}\right)\right) = 0.236$

b. $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 46.63\%$

c. $\omega_n = \sqrt{10^2 + 41.085^2} = 42.28$ rad/s

d. $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{10} = 0.4$ seconds

e. $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{41.085} = 0.076$ seconds

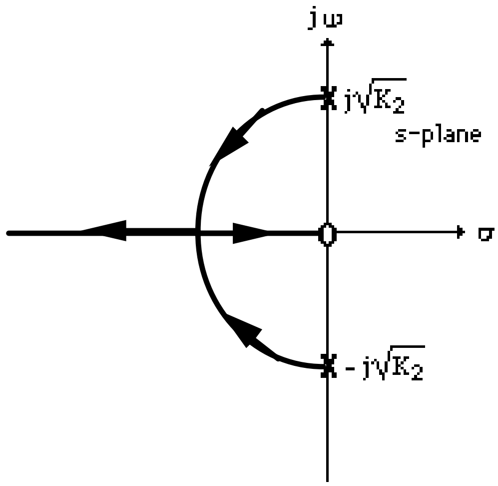
43.

Push K_2 to the right past the summing junction and find, $T(s) = \left(1 + \frac{K_1 s}{K_2}\right)\left(\frac{K_2}{s^2 + K_3 s + K_2}\right)$

$= \frac{K_1\left(s + \frac{K_2}{K_1}\right)}{s^2 + K_3 s + K_2}$. Changing form, $T(s) = \frac{\frac{K_1\left(s+\frac{K_2}{K_1}\right)}{s^2 + K_2}}{1 + \frac{K_3 s}{s^2 + K_2}}$. Thus, $G(s)H(s) = \frac{K_3 s}{s^2 + K_2}$. Sketching the

root locus,

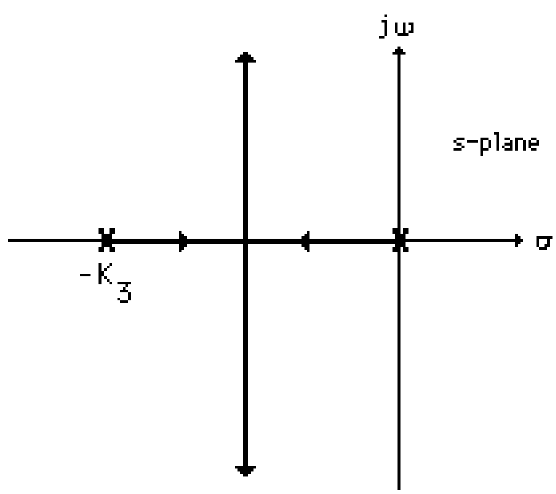
a.



b. $T(s) = \dfrac{\dfrac{K_1(s+\frac{K_2}{K_1})}{s^2+K_2}}{1+\dfrac{K_3 s}{s^2+K_2}} = \dfrac{K_1(s+\frac{K_2}{K_1})}{s^2+K_3 s+K_2}$. Therefore closed-loop zero at $-\dfrac{K_2}{K_1}$. Notice that the zero at the origin of the root locus is not a closed-loop zero.

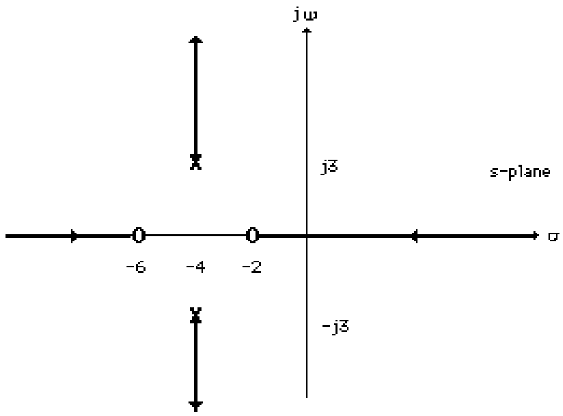
c. Push K_2 to the right past the summing junction and find, $T(s) = (1+\dfrac{K_1 s}{K_2})\,(\dfrac{K_2}{s^2+K_3 s+K_2})$

$= \dfrac{K_1\,(s+\frac{K_2}{K_1})}{s^2+K_3 s+K_2}$. Changing form, $T(s) = \dfrac{\dfrac{K_1(s+\frac{K_2}{K_1})}{s^2+K_3 s}}{1+\dfrac{K_2}{s^2+K_3 s}}$. Thus, $G(s)H(s) = \dfrac{K_2}{s^2+K_3 s}$. Sketching the root locus,

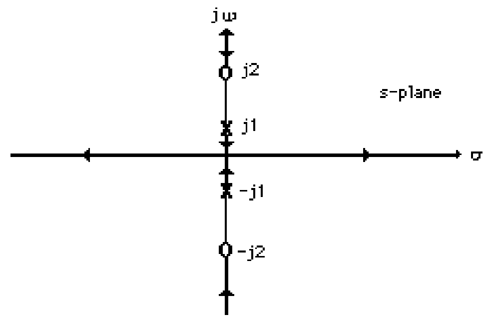


The closed-loop zero is at $-\dfrac{K_2}{K_1}$.

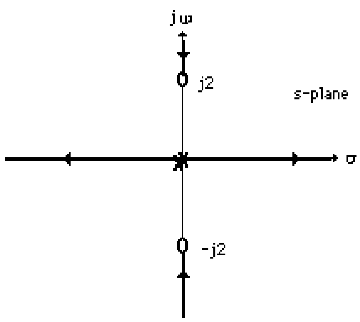
44.



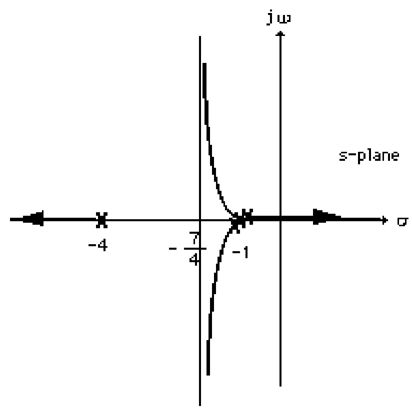
a



b



c



d

45.

a. Using Figure P8.15(a),

$$[Ms^2+(D+D_c)s+(K+K_c)]X(s) - [D_cs+K_c]X_a(s) = 0$$

Rearranging,

$$[Ms^2+Ds+K]X(s) = -[D_cs+K_c](X(s)-X_a(s)) \quad (1)$$

where $[D_cs+K_c](X(s)-X_a(s))$ can be thought of as the input to the plant.

For the active absorber,

$$(M_cs^2+D_cs+K_c)X_a(s) - (D_cs+K_c)X(s) = 0$$

or

$$M_cs^2X_a(s)+D_cs(X_a(s)-X(s))+K_c(X_a(s)-X(s)) = 0$$

Adding $-M_cs^2X(s)$ to both sides,

$$M_cs^2(X_a(s)-X(s))+D_cs(X_a(s)-X(s))+K_c(X_a(s)-X(s)) = -M_cs^2X(s)$$

Let $X_a(s)-X(s) = X_c(s)$ and $s^2X(s) = C(s)$ = plant output acceleration. Therefore,

$$M_cs^2X_c(s)+D_csX_c(s)+K_cX_c(s) = -M_cC(s)$$

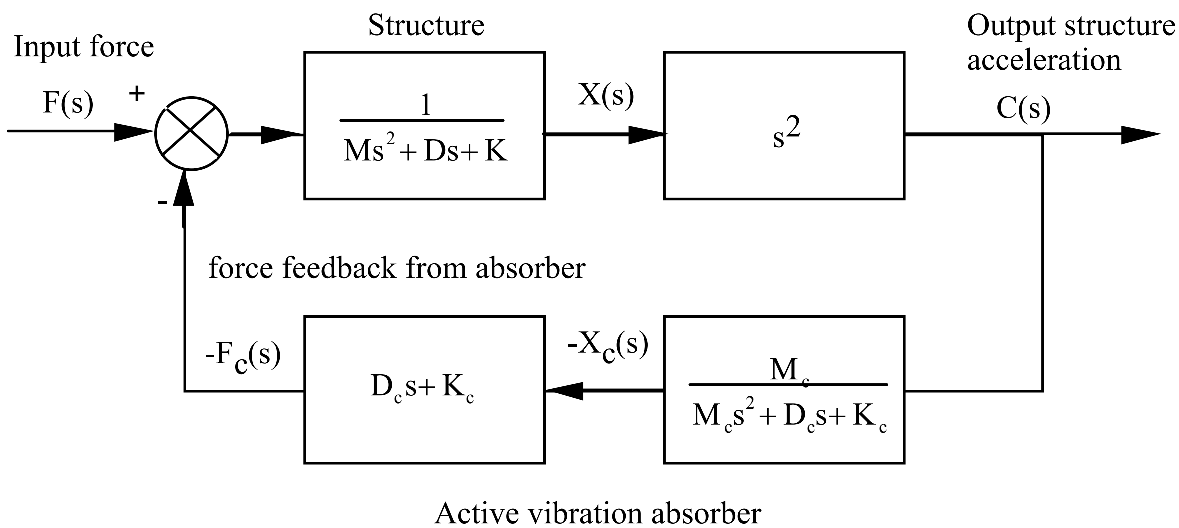
or

$$(M_cs^2+D_cs+K_c)X_c(s) = -M_cC(s) \qquad (2)$$

Using Eqs. (1) and (2), and $X_a(s)-X(s) = X_c(s)$,

$$\frac{X_c(s)}{C(s)} = \frac{-M_c}{M_cs^2+D_cs+K_c} \; ; \; \frac{X(s)}{X_c(s)} = \frac{D_cs+K_c}{Ms^2+Ds+K}$$

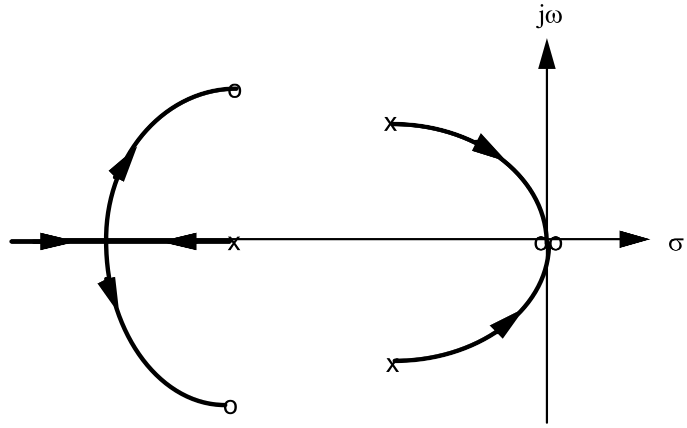
which suggests the following block diagram:



b. Substituting $M = D = K = D_c = K_c =1$ and redrawing the block diagram above to show X(s) as the output yields a block diagram with $G(s) = \frac{1}{s^2+s+1}$ and $H(s) = \frac{M_c s^2 (s+1)}{M_c s^2+s+1}$. To study the steady-state error, we create a unity-feedback system by subtracting unity from H(s). Thus $H_e(s) = H(s)-1 = \left(M_c s^3 - s - 1\right)\frac{1}{M_c s^2+s+1}$. The equivalent G(s) for this unity-feedback system is $G_e(s) = \frac{G}{1+G H_e}$ $= \frac{M_c s^2+s+1}{M_c s^4+2 M_c s^3+s^3+M_c s^2+2 s^2+s}$. Hence the equivalent unity-feedback system is Type 1 and will respond with zero steady-state error for a step force input.

c. Using $G_e(s)$ in part b, we find $T(s) = \frac{G_e}{1+G_e} = \frac{M_c s^2 + s + 1}{(s^2+2s+2)M_c s^2 + s^3 + 2s^2 + 2s + 1}$. Dividing numerator and denominator by s^3+2s^2+2s+1, $T(s) = \frac{\frac{M_c s^2+s+1}{s^3+2s^2+2s+1}}{\frac{(s^2+2s+2)M_c s^2}{s^3+2s^2+2s+1}+1}$. Thus, the system has the same root locus as a system with $G(s)H(s) = \frac{(s^2+2s+2)M_c s^2}{s^3+2s^2+2s+1} = \frac{(s^2+2s+2)M_c s^2}{(s+1)(s^2+s+1)}$.

Sketching the root locus,



SOLUTIONS TO DESIGN PROBLEMS

46.

a. For a settling time of 0.1 seconds, the real part of the dominant pole is $-\frac{4}{0.1} = -40$. Searching along the $\sigma = -40$ line for 180°, we find the point $-40 + j57.25$ with $20{,}000K = 2.046 \times 10^9$, or K = 102,300.

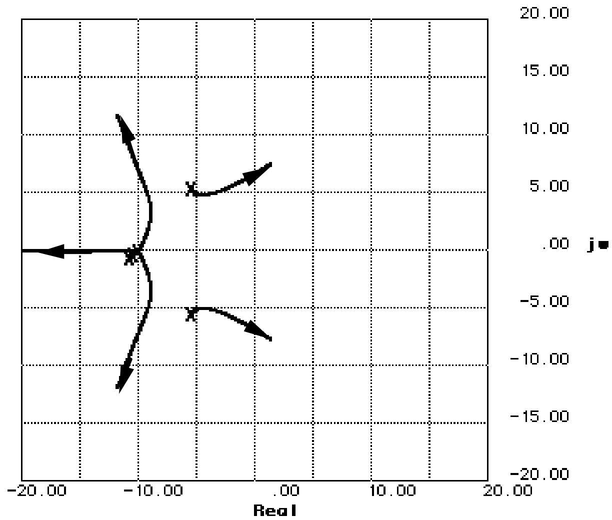
b. Since, for the dominant pole, $\tan^{-1}\left(\frac{57.25}{40}\right) = 55.058°$, $\zeta = \cos(55.058°) = 0.573$. Thus,

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 11.14\%$.

c. Searching the imaginary axis for 180°, we find $\omega = 169.03$ rad/s for $20{,}000K = 1.43 \times 10^{10}$. Hence, K = 715,000. Therefore, for stability, K < 715,000.

47.

$$G(s) = \frac{61.73K}{(s+10)^3(s^2 + 11.11s + 61.73)}$$



a. Root locus crosses the imaginary axis at ±j6.755 with 61.73K equal to 134892.8. Thus for oscillations, K = 2185.21.

b. From (a) the frequency of the oscillations is 6.755 rad/s.

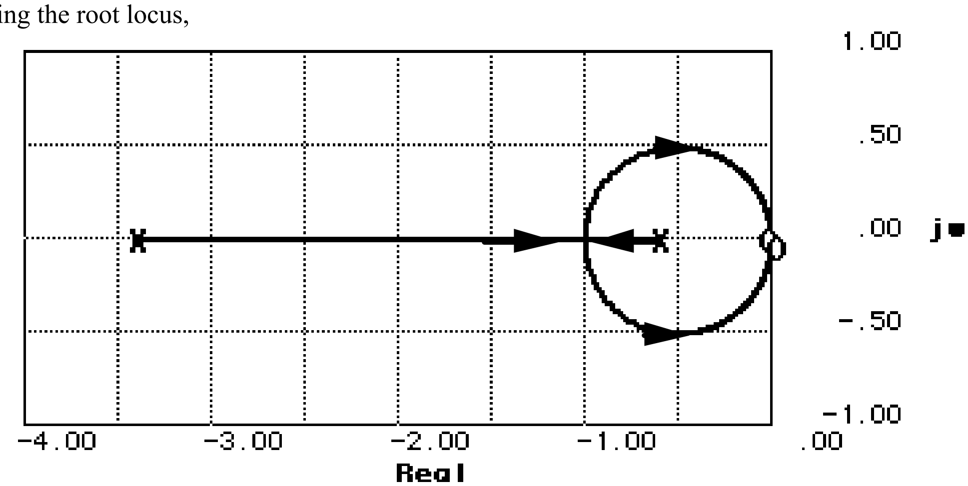
c. The root locus crosses the 20% overshoot line at $6\angle 117.126^{o}$ = - 2.736 + j5.34 with 61.73K = 23323.61. Thus, K = 377.83 and $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{2.736}$ = 1.462 seconds.

48.

a. Finding the transfer function with C_a as a parameter,

$$\frac{\ddot{Y}_m(s)}{Y_G(s)} = \frac{s^2(2s+2)}{(C_a+1)s^2+4s+2} = \frac{\frac{2s^2(s+1)}{s^2+4s+2}}{1+\frac{C_a s^2}{s^2+4s+2}}$$

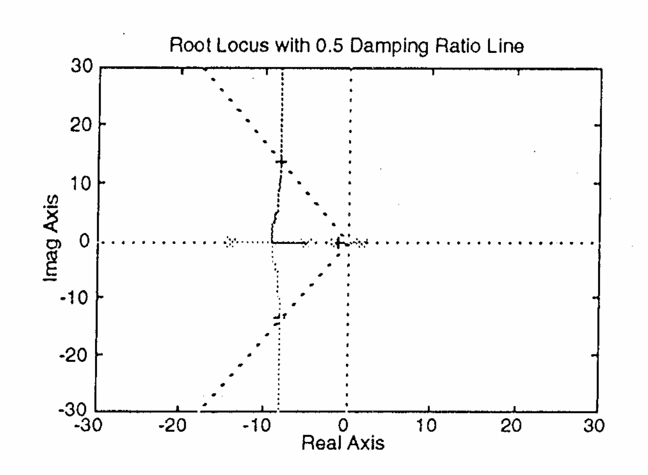
Plotting the root locus,



b. Since $2\zeta\omega_n = \frac{4}{C_a+1}$; $\omega_n^2 = \frac{2}{C_a+1}$, $\zeta^2 = \frac{2}{C_a+1} = 0.69^2$. Hence, $C_a = 3.2$.

49.

a.



b. The pole at 1.8 moves left and crosses the origin at a gain of 77.18. Hence, the system is stable for $K > 77.18$, where $K = -508K_2$. Hence, $K_2 < -0.152$.

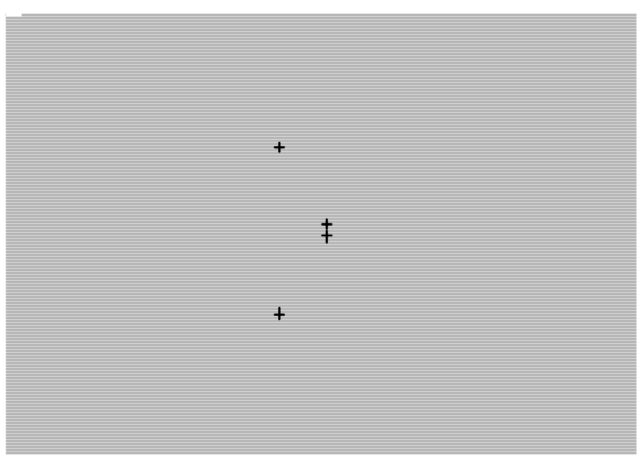
c. Search the $\zeta = 0.5$ ($\theta = 120^o$) damping ratio line for 180^o and find the point, $-8.044 + j13.932 = 16.087 \angle 120^o$ with a $K = -508K_2 = 240.497$. Thus, $K_2 = -0.473$.

d. Search the real axis between 1.8 and -1.6 for $K = 240.497$ and find the point -1.01.

Thus $G_e(s) = \frac{240.497K_1(s+1.6)}{s(s+1.01)(s+8.044+j13.932)(s+8.044-j13.932)} = \frac{240.497K_1(s+1.6)}{s(s+1.01)(s^2+16.088s+258.8066)}$.

Plotting the root locus and searching the $j\omega$ axis for 180^o we find $j15.792$ with $240.497K_1 = 4002.6$, or $K_1 = 16.643$.

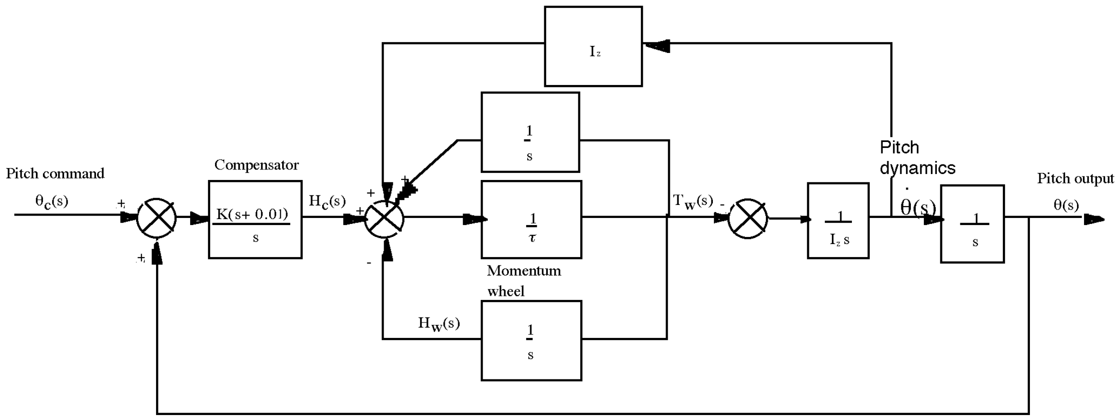
e.



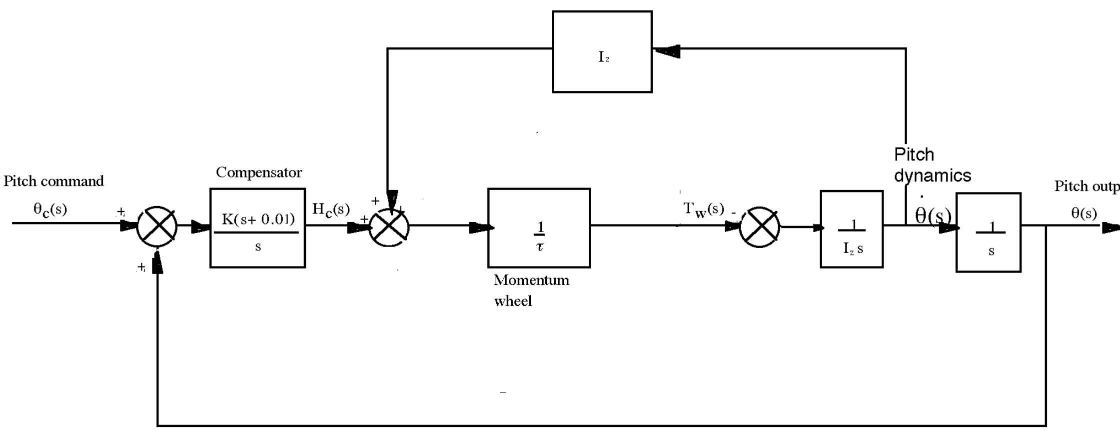
Search the $\zeta = 0.45$ ($\theta = 116.744^o$) damping ratio line for 180^o and find the point, -6.685 + j13.267 = 14.856 $\angle$ 116.744^o with a $K = 240.497K_1 = 621.546$. Thus, $K_1 = 2.584$.

50.

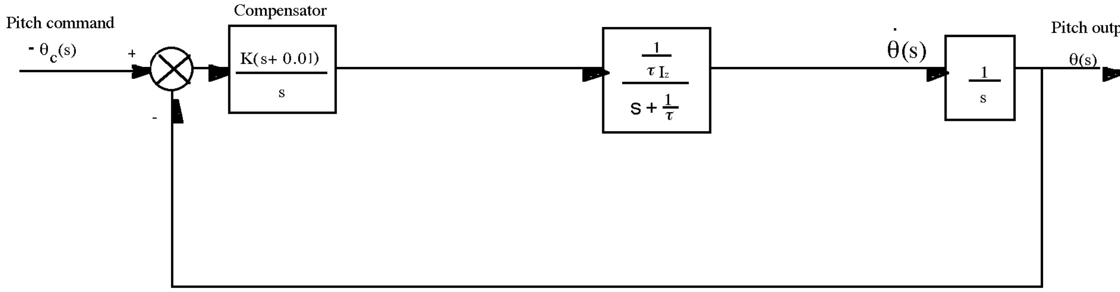
a. Update the block diagram to show the signals that form $H_{sys}(s)$.



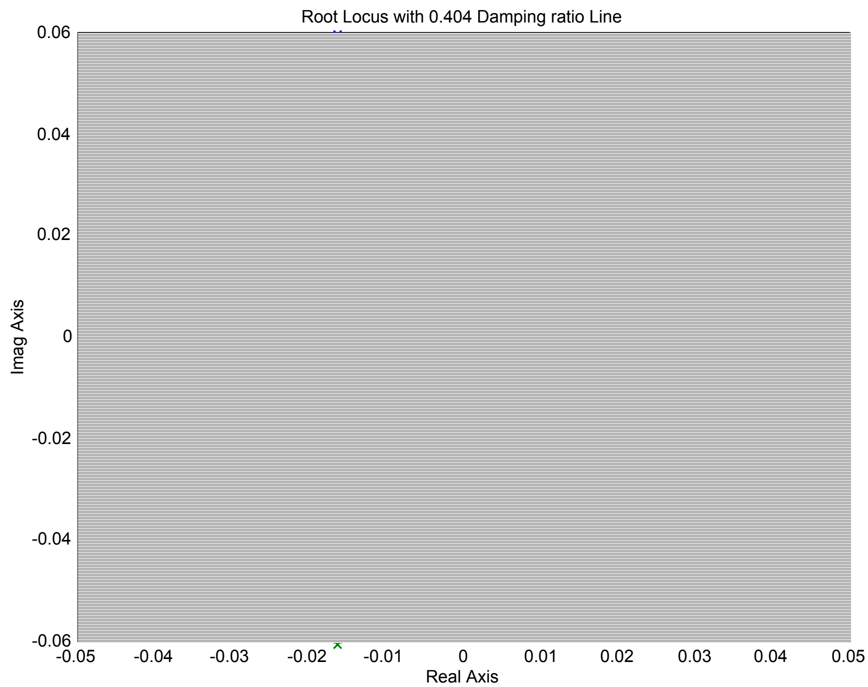
Perform block diagram reduction of the parallel paths from $T_W(s)$.



Reduce the momentum wheel assembly to a single block.



Substitute values and find $G_e(s) = \frac{4.514x10^{-6}K(s+0.01)}{s^2(s+0.043)}$. Plotting the root locus yields

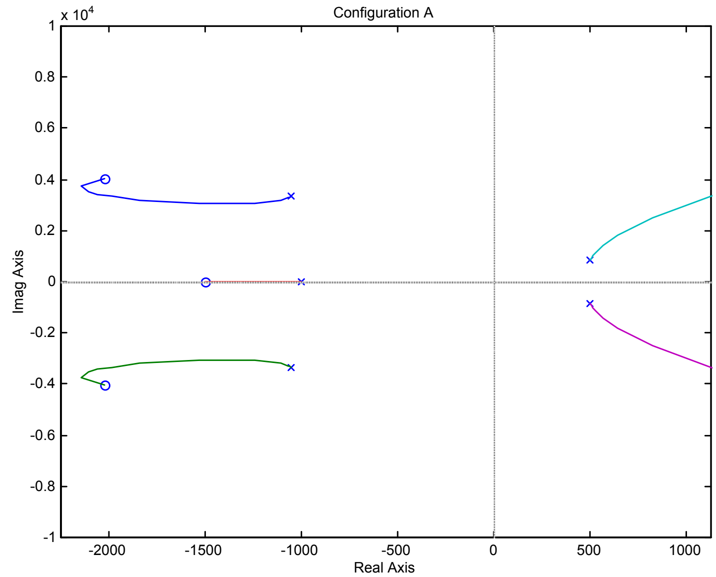


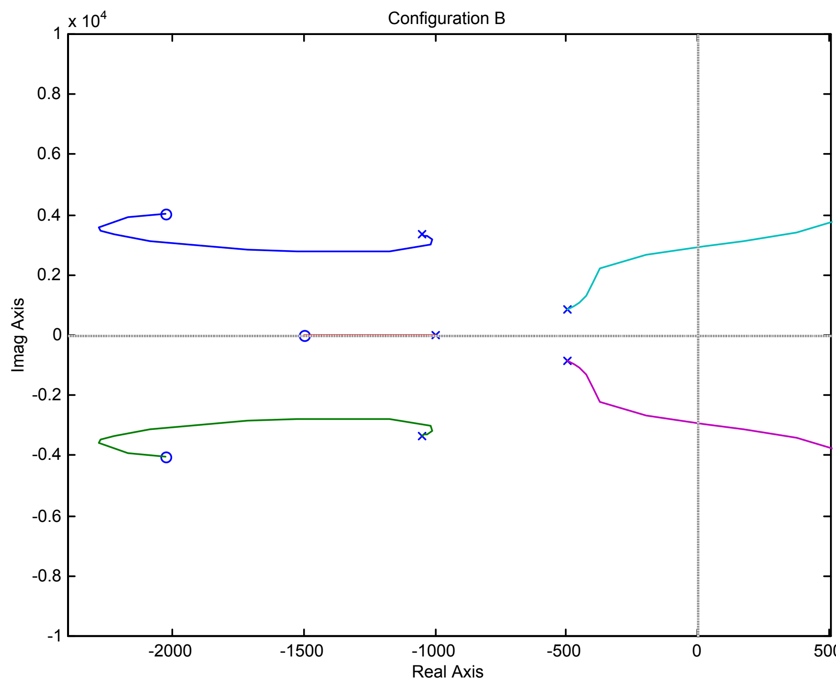
b. Searching the 25% overshoot line ($\zeta = 0.404$; $\theta = 113.8^o$) for 180^o yields -0.0153 + j0.0355 with a gain = 4.514E-6K = 0.0019. Thus, K = 420.9.

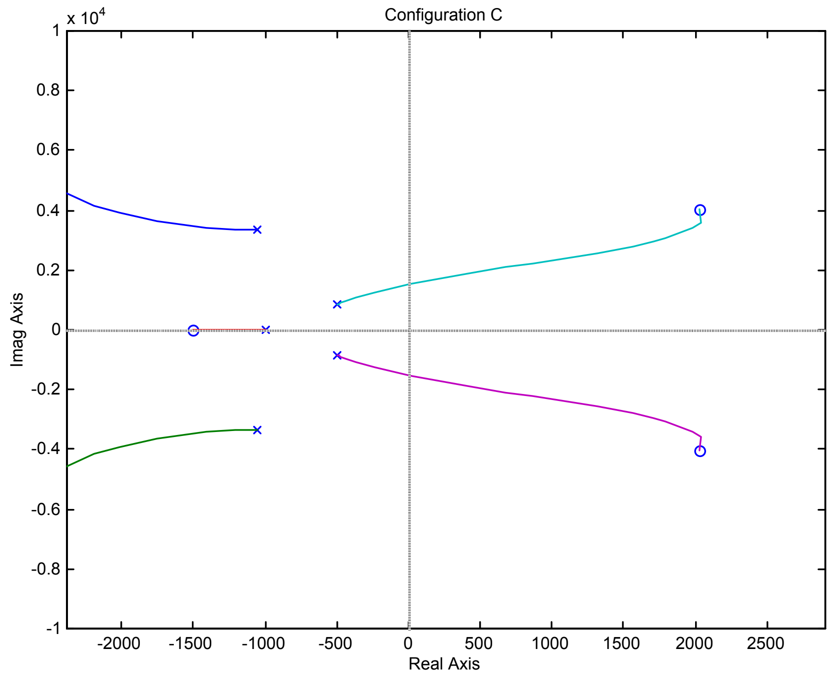
c. Searching the real axis between –0.025 and –0.043 for a gain of 0.0019. we find the third pole at -0.0125. Simulate the system. There is no pole-zero cancellation. A simulation shows approximately 95% overshoot. Thus, even though the compensator yields zero steady-state error, a system redesign for transient response is necessary using methods discussed in Chapter 9.

51.

a.


Configuration A
x 10^4
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Imag Axis
-2000
-1500
-1000
-500
0
500
1000
Real Axis


Configuration B
x 10^4
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Imag Axis
-2000
-1500
-1000
-500
0
500
Real Axis



b.

Configuration A: System is always unstable.

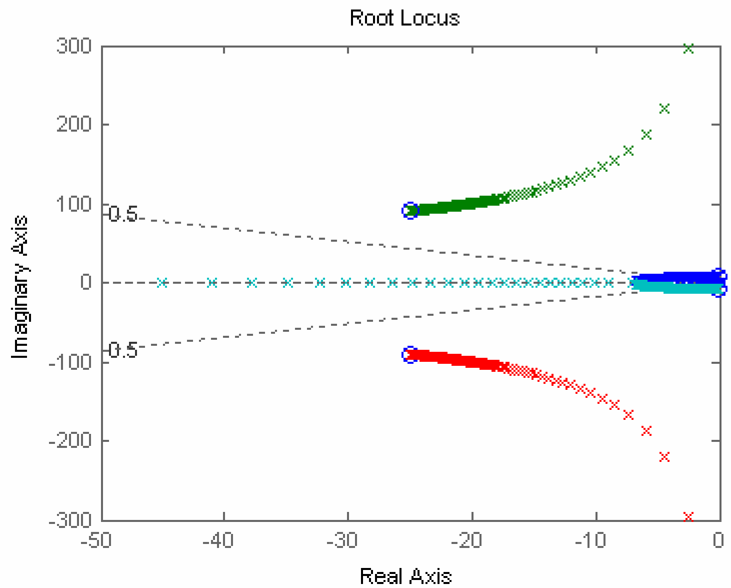
Configuration B: root locus crosses $j\omega$ axis at $j2897$ with a gain of 3.22×10^6. Thus, for stability, K $< 3.22 \times 10^6$.

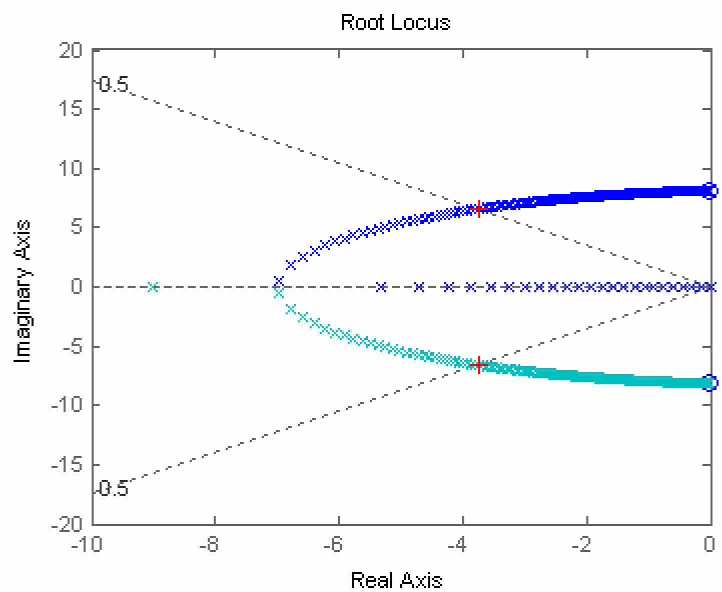
Configuration C: root locus crosses $j\omega$ axis at $j1531$ with a gain of 9.56×10^5. System is unstable at high gains. Thus, for stability, $9.56 \times 10^5 > K$.

52.

a. Using MATLAB and the Symbolic Math Toolbox, the open-loop expression that yields a root locus as a function of N^2 is

$$G_{dt}(s) = \frac{0.2284\text{x}10^7 N^2 (s^2 + 3.772\text{e-}05s + 66.27)(s^2 + 49.99s + 8789)}{s(s+45.12)(s^2 + 4.893s + 8.777\text{e}04)}$$





Program:

```
syms s N KLSS KHSS KG JR JG tel s
numGdt=3.92*N^2*KLSS*KHSS*KG*s;
denGdt=(N^2*KHSS*(JR*s^2+KLSS)*(JG*s^2*[tel*s+1]+KG*s)+JR*s^2*KLSS*[(JG*s^2
+KHSS)*(tel*s+1)+KG*s]);
Gdt=numGdt/denGdt;
'Gdt in General Terms'
pretty(Gdt)
'Values to Substitute'
KLSS=12.6e6
KHSS=301e3
KG=668
JR=190120
JG=3.8
tel=20e-3
numGdt=3.92*N^2*KLSS*KHSS*KG*s;
```

```
numGdt=vpa(numGdt,4);
denGdt=(N^2*KHSS*(JR*s^2+KLSS)*(JG*s^2*[tel*s+1]+KG*s)+JR*s^2*KLSS*[(JG*s^2
+KHSS)*(tel*s+1)+KG*s]);
denGdt=vpa(denGdt,4);
'Gdt with Values Substituted'
Gdt=numGdt/denGdt;
pretty(Gdt)
Gdt=expand(Gdt);
Gdt=vpa(Gdt,4);
'Gdt Different Form 1'
pretty(Gdt);
denGdt=collect(denGdt,N^2);
'Gdt Different Form 2'
Gdt=collect(Gdt,N^2);
pretty(Gdt)
[numGdt,denGdt]=numden(Gdt);
numGdt=numGdt/0.4349e10;
denGdt=denGdt/0.4349e10;
denGdt=expand(denGdt);
denGdt=collect(denGdt,N^2);
Gdt=vpa(numGdt/denGdt,4);
'Gdt Different Form 3'
pretty(Gdt)
'Putting into Form for RL as a Function of N^2 using previous results'
numGH=[1 49.99 8855 3313 582400];
denGH=[41.87 2094 0.3684e7 0.1658e9 0];
denGH=denGH/denGH(1)
GH=tf(numGH,denGH)
GHzpk=zpk(GH)
'Zeros of GH'
rootsnumGH=roots(numGH)
'Poles of GH'
rootsdenGH=roots(denGH)
K=0:1:10000;
rlocus(GH,K)
sgrid(0.5,0)
pause
axis([-10,0,-20,20])
[K,P]=rlocfind(GH)
```

Computer response:

```
ans =

Gdt in General Terms


  98  2                   /   2           2              2
  -- N  KLSS KHSS KG s  /  (N  KHSS (JR s  + KLSS) (JG s  (tel s + 1) + KG s)
  25                   /

               2            2
         + JR s  KLSS ((JG s  + KHSS) (tel s + 1) + KG s))

ans =

Values to Substitute


KLSS =

    12600000


KHSS =

      301000
```

```
KG =

   668


JR =

      190120


JG =

    3.8000


tel =

    0.0200


ans =

Gdt with Values Substituted


          16  2     /
  .9931 10   N  s  /  (301000.
                  /

         2           2           8          2
        N  (190100. s  + .1260 10 ) (3.800 s  (.02000 s + 1.) + 668. s)

                   13  2           2
         + .2396 10   s  ((3.800 s  + 301000.) (.02000 s + 1.) + 668. s))

ans =

Gdt Different Form 1


          16  2     /           10  2  5           12  2  4           14  2  3
  .9931 10   N  s  /  (.4349 10   N  s  + .2174 10   N  s  + .3851 10   N  s
                  /

                   14  2  2           16  2             12  5            13  4
         + .1441 10   N  s  + .2533 10   N  s + .1821 10   s  + .9105 10   s

                   17  3           18  2
         + .1602 10   s  + .7212 10   s )

ans =

Gdt Different Form 2


          16  2     /            10  5           12  4           14  3
  .9931 10   N  s  /  ((.4349 10   s  + .2174 10   s  + .3851 10   s
                  /

                   14  2           16      2            18  2           12  5
         + .1441 10   s  + .2533 10   s) N  + .7212 10   s  + .1821 10   s

                   13  4           17  3
         + .9105 10   s  + .1602 10   s )

ans =
```

```
Gdt Different Form 3

         7  2     /
  .2284 10  N  s  /  (
                 /

               5          4          3          2              2
      (1.000 s  + 49.99 s  + 8855. s  + 3313. s  + 582400. s) N

                  9  2          5          4           7  3
       + .1658 10  s  + 41.87 s  + 2094. s  + .3684 10  s )

ans =

Putting into Form for RL as a Function of N^2 using previous results


denGH =

  1.0e+006 *

  Columns 1 through 4

    0.0000    0.0001    0.0880    3.9599

  Column 5

         0


Transfer function:
s^4 + 49.99 s^3 + 8855 s^2 + 3313 s + 582400
--------------------------------------------
s^4 + 50.01 s^3 + 8.799e004 s^2 + 3.96e006 s


Zero/pole/gain:
 (s^2  + 66.27) (s^2  + 49.99s + 8789)
---------------------------------------
s (s+45.12) (s^2  + 4.893s + 8.777e004)


ans =

Zeros of GH


rootsnumGH =

 -24.9950 +90.3548i
 -24.9950 -90.3548i
  -0.0000 + 8.1404i
  -0.0000 - 8.1404i


ans =

Poles of GH


rootsdenGH =

  1.0e+002 *

        0
```

```
  -0.0245 + 2.9624i
  -0.0245 - 2.9624i
  -0.4512

Select a point in the graphics window

selected_point =

  -3.8230 + 6.5435i


K =

   51.5672

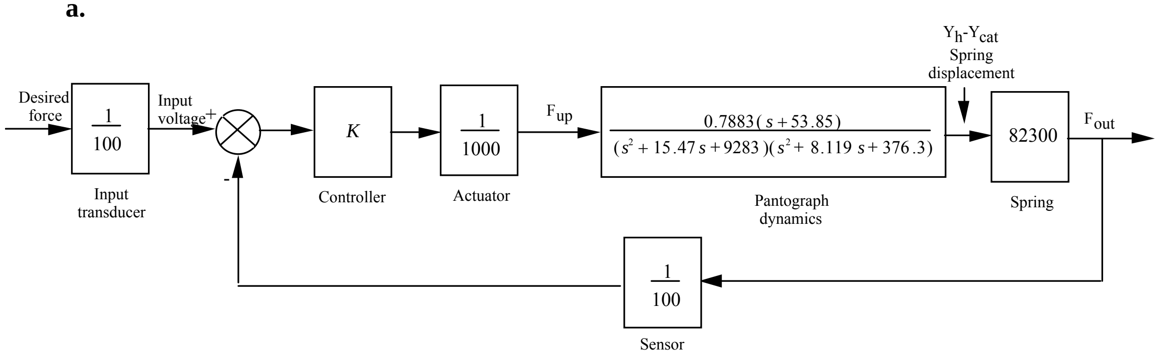

P =

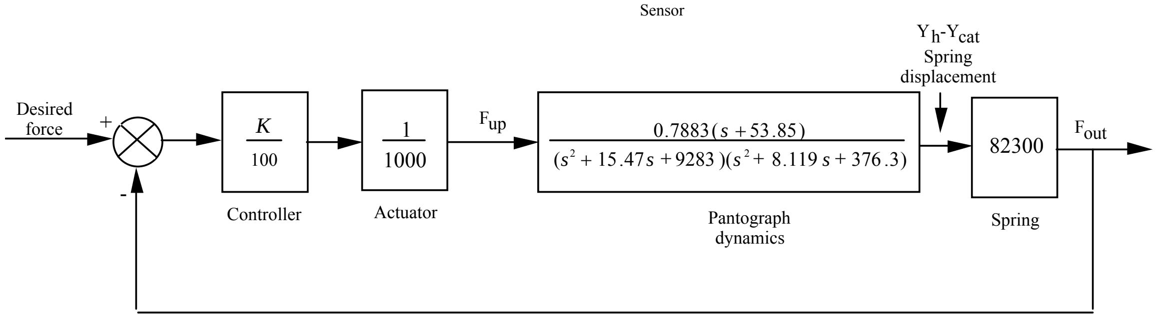
 -21.1798 +97.6282i
 -21.1798 -97.6282i
  -3.8154 + 6.5338i
  -3.8154 - 6.5338i
```

b. From the computer response, $K = 0.2284\text{x}10^7 N^2 = 49.6$. Therefore, N is approximately 5/1000.

53.

a.

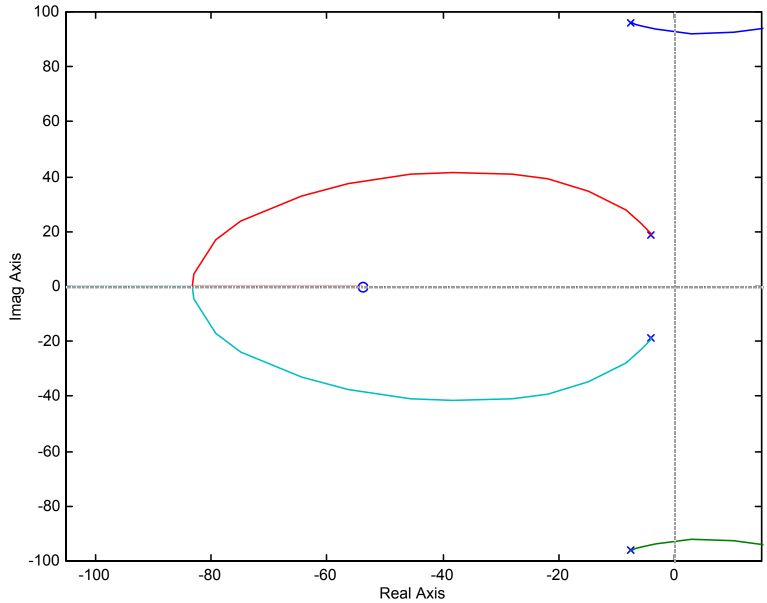




$$G(s) = \frac{Y_h(s) - Y_{cat}(s)}{F_{up}(s)} = \frac{0.7883(s+53.85)}{(s^2+15.47s+9283)(s^2+8.119s+376.3)}$$

Ge(s)=(K/100)*(1/1000)*G(s)*82.3e3

$$G_e(s) = \frac{0.6488K\ (s+53.85)}{(s^2 + 8.119s + 376.3)\ (s\text{^}2 + 15.47s + 9283)}$$



b. 38% overshoot yields $\zeta = 0.294$. The $\zeta = 0.294$ line intersects the root locus at $-9 + j27.16$. Here, $K_e = 7.179 \times 10^4$. Thus $K = K_e/0.6488$, or $K = 1.107 \times 10^5$.

c. $T_s = 4/Re = 4/9 = 0.44$ s; $T_p = \pi/Im = \pi/27.16 = 0.116$ s

d. Nondominant closed-loop poles are located at $-3.4 \pm j93.94$. Thus poles are closer to the imaginary axis than the dominant poles. Second order approximation not valid.

e.

Program:

```
syms s
numg=(s+53.85);
deng=(s^2+15.47*s+9283)*(s^2+8.119*s+376.3);
numg=sym2poly(numg);
deng=sym2poly(deng);
G=tf(numg,deng)
K=7.179e4
Ke=0.6488*K
T=feedback(Ke*G,1)
step(T)
```

Computer response:

```
Transfer function:

                    s + 53.85
-------------------------------------------------
s^4 + 23.59 s^3 + 9785 s^2 + 8.119e004 s

                                       + 3.493e006


K =

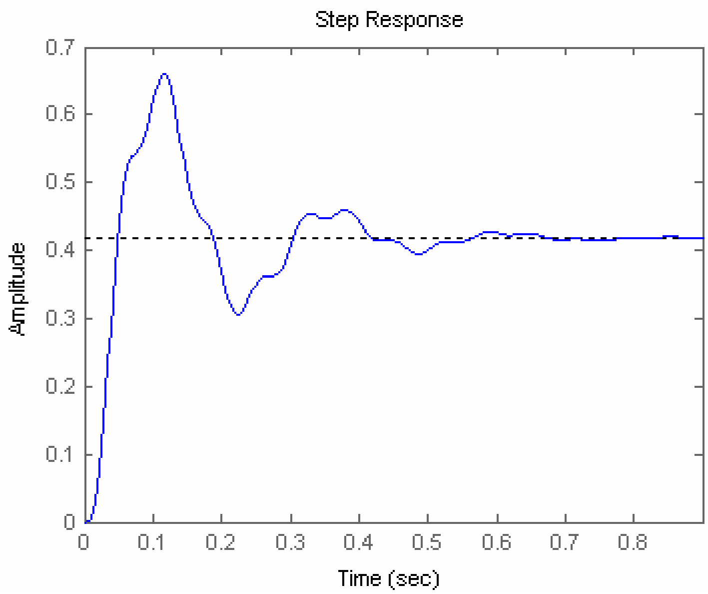
       71790


Ke =
```

```
  4.6577e+004

Transfer function:
             4.658e004 s + 2.508e006
-------------------------------------------------
s^4 + 23.59 s^3 + 9785 s^2 + 1.278e005 s

                                      + 6.001e006
```



$T_p = 0.12$ s, $T_s = 0.6$ s, $\%OS = \dfrac{0.66 - 0.42}{0.42} = 57.1\%$.

N I N E

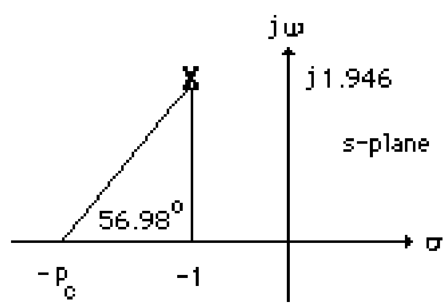
Design via Root Locus

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Lag-Lead Compensation

a. Uncompensated: From the Chapter 8 Case Study Challenge, $G(s) = \frac{76.39K}{s(s+150)(s+1.32)} = \frac{7194.23}{s(s+150)(s+1.32)}$ with the dominant poles at $-0.5 \pm j6.9$. Hence, $\zeta = \cos(\tan^{-1}\frac{6.9}{0.5}) = 0.0723$, or $\%OS = 79.63\%$ and $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.5} = 8$ seconds. Also, $K_v = \frac{7194.23}{150 \times 1.32} = 36.33$.

b. Lead-Compensated: Reducing the percent overshoot by a factor of 4 yields, $\%OS = \frac{79.63}{4} = 19.91\%$, or $\zeta = 0.457$. Reducing the settling time by a factor of 2 yields, $T_s = \frac{8}{2} = 4$. Improving K_v by 2 yields $K_v = 72.66$. Using $T_s = \frac{4}{\zeta\omega_n} = 4$, $\zeta\omega_n = 1$, from which $\omega_n = 2.188$ rad/s. Thus, the design point equals $-\zeta\omega_n + j\,\omega_n\sqrt{1-\zeta^2} = -1 + j1.946$. Using the system's original poles and assuming a lead compensator zero at -1.5, the summation of the system's poles and the lead compensator zero to the design point is -123.017^o . Thus, the compensator pole must contribute $123.017^o - 180^o = -56.98^o$. Using the geometry below, $\frac{1.946}{p_c - 1} = \tan 56.98^o$, or $p_c = 2.26$.



Adding this pole to the system poles and the compensator zero yields 76.39K = 741.88 at -1+j1.946. Hence the lead-compensated open-loop transfer function is $G_{Lead\text{-}comp}(s) =$

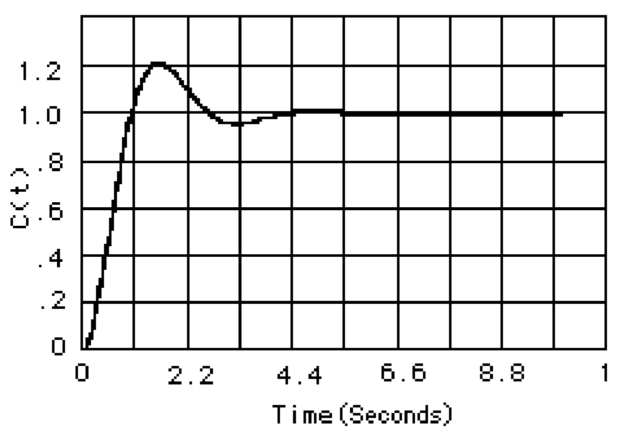
$\frac{741.88(s+1.5)}{s(s+150)(s+1.32)(s+2.26)}$. Searching the real axis segments of the root locus yields higher-order poles at greater than -150 and at -1.55. The response should be simulated since there may not be pole/zero cancellation. The lead-compensated step response is shown below.



Lead compensated

Since the settling time and percent overshoot meet the transient requirements, proceed with the lag compensator. The lead-compensated system has $K_v = \frac{741.88 \text{ x } 1.5}{150 \text{ x } 1.32 \text{ x } 2.26} = 2.487$. Since we want $K_v = 72.66$, an improvement of $\frac{72.66}{2.487} = 29.22$ is required. Select $G(s)_{Lag} = \frac{s+0.002922}{s+0.0001}$ to improve the steady-state error by 29.22. A simulation of the lag-lead compensated system,

$G_{Lag\text{-}lead\text{-}comp}(s) = \frac{741.88(s+1.5)(s+0.002922)}{s(s+150)(s+1.32)(s+2.26)(s+0.0001)}$ is shown below.



Lag-lead compensated

UFSS Vehicle: Lead and Feedback Compensation

Minor loop: Open-loop transfer function $G(s)H(s) = \frac{0.25K_2(s+0.437)}{(s+2)(s+1.29)(s+0.193)}$; Closed-loop transfer function: $T_{ML}(s) = \frac{0.25K_2(s+0.437)}{s(s^3+...)}$. Searching along the 126.87° line ($\zeta = 0.6$), find the dominant second-order poles at -1.554 ± j2.072 with $0.25K_2 = 4.7$. Thus $K_2 = 18.8$. Searching the real axis segment of the root locus for a gain of 4.7 yields a 3rd pole at -0.379.

Major loop: The unity feedback, open-loop transfer function found by using the minor-loop closed-loop poles is $G_{ML}(s) = \frac{-0.25K_1(s+0.437)}{s(s+0.379)(s+1.554+j2.072)(s+1.554-j2.072)}$. Searching along the 120° line ($\zeta = 0.5$), find the dominant second-order poles at -1.069±j1.85 with $0.25K_1 = 4.55$. Thus $K_1 = 18.2$. Searching the real axis segment of the root locus for a gain of 4.55 yields a 3rd pole at -0.53 and a 4th pole at -0.815.

ANSWERS TO REVIEW QUESTIONS

1. Chapter 8: Design via gain adjustment. Chapter 9: Design via cascaded or feedback filters

2. A. Permits design for transient responses not on original root locus and unattainable through simple gain adjustments. B. Transient response and steady-state error specifications can be met separately and independently without the need for tradeoffs

3. PI or lag compensation

4. PD or lead compensation

5. PID or lag-lead compensation

6. A pole is placed on or near the origin to increase or nearly increase the system type, and the zero is placed near the pole in order not to change the transient response.

7. The zero is placed closer to the imaginary axis than the pole. The total contribution of the pole and zero along with the previous poles and zeros must yield 180^0 at the design point. Placing the zero closer to the imaginary axis tends to speed up a slow response.

8. A PD controller yields a single zero, while a lead network yields a zero and a pole. The zero is closer to the imaginary axis.

9. Further out along the same radial line drawn from the origin to the uncompensated poles

10. The PI controller places a pole right at the origin, thus increasing the system type and driving the error to zero. A lag network places the pole only close to the origin yielding improvement but no zero error.

11. The transient response is approximately the same as the uncompensated system, except after the original settling time has passed. A slow movement toward the new final value is noticed.

12. 25 times; the improvement equals the ratio of the zero location to the pole location.

13. No; the feedback compensator's zero is not a zero of the closed-loop system.

14. A. Response of inner loops can be separately designed; B. Faster responses possible; C. Amplification may not be necessary since signal goes from high amplitude to low.

SOLUTIONS TO PROBLEMS

1.

Uncompensated system: Search along the ζ = 0.5 line and find the operating point is at -1.5356 ± j2.6598 with K = 73.09. Hence, $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100$ = 16.3%; $T_s = \frac{4}{1.5356}$ = 2.6 seconds; $K_p = \frac{73.09}{30}$ =2.44. A higher-order pole is located at -10.9285.

Compensated: Add a pole at the origin and a zero at -0.1 to form a PI controller. Search along the ζ= 0.5 line and find the operating point is at -1.5072 ± j2.6106 with K = 72.23. Hence, the estimated performance specifications for the compensated system are: $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100$ = 16.3%; $T_s = \frac{4}{1.5072}$ = 2.65 seconds; $K_p = \infty$. Higher-order poles are located at -0.0728 and -10.9125. The compensated system should be simulated to ensure effective pole/zero cancellation.

2.

a. Insert a cascade compensator, such as $G_c(s) = \frac{s+0.01}{s}$.

b.

Program:

```
K=1
G1=zpk([],[0,-2,-5],K)  %G1=1/s(s+2)(s+5)
Gc=zpk([-0.01],[0],1)   %Gc=(s+0.01)/s
G=G1*Gc
rlocus(G)
T=feedback(G,1)
T1=tf(1,[1,0])          %Form 1/s to integrate step input
T2=T*T1
t=0:0.1:200;
step(T1,T2,t)           %Show input ramp and ramp response
```

Computer response:

```
K =

     1


Zero/pole/gain:
      1
-------------
s (s+2) (s+5)
```

```
Zero/pole/gain:
(s+0.01)
--------
   s


Zero/pole/gain:
   (s+0.01)
---------------
s^2 (s+2) (s+5)

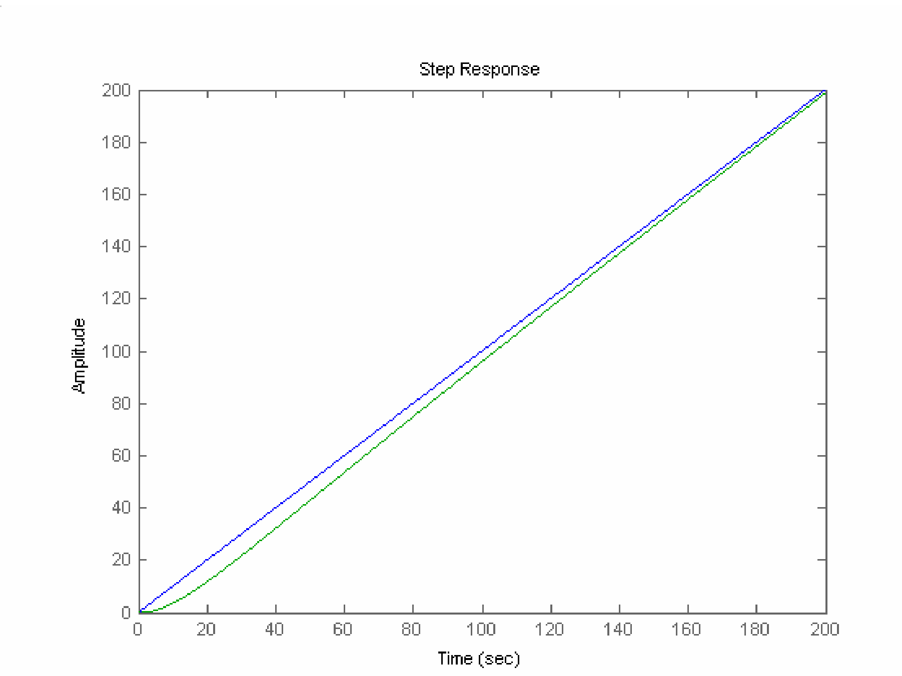

Zero/pole/gain:

                 (s+0.01)
------------------------------------------
(s+5.064) (s+1.829) (s+0.09593)
                                 (s+0.01126)


Transfer function:
1
-
s


Zero/pole/gain:

                 (s+0.01)
------------------------------------------
s (s+5.064) (s+1.829) (s+0.09593)
                                 (s+0.01126)
```



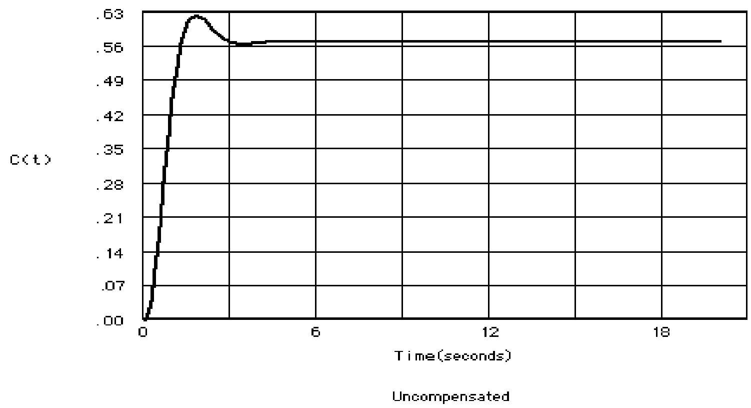
3.

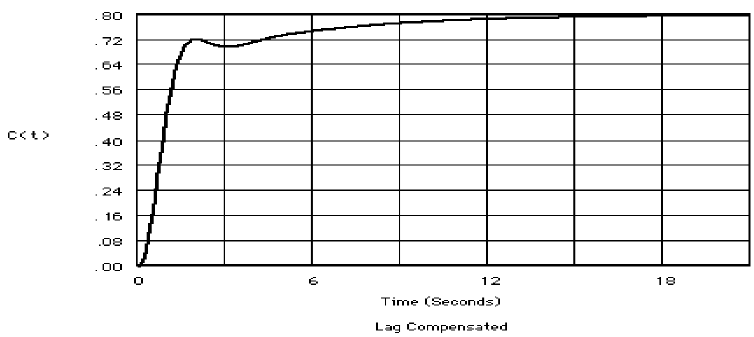
a. Searching along the 126.16° line (10% overshoot, $\zeta = 0.59$), find the operating point at

-1.4 + j1.92 with K = 20. Hence, $K_p = \frac{20}{1 \times 5 \times 3} = 1.333$.

b. A 3x improvement will yield $K_p = 4$. Use a lag compensator, $G_c(s) = \frac{s+0.3}{s+0.1}$.

c.





4.

a. Searching along the 126.16° line (10% overshoot, $\zeta = 0.59$), find the operating point at

-1.009 + j1.381 with $K = 17.5$. Hence, $K_v = \frac{17.5}{5 \times 3} = 1.1667$.

b. A 3.429x improvement will yield $K_v = 4$. Use a lag compensator, $G_c(s) = \frac{s+0.3429}{s+0.1}$.

c.

Program:
```
K=17.5
G=zpk([],[0,-3,-5],K)
Gc=zpk([-0.3429],[-0.1],1)
Ge=G*Gc;
T1=feedback(G,1);
T2=feedback(Ge,1);
T3=tf(1,[1,0]);          %Form 1/s to integrate step input
T4=T1*T3;
T5=T2*T3;
t=0:0.1:20;
step(T3,T4,T5,t)         %Show input ramp and ramp responses
```

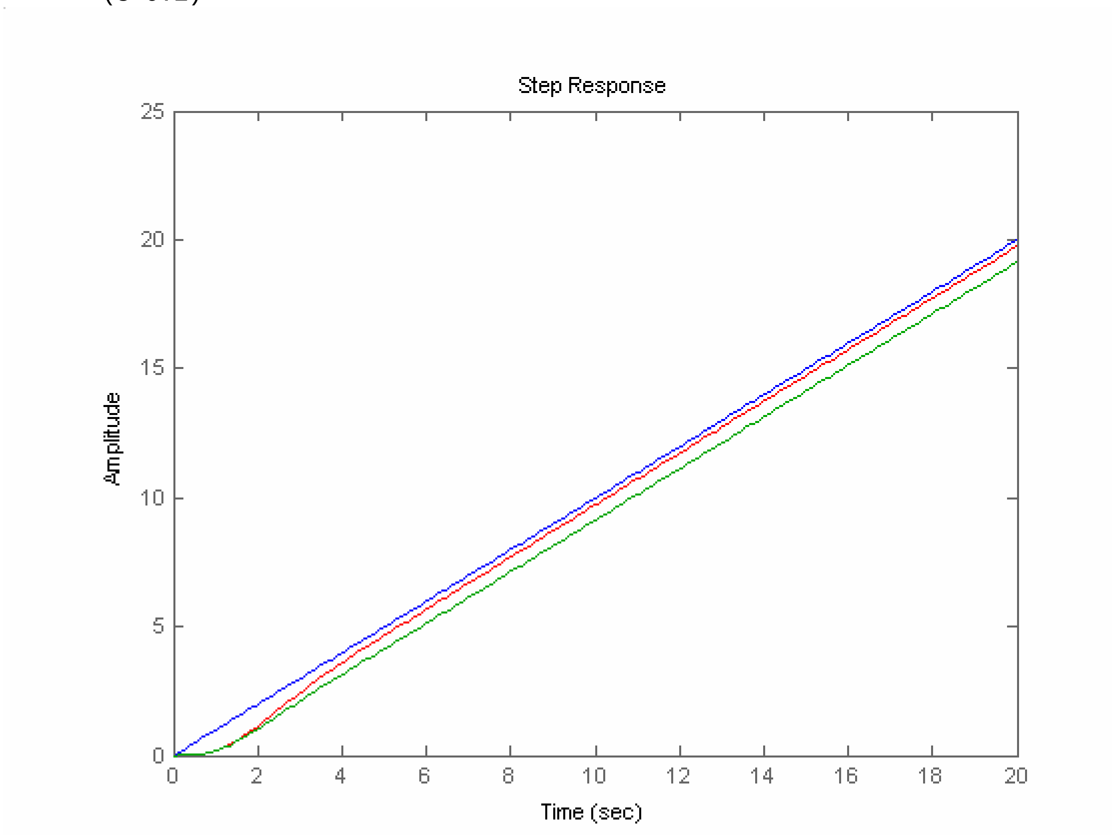
Computer response:

```
K =

   17.5000


Zero/pole/gain:
    17.5
-------------
s (s+3) (s+5)

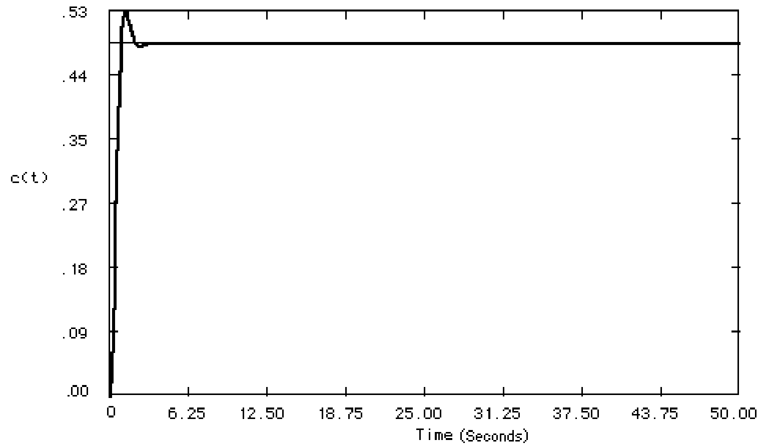

Zero/pole/gain:
(s+0.3429)
----------
 (s+0.1)
```



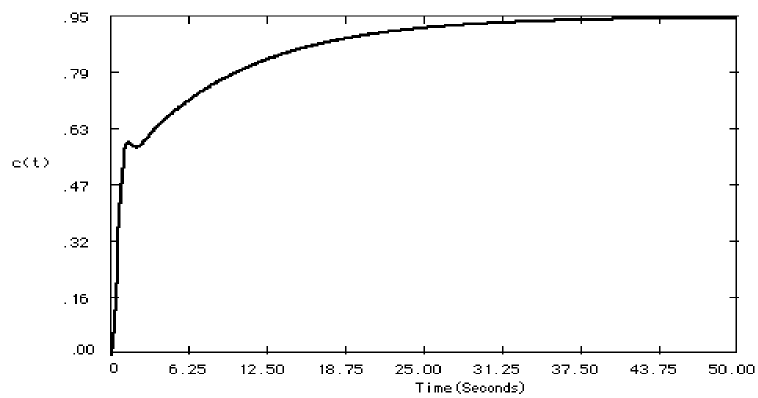
5.

a. Uncompensated: Searching along the 126.16^{o} line (10% overshoot, $\zeta = 0.59$), find the operating point at -2.03 + j2.77 with K = 45.72. Hence, $K_p = \frac{45.72}{2 \text{ x } 4 \text{ x } 6} = 0.9525$. An improvement of $\frac{20}{0.9525} = 20.1$ is required. Let $G_c(s) = \frac{0.201}{0.01}$. Compensated: Searching along the 126.16^{o} line (10% overshoot, $\zeta = 0.59$), find the operating point at - 1.99+j2.72 with K = 46.05. Hence, $K_p = \frac{46.05 \text{ x } 0.201}{2 \text{ x } 4 \text{ x } 6 \text{ x } 0.01} = 19.28$.

b.



Uncompensated



Lag Compensated

c. From (b), about 28 seconds

6.

Uncompensated: Searching along the 135° line (ζ = 0.707), find the operating point at

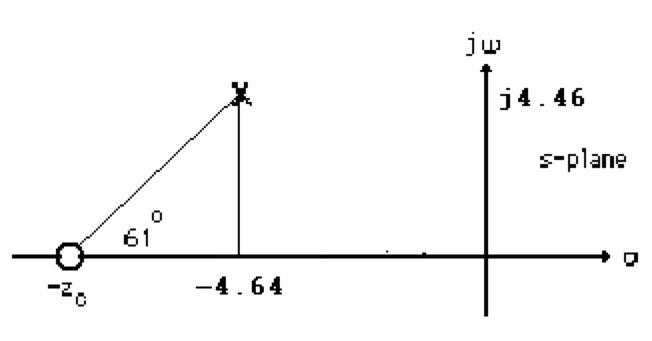
-2.32 + j2.32 with K = 4.6045. Hence, $K_p = \frac{4.6045}{30} = 0.153$; $T_s = \frac{4}{2.32} = 1.724$ seconds; $T_p =$

$\frac{\pi}{2.32} = 1.354$ seconds; $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 4.33\%$;

$\omega_n = \sqrt{2.32^2 + 2.32^2} = 3.28$ rad/s; higher-order pole at -5.366.

Compensated: To reduce the settling time by a factor of 2, the closed-loop poles should be – 4.64 ± j4.64. The summation of angles to this point is 119° . Hence, the contribution of the compensating zero should be 180° -119° =61° . Using the geometry shown below,

$\frac{4.64}{z_c - 4.64} = \tan(61°)$. Or, $z_c = 7.21$.



After adding the compensator zero, the gain at -4.64+j4.64 is K = 4.77. Hence,

$K_p = \frac{4.77x6x7.21}{2x3x5} = 6.88$. $T_s = \frac{4}{4.64} = 0.86$ second; $T_p = \frac{\pi}{4.64} = 0.677$ second;

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 4.33\%$; $\omega_n = \sqrt{4.64^2 + 4.64^2} = 6.56$ rad/s; higher-order pole at -5.49. The problem with the design is that there is steady-state error, and no effective pole/zero cancellation. The design should be simulated to be sure the transient requirements are met.

7.

Program:

```
clf
'Uncompensated System'
numg=[1 6];
deng=poly([-2 -3 -5]);
'G(s)'
G=tf(numg,deng);
Gzpk=zpk(G)
rlocus(G,0:1:100)
z=0.707;
pos=exp(-pi*z/sqrt(1-z^2))*100;
sgrid(z,0)
title(['Uncompensated Root Locus with ' , num2str(z), ' Damping Ratio
Line'])
[K,p]=rlocfind(G);  %Allows input by selecting point on graphic
'Closed-loop poles = '
p
i=input('Give pole number that is operating point   ');
'Summary of estimated specifications'
operatingpoint=p(i)
gain=K
estimated_settling_time=4/abs(real(p(i)))
estimated_peak_time=pi/abs(imag(p(i)))
estimated_percent_overshoot=pos
estimated_damping_ratio=z
estimated_natural_frequency=sqrt(real(p(i))^2+imag(p(i))^2)
Kp=dcgain(K*G)
'T(s)'
T=feedback(K*G,1)
'Press any key to continue and obtain the step response'
pause
step(T)
title(['Step Response for Uncompensated System with  ' , num2str(z),...
' Damping Ratio'])
'Press any key to go to PD compensation'
pause
'Compensated system'
```

```
done=1;
while done>0
a=input('Enter a Test PD Compensator, (s+a). a =     ')
numc=[1 a];
'Gc(s)'
GGc=tf(conv(numg,numc),deng);
GGczpk=zpk(GGc)
wn=4/[(estimated_settling_time/2)*z];
rlocus(GGc)
sgrid(z,wn)
title(['PD Compensated Root Locus with ' , num2str(z),...
' Damping Ratio Line', 'PD Zero at ', num2str(a), ', and Required Wn'])
done=input('Are you done? (y=0,n=1)  ');
end
[K,p]=rlocfind(GGc);  %Allows input by selecting point on graphic
'Closed-loop poles = '
p
i=input('Give pole number that is operating point   ');
'Summary of estimated specifications'
operatingpoint=p(i)
gain=K
estimated_settling_time=4/abs(real(p(i)))
estimated_peak_time=pi/abs(imag(p(i)))
estimated_percent_overshoot=pos
estimated_damping_ratio=z
estimated_natural_frequency=sqrt(real(p(i))^2+imag(p(i))^2)
Kp=dcgain(K*GGc)
'T(s)'
T=feedback(K*GGc,1)
'Press any key to continue and obtain the step response'
pause
step(T)
title(['Step Response for Compensated System with  ' , num2str(z),...
' Damping Ratio'])
```

Computer response:

```
ans =

Uncompensated System


ans =

G(s)


Zero/pole/gain:
      (s+6)
-----------------
(s+5) (s+3) (s+2)

Select a point in the graphics window

selected_point =

  -2.3104 + 2.2826i


ans =

Closed-loop poles =


p =

  -5.3603
  -2.3199 + 2.2835i
```

```
  -2.3199 - 2.2835i

Give pole number that is operating point   2

ans =

Summary of estimated specifications

operatingpoint =

  -2.3199 + 2.2835i

gain =

    4.4662

estimated_settling_time =

    1.7242

estimated_peak_time =

    1.3758

estimated_percent_overshoot =

    4.3255

estimated_damping_ratio =

    0.7070

estimated_natural_frequency =

    3.2552

Kp =

    0.8932

ans =

T(s)

Transfer function:
       4.466 s + 26.8
-----------------------------
s^3 + 10 s^2 + 35.47 s + 56.8

ans =

Press any key to continue and obtain the step response

ans =
```

```
Press any key to go to PD compensation


ans =

Compensated system

Enter a Test PD Compensator, (s+a). a =     6

a =

     6


ans =

Gc(s)


Zero/pole/gain:
     (s+6)^2
-----------------
(s+5) (s+3) (s+2)

Are you done? (y=0,n=1)  1
Enter a Test PD Compensator, (s+a). a =     7.1

a =

    7.1000


ans =

Gc(s)


Zero/pole/gain:
  (s+7.1) (s+6)
-----------------
(s+5) (s+3) (s+2)

Are you done? (y=0,n=1)  0
Select a point in the graphics window

selected_point =

  -4.6607 + 4.5423i


ans =

Closed-loop poles =


p =

  -4.6381 + 4.5755i
  -4.6381 - 4.5755i
  -5.4735

Give pole number that is operating point   1

ans =

Summary of estimated specifications
```

```
operatingpoint =

  -4.6381 + 4.5755i

gain =

    4.7496

estimated_settling_time =

    0.8624

estimated_peak_time =

    0.6866

estimated_percent_overshoot =

    4.3255

estimated_damping_ratio =

    0.7070

estimated_natural_frequency =

    6.5151

Kp =

    6.7444

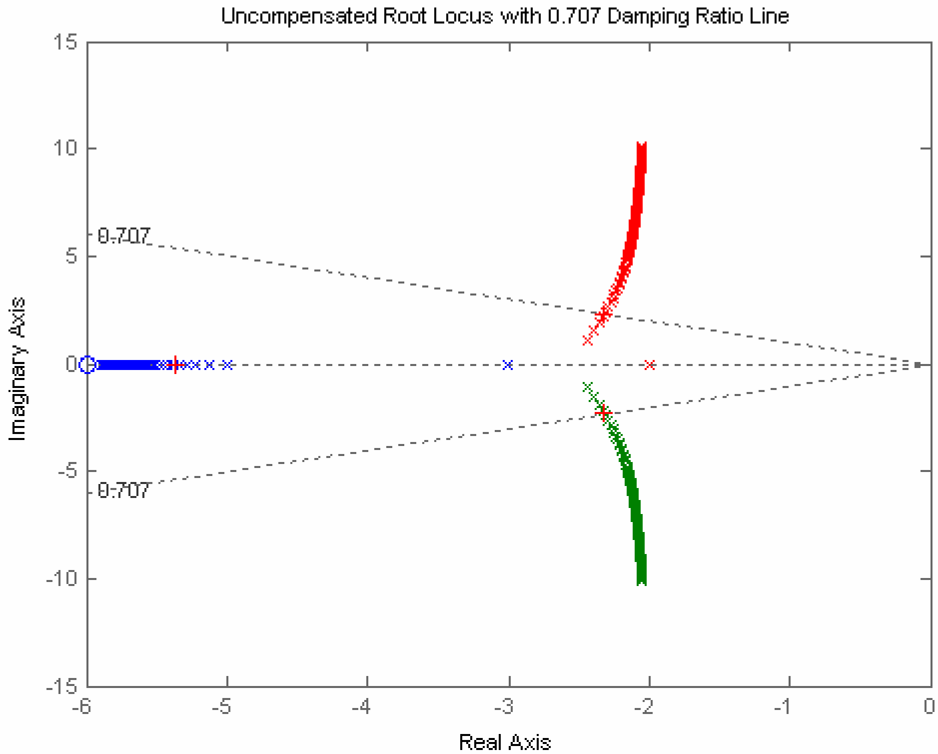
ans =

T(s)

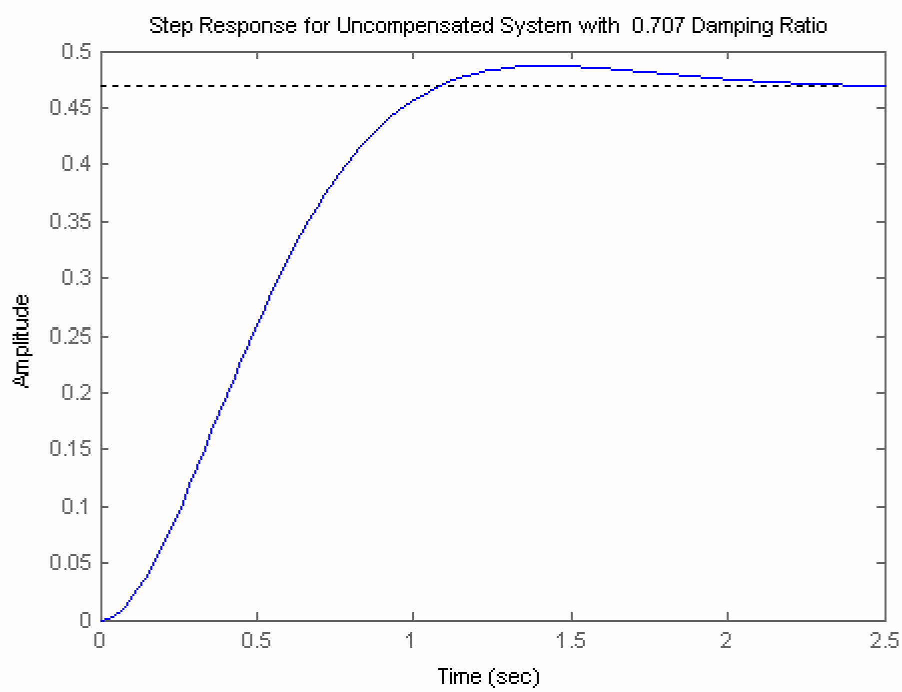
Transfer function:
   4.75 s^2 + 62.22 s + 202.3
---------------------------------
s^3 + 14.75 s^2 + 93.22 s + 232.3

ans =

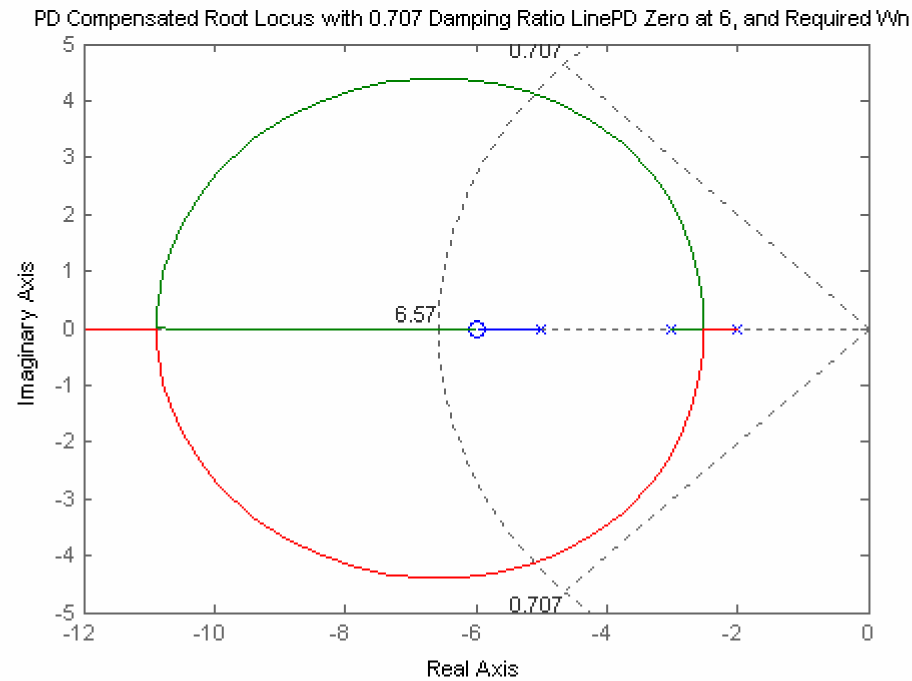
Press any key to continue and obtain the step response
```

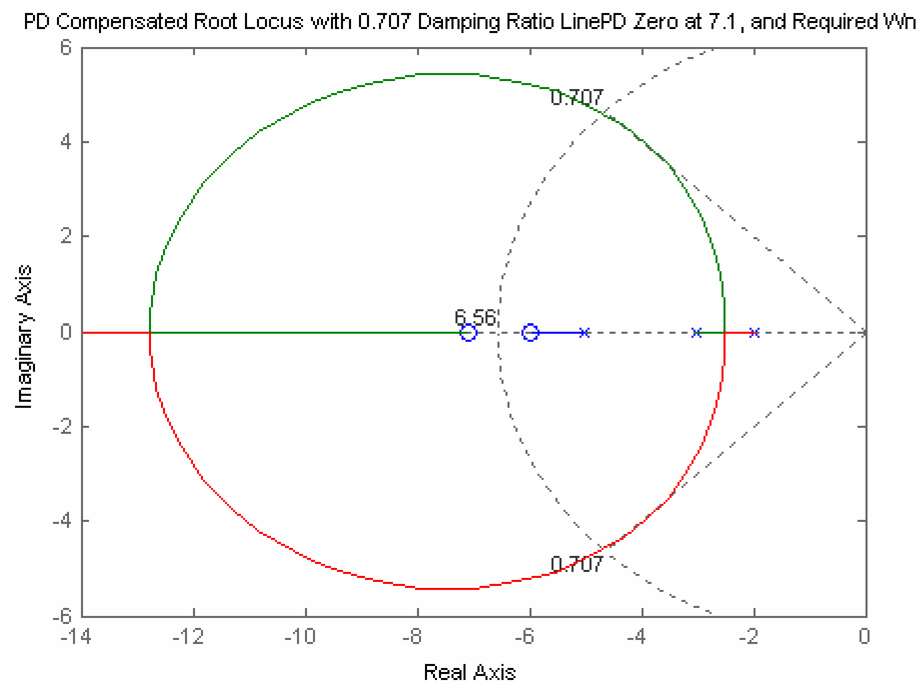


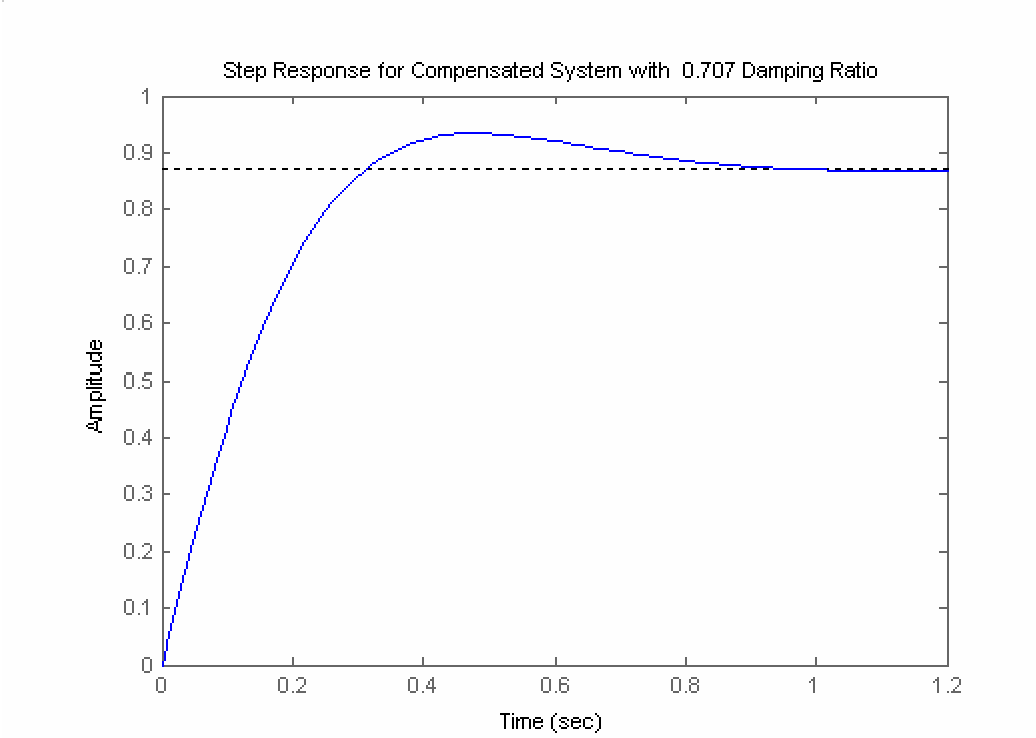
Uncompensated Root Locus with 0.707 Damping Ratio Line
Imaginary Axis
Real Axis
0.707
0.707
15
10
5
0
-5
-10
-15
-6
-5
-4
-3
-2
-1
0



Step Response for Uncompensated System with 0.707 Damping Ratio
Amplitude
Time (sec)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2
2.5

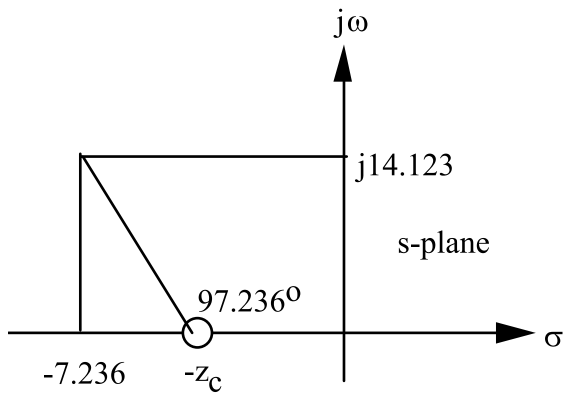

PD Compensated Root Locus with 0.707 Damping Ratio LinePD Zero at 6, and Required Wn
0.707
6.57
0.707
Imaginary Axis
Real Axis
5
4
3
2
1
0
-1
-2
-3
-4
-5
-12
-10
-8
-6
-4
-2
0


PD Compensated Root Locus with 0.707 Damping Ratio LinePD Zero at 7.1, and Required Wn
0.707
6.56
0.707
Imaginary Axis
Real Axis
6
4
2
0
-2
-4
-6
-14
-12
-10
-8
-6
-4
-2
0



8.

The uncompensated system performance is summarized in Table 9.8 in the text. To improve settling time by 4, the dominant poles need to be at -7.236 ± j14.123. Summing the angles from the open-loop poles to the design point yields -277.326°. Thus, the zero must contribute 277.326° - 180° = 97.326°. Using the geometry below,



$\frac{14.123}{7.236 - z_c}$ = tan(180-97.326). Thus, z_c = 5.42. Adding the zero and evaluating the gain at the design point yields K = 256.819. Summarizing results:

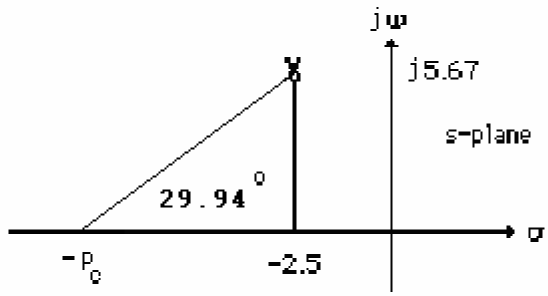
	Uncompensated	Compensated
Plant and compensator	$\frac{K}{s(s+5)(s+15)}$	$\frac{K(s+5.42)}{s(s+5)(s+15)}$
Feedback	1	1
Dominant poles	-1.809 ± j3.531	-7.236 ± j14.123
K	257.841	256.819
ζ	0.456	0.456
ω_n	3.97	15.88
%OS	20	20
T_s	2.21	0.55
T_p	0.89	0.22
K_v	3.44	18.559
$e(\infty)$ (ramp)	0.29	0.0539
Other poles	-16.4	-5.528
Zero	None	-5.42
Comments	Second-order approx. OK	Second-order approx. OK by assuming pole/zero cancellation.

9.

a. $\zeta\omega_n = \frac{4}{T_s} = 2.5$; $\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.404$. Thus, $\omega_n = 6.188$ rad/s and the operating point is $-2.5 \pm j5.67$.

b. Summation of angles including the compensating zero is -150.06^o. Therefore, the compensator pole must contribute $150.06^o - 180^o = -29.94^o$.

c. Using the geometry shown below, $\frac{5.67}{p_c - 2.5} = \tan 29.94^o$. Thus, $p_c = 12.34$.

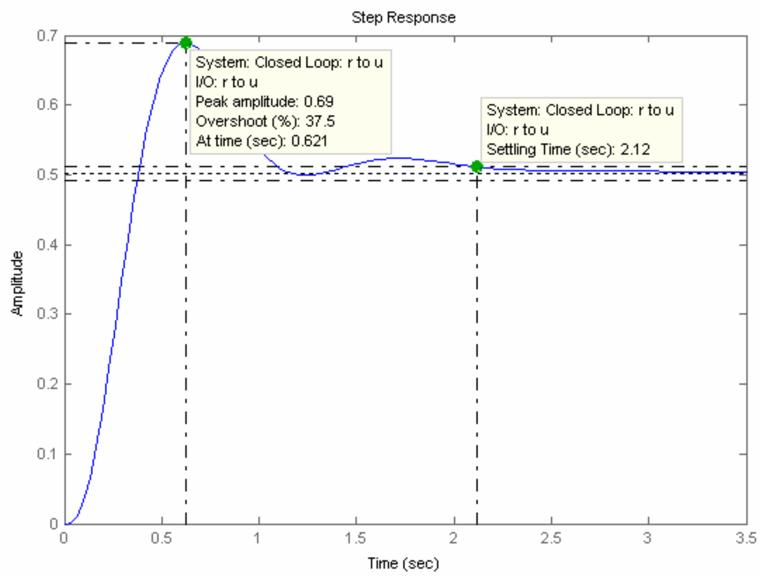


d. Adding the compensator pole and using -2.5 + j5.67 as the test point, K = 357.09.

e. Searching the real axis segments for K = 1049.41, we find higher-order poles at -15.15, and -1.186.

f. Pole at -15.15 is more than 5 times further from the imaginary axis than the dominant poles. Pole at -1.186 may not cancel the zero at -1

g.



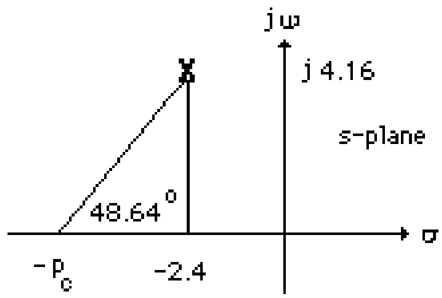
A simulation of the system shows a percent overshoot of 37.5% and a settling time of 2.12 seconds. Thus, the specifications were not met because pole-zero cancellation was not achieved. A redesign is required.

10.

a. $\zeta\omega_n = \frac{4}{T_s} = 2.4$; $\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.5$. Thus, $\omega_n = 4.799$ rad/s and the operating point is -2.4 ± j4.16.

b. Summation of angles including the compensating zero is -131.36^o. Therefore, the compensator pole must contribute $180^o - 131.36^o = -48.64^o$. Using the geometry shown below, $\frac{4.16}{p_c - 2.4}$ =

tan 48.64°. Thus, p_c = 6.06.



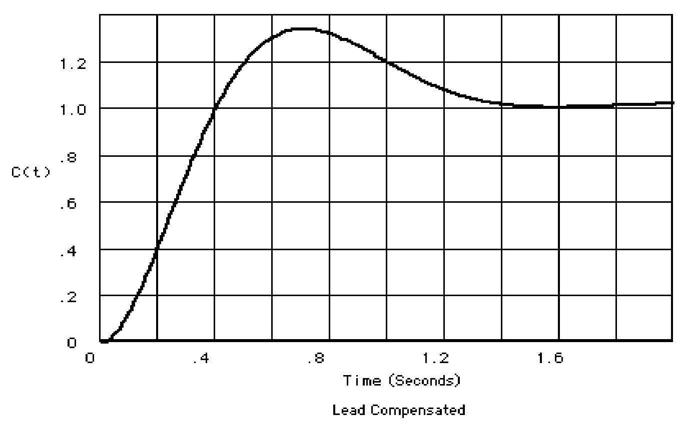
c. Adding the compensator pole and using -2.4 + j4.16 as the test point, K = 29.117.

d. Searching the real axis segments for K = 29.117, we find a higher-order pole at -1.263.

e. Pole at -1.263 is near the zero at -1. Simulate to ensure accuracy of results.

f. $K_a = \frac{29.117}{6.06} = 4.8$

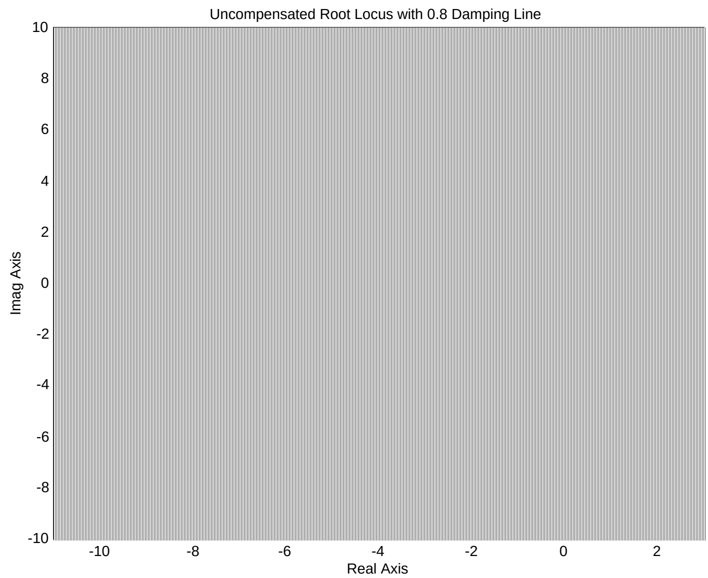
g.



From the plot, T_s = 1.4 seconds; T_p = 0.68 seconds; %OS = 35%.

11.

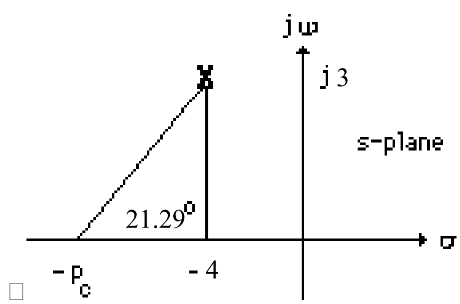
a.



b. and **c.** Searching along the $\zeta = 0.8$ line (143.13°), find the operating point at –2.682 + j2.012 with K = 35.66.

d. Since $\zeta\omega_n = \frac{4}{T_s}$, the real part of the compensated dominant pole is -4. The imaginary part is

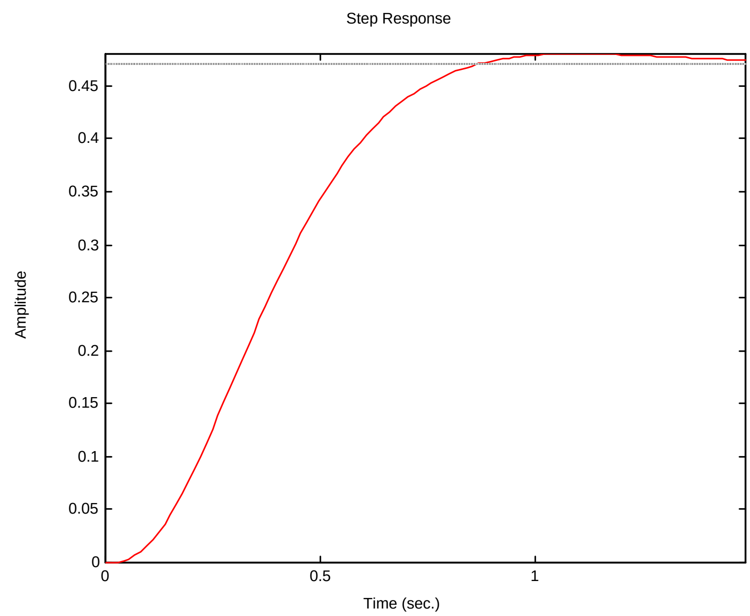
4 tan (180°-143.13°) = 3. Using the uncompensated system's poles and zeros along with the compensator zero at - 4.5, the summation of angles to the design point, -4 + j3 is –158.71°. Thus, the contribution of the compensator pole must be 158.71° - 180° = -21.29°. Using the following geometry, $\frac{3}{p_c - 4}$ = tan 21.29°, or pc = 11.7.



Adding the compensator pole and using – 4 + j3 as the test point, K = 172.92.

e. Compensated: Searching the real axis segments for K = 172.92, we find higher-order poles at 14.19, and approximately at –5.26 ± j0.553. Since there is no pole/zero cancellation with the zeros at -6 and –4.5, the system should be simulated to check the settling time.

f.



The graph shows about 2% overshoot and a 0.8 second settling time compared to a desired 1.52% overshoot and a settling time of 1 second.

12.

Program:

```
clf
numg=[1 6];
deng=poly([-2 -4 -7 -8]);
'G(s)'
G=tf(numg,deng);
Gzpk=zpk(G)
rlocus(G)
z=0.8;
pos=exp(-pi*z/sqrt(1-z^2))*100;
sgrid(z,0)
title(['Uncompensated Root Locus with ' , num2str(z), ' Damping Ratio Line'])
[K,p]=rlocfind(G);
'Closed-loop poles = '
p
i=input('Give pole number that is operating point   ');
'Summary of estimated specifications'
operatingpoint=p(i)
gain=K
estimated_settling_time=4/abs(real(p(i)))
estimated_peak_time=pi/abs(imag(p(i)))
estimated_percent_overshoot=pos
estimated_damping_ratio=z
estimated_natural_frequency=sqrt(real(p(i))^2+imag(p(i))^2)
Kp=K*numg(max(size(numg)))/deng(max(size(deng)))
'T(s)'
T=feedback(K*G,1)
'Press any key to continue and obtain the step response'
pause
step(T)
title(['Step Response for Uncompensated System with  ' , num2str(z),...
' Damping Ratio'])
'Press any key to go to Lead compensation'
pause
```

```
'Compensated system'
b=4.5;
'Lead Zero at -4.5 '
done=1;
while done>0
a=input('Enter a Test Lead Compensator Pole, (s+a). a =     ');
'Gc(s)'
Gc=tf([1 b],[1 a])
GGc=G*Gc;
[numggc,denggc]=tfdata(GGc,'v');
'G(s)Gc(s)'
GGczpk=zpk(GGc)
wn=4/((1)*z);
rlocus(GGc);
sgrid(z,wn)
title(['Lead Compensated Root Locus with ' , num2str(z),...
' Damping Ratio Line, Lead Pole at  ', num2str(-a), ', and Required Wn'])
done=input('Are you done? (y=0,n=1)  ');
end
[K,p]=rlocfind(GGc);  %Allows input by selecting point on graphic
'Closed-loop poles = '
p
i=input('Give pole number that is operating point   ');
'Summary of estimated specifications'
operatingpoint=p(i)
gain=K
estimated_settling_time=4/abs(real(p(i)))
estimated_peak_time=pi/abs(imag(p(i)))
estimated_percent_overshoot=pos
estimated_damping_ratio=z
estimated_natural_frequency=sqrt(real(p(i))^2+imag(p(i))^2)
Kp=dcgain(K*GGc)
'T(s)'
T=feedback(K*GGc,1)
'Press any key to continue and obtain the step response'
pause
step(T)
title(['Step Response for Compensated System with  ' , num2str(z),...
' Damping Ratio'])
```

Computer response:

```
ans =

G(s)


Zero/pole/gain:
         (s+6)
-----------------------
(s+8) (s+7) (s+4) (s+2)

Select a point in the graphics window

selected_point =

  -2.7062 + 2.0053i


ans =

Closed-loop poles =


p =

  -9.3056
  -6.3230
```

```
  -2.6857 + 2.0000i
  -2.6857 - 2.0000i

Give pole number that is operating point   3

ans =

Summary of estimated specifications


operatingpoint =

  -2.6857 + 2.0000i


gain =

   35.2956


estimated_settling_time =

    1.4894


estimated_peak_time =

    1.5708


estimated_percent_overshoot =

    1.5165


estimated_damping_ratio =

    0.8000


estimated_natural_frequency =

    3.3486


Kp =

    0.4727


ans =

T(s)


Transfer function:
             35.3 s + 211.8
----------------------------------------
s^4 + 21 s^3 + 154 s^2 + 491.3 s + 659.8


ans =

Press any key to continue and obtain the step response


ans =
```

```
Press any key to go to Lead compensation


ans =

Compensated system


ans =

Lead Zero at -4.5

Enter a Test Lead Compensator Pole, (s+a). a =     10

ans =

Gc(s)


Transfer function:
s + 4.5
-------
s + 10


ans =

G(s)Gc(s)


Zero/pole/gain:
        (s+6) (s+4.5)
------------------------------
(s+10) (s+8) (s+7) (s+4) (s+2)

Are you done? (y=0,n=1)  1
Enter a Test Lead Compensator Pole, (s+a). a =     11.7

ans =

Gc(s)


Transfer function:
s + 4.5
--------
s + 11.7


ans =

G(s)Gc(s)


Zero/pole/gain:
         (s+6) (s+4.5)
--------------------------------
(s+11.7) (s+8) (s+7) (s+4) (s+2)

Are you done? (y=0,n=1)  0
Select a point in the graphics window

selected_point =

  -3.9885 + 3.0882i
```

```
ans =

Closed-loop poles =


p =

 -14.2326
  -3.9797 + 3.0860i
  -3.9797 - 3.0860i
  -5.2540 + 0.5076i
  -5.2540 - 0.5076i

Give pole number that is operating point   2

ans =

Summary of estimated specifications


operatingpoint =

  -3.9797 + 3.0860i


gain =

  178.3530


estimated_settling_time =

    1.0051


estimated_peak_time =

    1.0180


estimated_percent_overshoot =

    1.5165


estimated_damping_ratio =

    0.8000


estimated_natural_frequency =

    5.0360


Kp =

    0.9187


ans =

T(s)

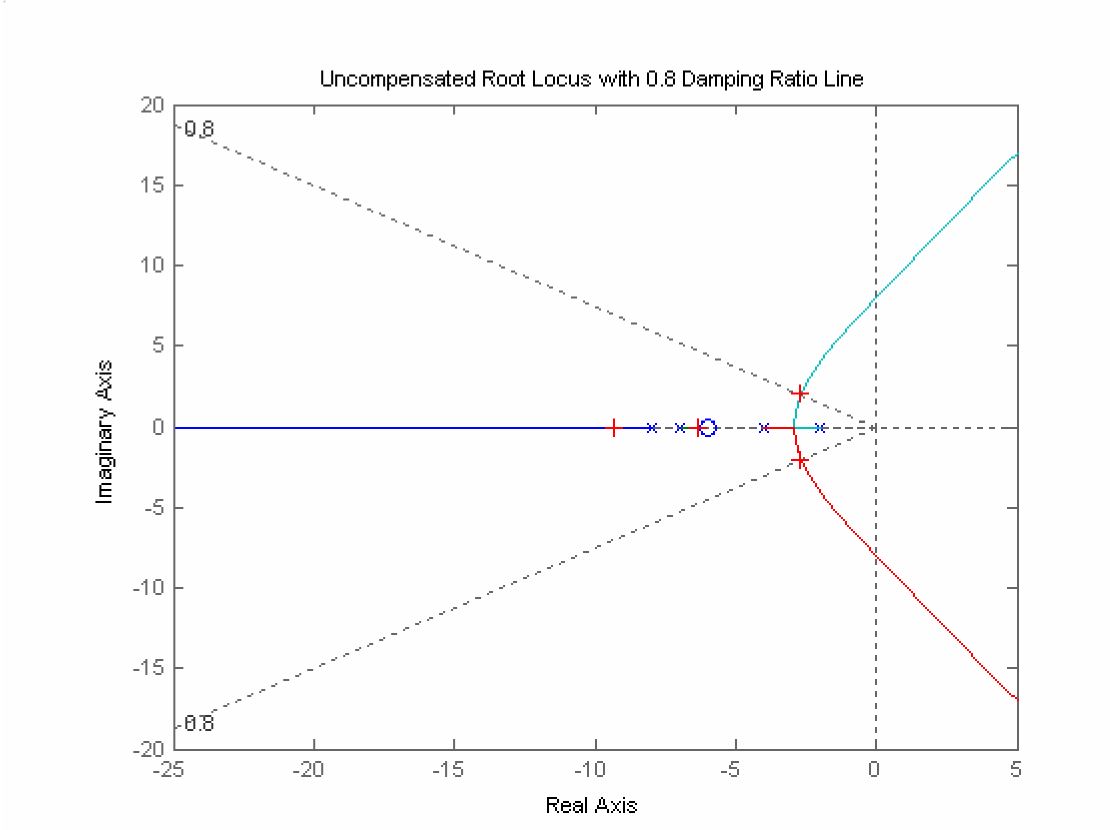

Transfer function:
```

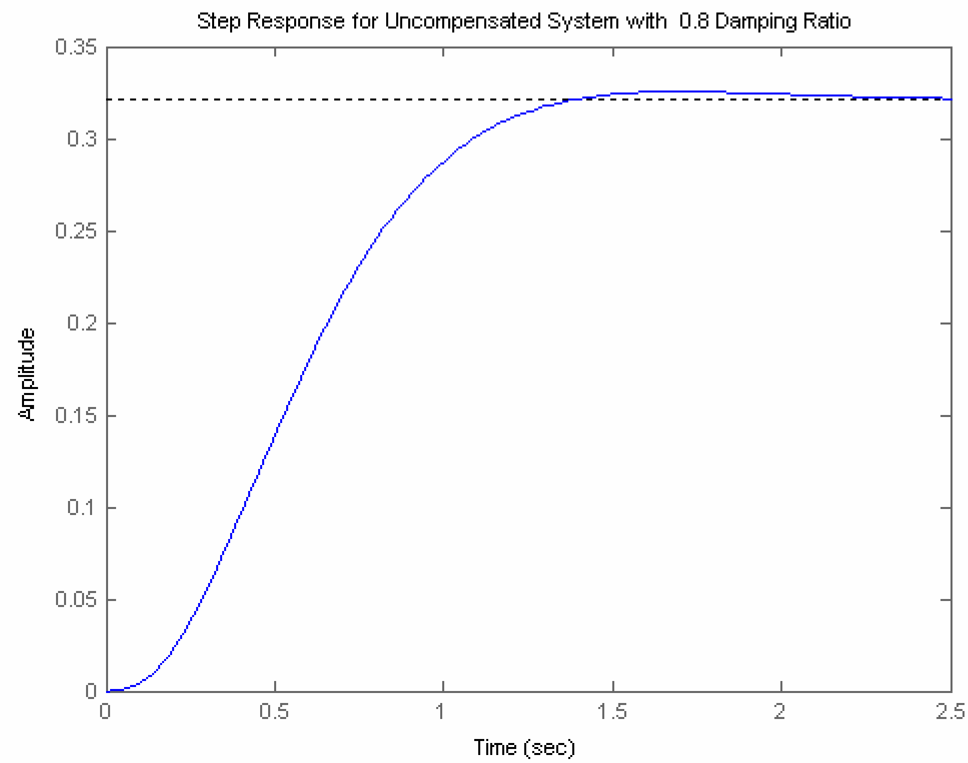
```
        178.4 s^2 + 1873 s + 4816
------------------------------------------
s^5 + 32.7 s^4 + 399.7 s^3 + 2436 s^2

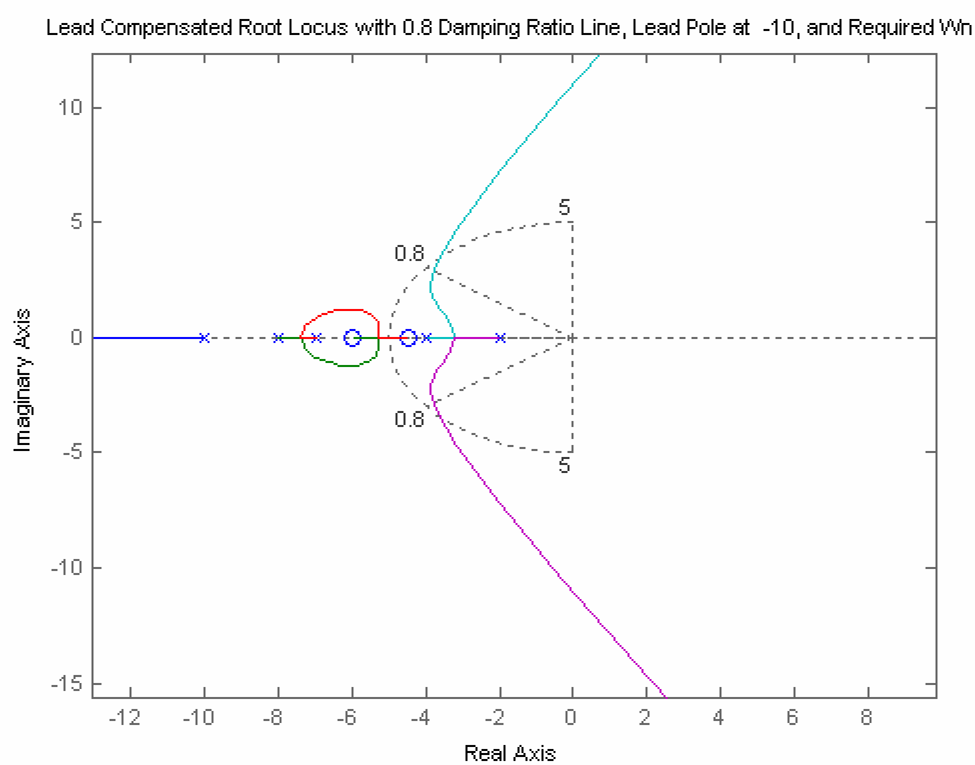
                      + 7656 s + 1.006e004

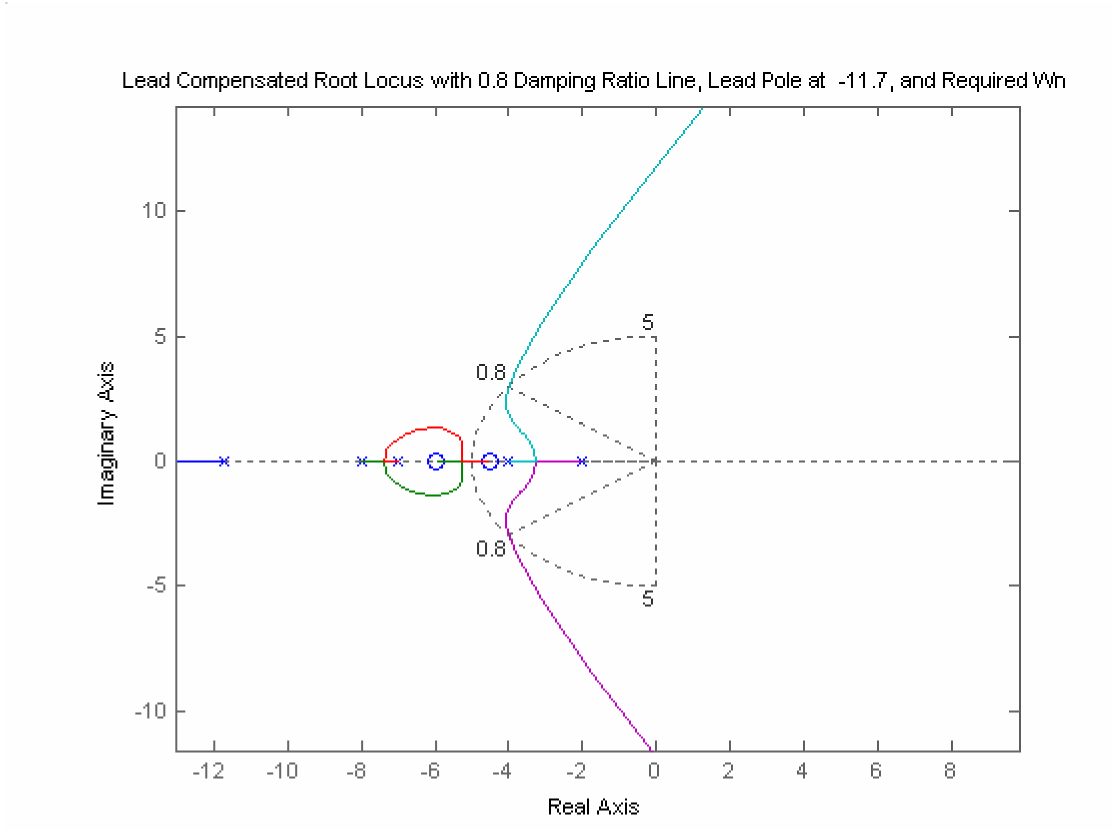

ans =

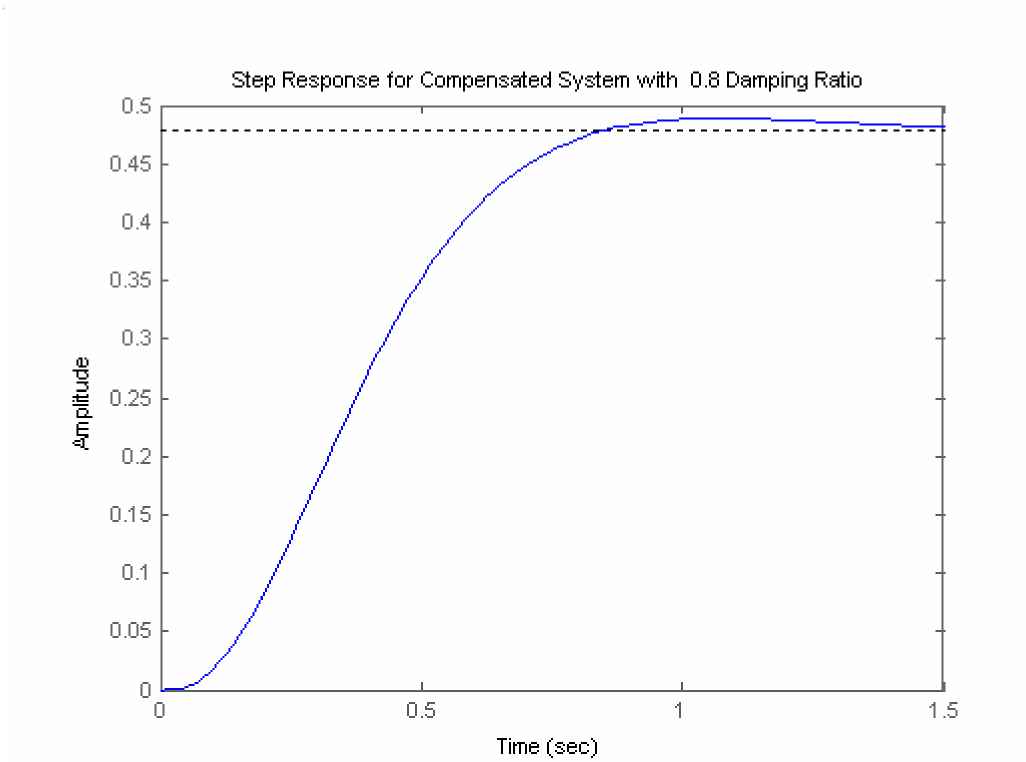
Press any key to continue and obtain the step
response
```




Step Response for Uncompensated System with 0.8 Damping Ratio
Amplitude
Time (sec)
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2
2.5


Lead Compensated Root Locus with 0.8 Damping Ratio Line, Lead Pole at -10, and Required Wn
Imaginary Axis
Real Axis
10
5
0
-5
-10
-15
-12
-10
-8
-6
-4
-2
0
2
4
6
8
0.8
0.8
5
5





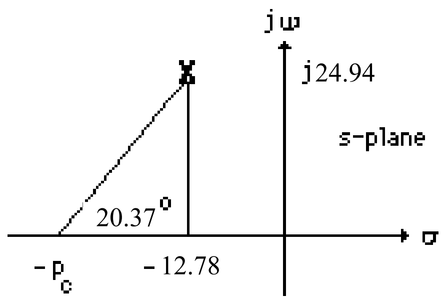
13.

a. Searching along the 117.13o line (%OS = 20%; ζ = 0.456), find the operating point at

-6.39 + j12.47 with K = 9273. Searching along the real axis for K = 9273, we find a higher-order pole at –47.22. Thus, $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{6.39} = 0.626$ second.

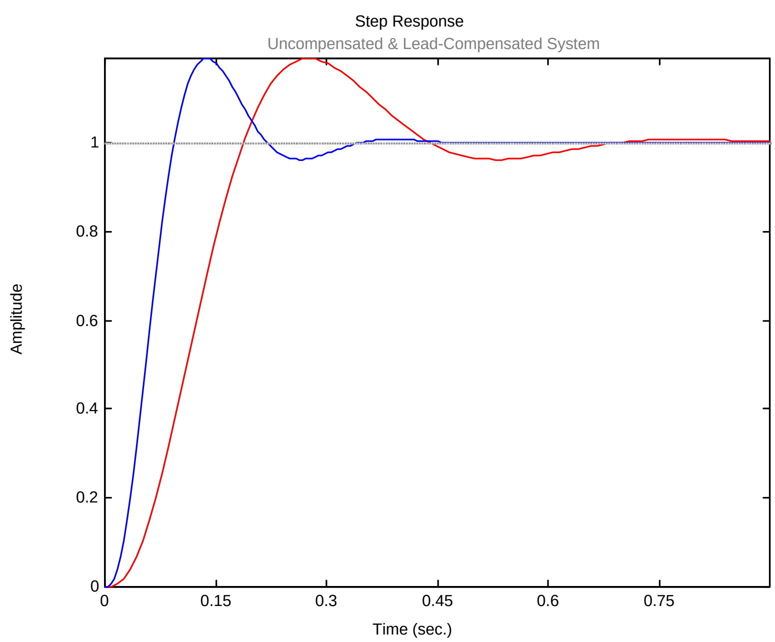
b. For the settling time to decrease by a factor of 2, Re = $-\zeta\omega_n$ = -6.39 x 2 = -12.78. The imaginary part is Im = -12.78 tan 117.13° = 24.94. Hence, the compensated closed-loop poles are -12.78 ± j24.94. A settling time of 0.313 second would result.

c. Assume a compensator zero at -20. Using the uncompensated system's poles along with the compensator zero, the summation of angles to the design point, -12.78 ± j24.94 is –159.63°. Thus, the contribution of the compensator pole must be 159.63°-180° = -20.37°. Using the following geometry, $\frac{24.94}{p_c - 12.78}$ = tan 20.37°, or p_c = 79.95.



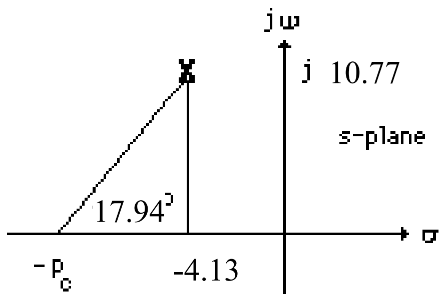
Adding the compensator pole and using --12.78 ± j24.94 as the test point, K = 74130.

d.



14.

a. Searching along the 110.97° line ($\%OS$ = 30%; ζ= 0.358), find the operating point at -2.065 + j5.388 with K = 366.8. Searching along the real axis for K = 366.8, we find a higher-order pole at –16.87. Thus, $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{2.065} = 1.937$ seconds. For the settling time to decrease by a factor of 2, Re = -$\zeta\omega_n$ = -2.065 x 2 = - 4.13. The imaginary part is – 4.13 tan 110.97^0 = 10.77. Hence, the compensated dominant poles are – 4.13 ± j10.77. The compensator zero is at -7. Using the uncompensated system's poles along with the compensator zero, the summation of angles to the design point, – 4.13 ± j10.77 is –162.06°. Thus, the contribution of the compensator pole must be – 162.06° - 180° = -17.94°. Using the following geometry, $\frac{10.77}{p_c - 4.13}$ = tan 17.94°, or p_c = 37.4.



Adding the compensator pole and using – 4.13 ± j10.77 as the test point, K = 5443.

b. Searching the real axis segments for K = 5443 yields higher-order poles at approximately –8.12 and –42.02. The pole at –42.02 can be neglected since it is more than five times further from the imaginary axis than the dominant pair. The pole at –8.12 may not be canceling the zero at -7. Hence, simulate to be sure the requirements are met.

c.

Program:

```
'Uncompensated System G1(s)'
numg1=1;
deng1=poly([-15 (-3+2*j) (-3-2*j)]);
G1=tf(numg1,deng1)
G1zpk=zpk(G1)
K1=366.8
'T1(s)'
T1=feedback(K1*G1,1);
T1zpk=zpk(T1)
'Compensator Gc(s)'
numc=[1 7];
denc=[1 37.4];
Gc=tf(numc,denc)
'Compensated System G2(s) = G1(s)Gc(s)'
K2=5443
G2=G1*Gc;
G2zpk=zpk(G2)
'T2(s)'
T2=feedback(K2*G2,1);
T2zpk=zpk(T2)
step(T1,T2)
title(['Uncompensated and Lead Compensated Systems'])
```

Computer response:

```
ans =

Uncompensated System G1(s)


Transfer function:
            1
--------------------------
s^3 + 21 s^2 + 103 s + 195


Zero/pole/gain:
           1
----------------------
(s+15) (s^2  + 6s + 13)


K1 =

  366.8000


ans =

T1(s)


Zero/pole/gain:
              366.8
--------------------------------
(s+16.87) (s^2  + 4.132s + 33.31)


ans =

Compensator Gc(s)


Transfer function:
 s + 7
--------
s + 37.4


ans =

Compensated System G2(s) = G1(s)Gc(s)


K2 =

        5443

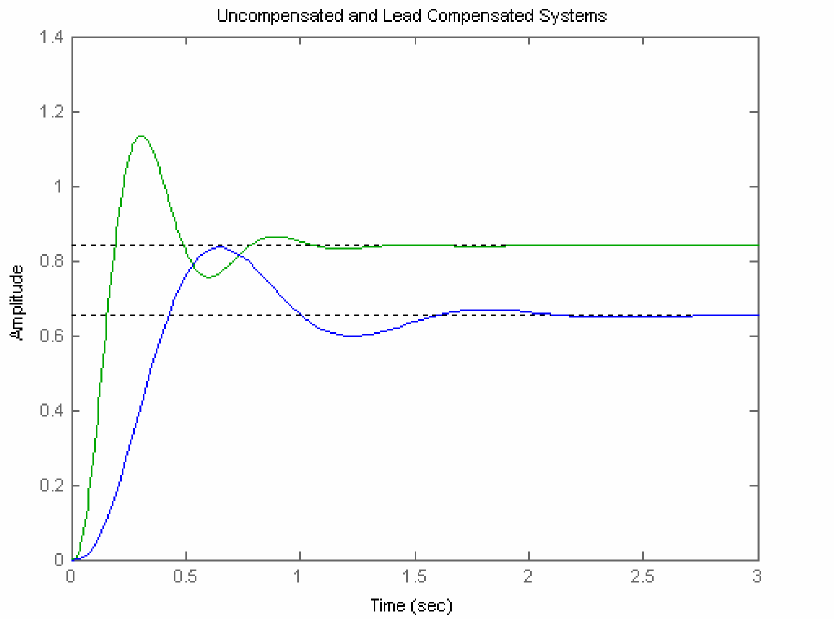

Zero/pole/gain:
             (s+7)
-------------------------------
(s+37.4) (s+15) (s^2  + 6s + 13)


ans =

T2(s)


Zero/pole/gain:
```

```
                   5443 (s+7)
-------------------------------------------
(s+42.02) (s+8.118) (s^2  + 8.261s + 133.1)
```

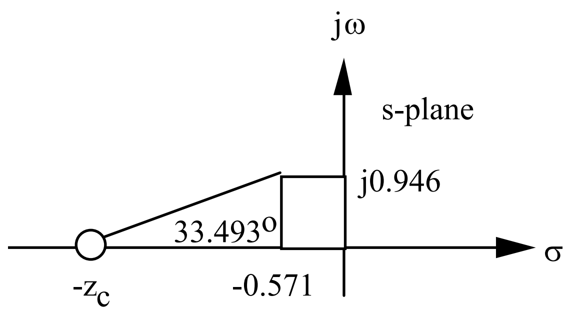


15.

a. Searching the 15% overshoot line (121.127°) for 180° yields -0.372 + j0.615. Hence, $T_s = \frac{4}{\sigma_d} = \frac{4}{0.372} = 10.75$ seconds.

b. For 7 seconds settling time, $\sigma_d = \frac{4}{T_s} = \frac{4}{7} = 0.571$. $\omega_d = 0.571 \tan(180^\circ - 121.127^\circ) = 0.946$.

Therefore, the design point is -0.571 + j0.946. Summing the angles of the uncompensated system's poles as well as the compensator pole at -15 yields -213.493°. Therefore, the compensator zero must contribute (213.493° - 180°) = 33.493°. Using the geometry below,



$\frac{0.946}{z_c - 0.571} = \tan(33.493^\circ)$. Hence, $z_c = 2$. The compensated open-loop transfer function is

$\frac{K(s+2)}{s(s+1)(s^2+10s+26)(s+15)}$. Evaluating the gain for this function at the point, -0.571 + j0.946 yields K = 207.512.

c.

Program:

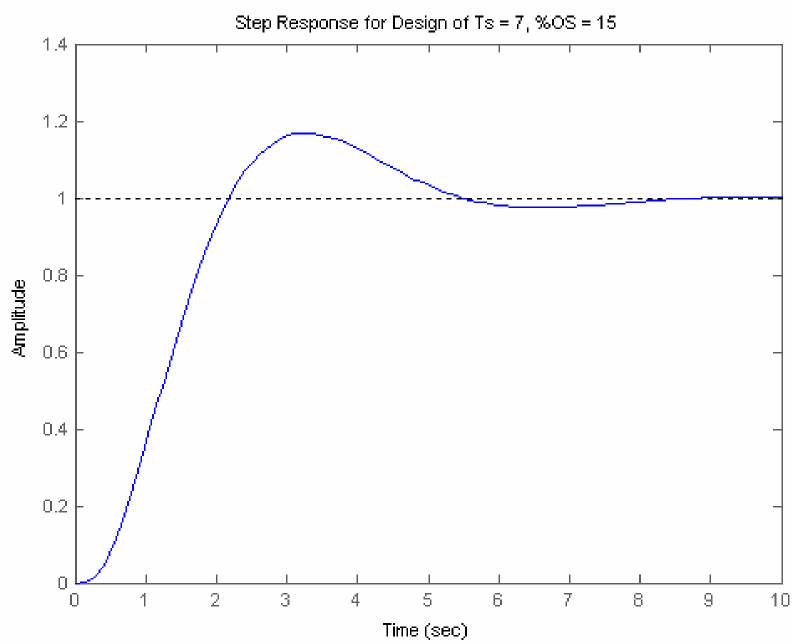
```
numg= 207.512*[1 2];
r=roots([1,10,26]);
deng=poly([0 ,-1, r(1),r(2),-15]);
'G(s)'
G=tf(numg,deng);
Gzpk=zpk(G)
T=feedback(G,1);
step(T)
title(['Step Response for Design of Ts = 7, %OS = 15'])
```

Computer response:

```
ans =

G(s)

Zero/pole/gain:
                  207.512 (s+2)
          -------------------------------
          s (s+15) (s+1) (s^2 + 10s + 26)
```

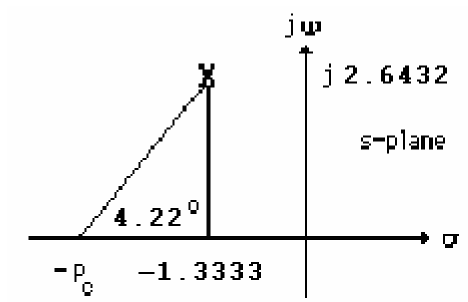


16.

a. From 20.5% overshoot evaluate $\zeta = 0.45$. Also, since $\zeta\omega_n = \frac{4}{T_s} = \frac{4}{3}$, $\omega_n = 2.963$. The compensated dominant poles are located at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ = - 1.3333 ± j2.6432. Assuming the compensator zero at -0.02, the contribution of open-loop poles and the compensator zero to the design point, - 1.3333 ± j2.6432 is -175.78^o. Hence, the compensator pole must contribute

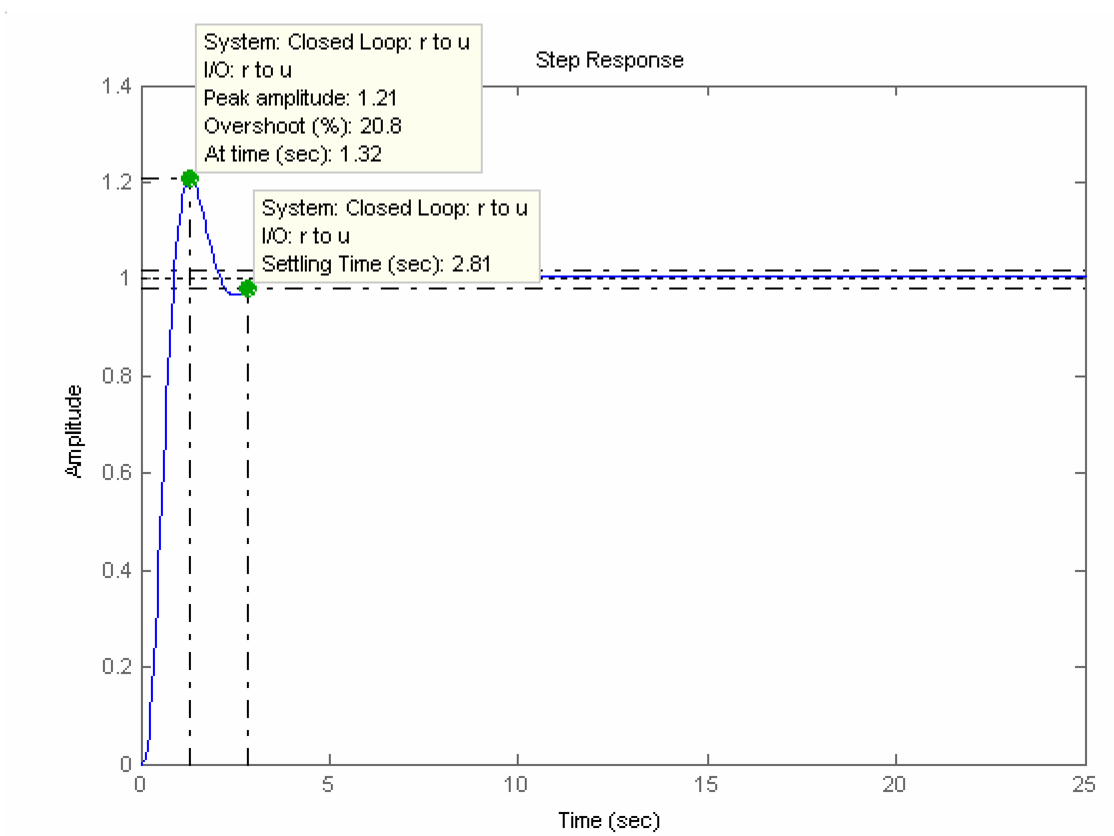
175.78º - 180º = -4.22º. Using the following geometry, $\frac{2.6432}{p_c - 1.3333} = \tan 4.22^{\circ}$, or $p_c = 37.16$

Adding the pole to the system, $K = 4401.52$ at the design point..



b. Searching along the real axis segments of the root locus for $K = 4401.52$, we find higher-order poles at -0.0202, -13.46, and -37.02. There is pole/zero cancellation at -0.02. Also, the poles at , -13.46, and -37.02 are at least 5 times the design point's real part. Thus, the second-order approximation is valid.

c.

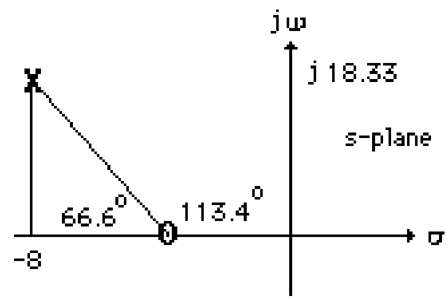


From the plot, T_s = 2.81 seconds, and %OS = 20.8%. Thus, the requirements are met.

17.

a. $\zeta\omega_n = \frac{4}{T_s} = \frac{4}{0.5} = 8$. Since $\zeta = 0.4$, $\omega_n = 20$. Therefore the compensated closed-loop poles are located at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -8 \pm j18.33$.

b. Using the system's poles along with the compensator's pole at -15, the sum of angles to the test point $-8 \pm j18.33$ is -293.4^o. Therefore, the compensator's zero must contribute $293.4^o - 180^o = 113.4^o$. Using the following geometry, $\frac{18.33}{8 - z_c} = \tan 66.6^o$, or $z_c = 0.0679$.



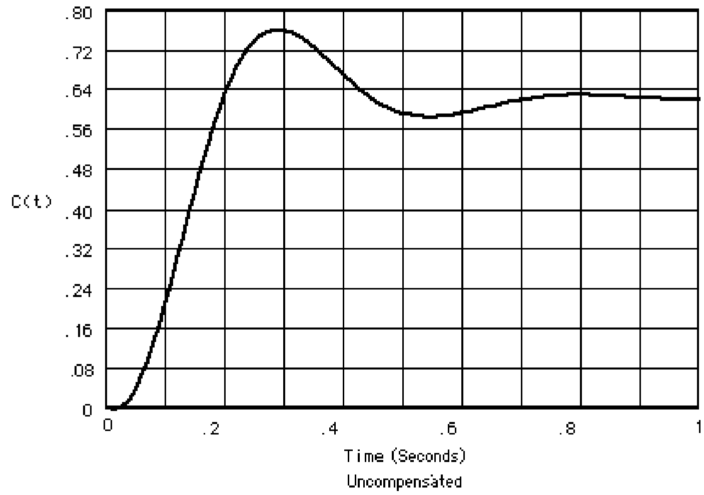
c. Adding the compensator zero and using $-8 \pm j18.33$ as the test point, K = 7297.

d. Making a second-order assumption, the predicted performance is as follows:

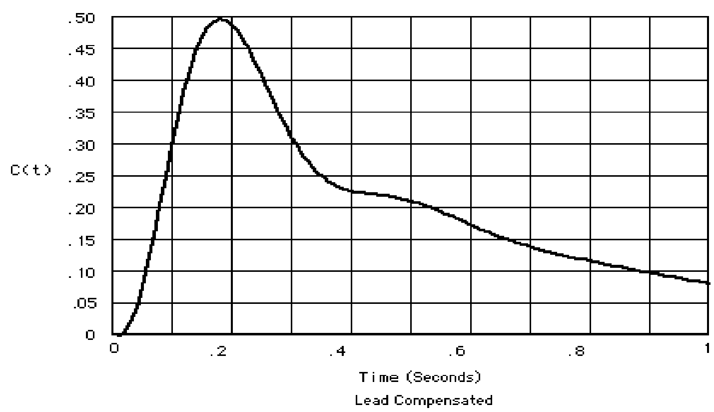
Uncompensated: Searching along the 133.58^o line ($\zeta = 0.4$), find the uncompensated closed-loop pole at $-5.43 + j12.45$ with K = 3353. Hence, $T_s = \frac{4}{\zeta\omega_n} = 0.74$ seconds; $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 25.38\%$; $K_p = \frac{3353}{101x20} = 1.66$. Checking the second-order assumption by searching the real axis segments of the root locus for K = 3353, we find a higher-order pole at -29.13. Since this pole is more than five times further from the imaginary axis than the dominant pair, the second order assumption is reasonable.

Compensated: Using the compensated dominant pole location, $-8 \pm j18.33$, $T_s = \frac{4}{\zeta\omega_n} = 0.5$ seconds; $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 25.38\%$; $K_p = \frac{7297x0.0679}{101x20x15} = 0.016$. Checking the second-order assumption by searching the real axis segments of the root locus for K = 7297, we find higher-order poles at -2.086 and -36.91. The poles are not five times further from the imaginary axis nor do they yield pole/zero cancellation. The second-order assumption is not valid.

e.



The uncompensated system exhibits a steady-state error of 0.38, a percent overshoot of 22.5%, and a settling time of 0.78 seconds.



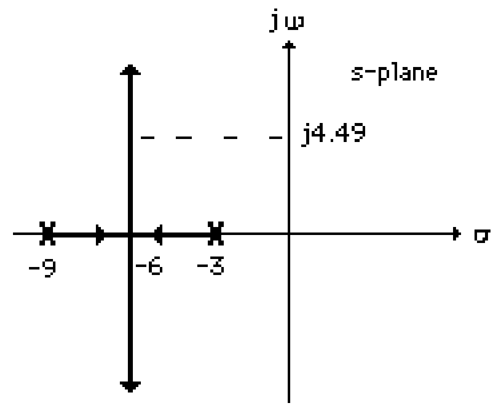
Since there is no pole/zero cancellation the closed-loop zero near the origin produces a large steady-state error. The student should be asked to find a viable design solution to this problem by choosing the compensator zero further from the origin. For example, placing the compensator zero at -20 yields a compensator pole at -90.75 and a gain of 28730. This design yields a valid second-order approximation.

18.

a. Since %OS = 1.5%, $\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.8$. Since $T_s = \frac{4}{\zeta\omega_n} = \frac{2}{3}$ second,

$\omega_n = 7.49$ rad/s. Hence, the location of the closed-loop poles must be $-6 \pm j4.49$. The summation of angles from open-loop poles to $-6 \pm j4.49$ is -226.3^o. Therefore, the design point is not on the root locus.

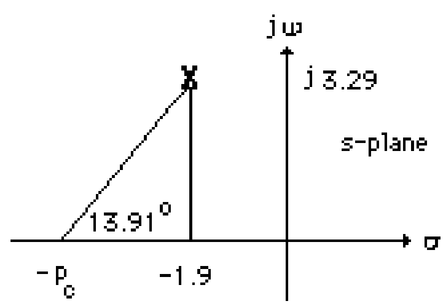
b. A compensator whose angular contribution is $226.3^o - 180^o = 46.3^o$ is required. Assume a compensator zero at -5 canceling the pole. Thus, the breakaway from the real axis will be at the required -6 if the compensator pole is at -9 as shown below.



Adding the compensator pole and zero to the system poles, the gain at the design point is found to be 29.16. Summarizing the results: $G_c(s) = \frac{s+5}{s+9}$ with K = 29.16.

19.

Lead compensator design: Searching along the 120^o line ($\zeta = 0.5$), find the operating point at $-1.531 + j2.652$ with K = 354.5. Thus, $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1.531} = 2.61$ seconds. For the settling time to decrease by 0.5 second, $T_s = 2.11$ seconds, or $Re = -\zeta\omega_n = -\frac{4}{2.11} = -1.9$. The imaginary part is $-1.9 \tan 60^o = 3.29$. Hence, the compensated dominant poles are $-1.9 \pm j3.29$. The compensator zero is at -5. Using the uncompensated system's poles along with the compensator zero, the summation of angles to the design point, $-1.9 \pm j3.29$ is -166.09^o. Thus, the contribution of the compensator pole must be $166.09^o - 180^o = -13.91^o$. Using the following geometry, $\frac{3.29}{p_c - 1.9} = \tan 13.91^o$, or $p_c = 15.18$.



Adding the compensator pole and using $-1.9 \pm j3.29$ as the test point, K = 1417.

Computer simulations yield the following: Uncompensated: $T_s = 3$ seconds, %OS = 14.6%.

Compensated: T_s = 2.3 seconds, %OS = 15.3%.

Lag compensator design: The lead compensated open-loop transfer function is

$G_{LC}(s) = \dfrac{1417(s+5)}{(s+2)(s+4)(s+6)(s+8)(s+15.18)}$. The uncompensated

K_p = 354.5/(2 x 4 x 6 x 8) = 0.923. Hence, the uncompensated steady-state error is $\dfrac{1}{1+K_p}$ = 0.52.

Since we want 30 times improvement, the lag-lead compensated system must have a steady-state error of 0.52/30 = 0.017. The lead compensated K_p = 1417*5/(2*4*6*8*15.18) = 1.215. Hence, the lead-compensated error is $\dfrac{1}{1+K_p}$ = 0.451. Thus, the lag compensator must improve the lead-compensated error by 0.451/0.017 = 26.529 times. Thus 0.451/ ($\dfrac{1}{1+K_{pllc}}$) = 26.529, where K_{pllc} = 57.823 is the lead-lag compensated system's position constant. Thus, the improvement in K_p from the lead to the lag-lead compensated system is 57.823/1.215 = 47.59. Use a lag compensator, whose zero is 47.59 times farther than its pole, or $G_{lag} = \dfrac{(s+0.04759)}{(s+0.001)}$. Thus, the lead-lag compensated open-loop transfer function is $G_{LLC}(s) = \dfrac{1417(s+5)(s+0.04759)}{(s+2)(s+4)(s+6)(s+8)(s+15.18)(s+0.001)}$.

20.

Program:

```
numg=1;
deng=poly([-2 -4 -6 -8]);
'G(s)'
G=tf(numg,deng);
Gzpk=zpk(G)
rlocus(G,0:5:500)
z=0.5;
pos=exp(-pi*z/sqrt(1-z^2))*100;
sgrid(z,0)
title(['Uncompensated Root Locus with ' , num2str(z), ' Damping Ratio
Line'])
[K,p]=rlocfind(G);  %Allows input by selecting point on graphic
'Closed-loop poles = '
p
i=input('Give pole number that is operating point   ');
'Summary of estimated specifications for uncompensated system'
operatingpoint=p(i)
gain=K
estimated_settling_time=4/abs(real(p(i)))
estimated_peak_time=pi/abs(imag(p(i)))
estimated_percent_overshoot=pos
estimated_damping_ratio=z
estimated_natural_frequency=sqrt(real(p(i))^2+imag(p(i))^2)
Kpo=dcgain(K*G)
T=feedback(K*G,1);
'Press any key to continue and obtain the step response'
pause
step(T)

whitebg('w')
title(['Step Response for Uncompensated System with  ' , num2str(z),...
' Damping Ratio'],'color','black')
'Press any key to go to Lead compensation'
```

```
pause
'Compensated system'
b=5;
'Lead Zero at -b '
done=1;
while done>0
a=input('Enter a Test Lead Compensator Pole, (s+a). a =     ');
numgglead=[1 b];
dengglead=conv([1 a],poly([-2 -4 -6 -8]));
'G(s)Glead(s)'
GGlead=tf(numgglead,dengglead);
GGleadzpk=zpk(GGlead)
wn=4/((estimated_settling_time-0.5)*z);
clf
rlocus(GGlead,0:10:2000)
sgrid(z,wn)
axis([-10 0 -5 5])
title(['Lead Compensated Root Locus with ' , num2str(z),...
' Damping Ratio Line, Lead Pole at  ', num2str(-a), ', and Required Wn'])
done=input('Are you done? (y=0,n=1)  ');
end
[K,p]=rlocfind(GGlead);  %Allows input by selecting point on graphic
'Closed-loop poles = '
p
i=input('Give pole number that is operating point   ');
'Summary of estimated specifications for lead-compensated system'
operatingpoint=p(i)
gain=K
estimated_settling_time=4/abs(real(p(i)))
estimated_peak_time=pi/abs(imag(p(i)))
estimated_percent_overshoot=pos
estimated_damping_ratio=z
estimated_natural_frequency=sqrt(real(p(i))^2+imag(p(i))^2)
Kplead=dcgain(K*GGlead)
T=feedback(K*GGlead,1);
'Press any key to continue and obtain the step response'
pause
step(T)

whitebg('w')
title(['Step Response for Lead Compensated System with  ' , num2str(z),...
' Damping Ratio'],'color','black')
'Press any key to continue and design lag compensation'
pause
'Improvement in steady-state error with lead compensator is'
error_improvement=(1+Kplead)/(1+Kpo)
additional_error_improvement=30/error_improvement
Kpnn=additional_error_improvement*(1+Kplead)-1
pc=0.001
zc=pc*(Kpnn/Kplead)
numggleadlag=conv(numgglead,[1 zc]);
dengleadlag=conv(dengglead,[1 pc]);
'G(s)Glead(s)Glag(s)'
GGleadGlag=tf(numggleadlag,denggleadlag);
GGleadGlagzpk=zpk(GGleadGlag)
rlocus(GGleadGlag,0:10:2000)
z=0.5;
pos=exp(-pi*z/sqrt(1-z^2))*100;
sgrid(z,0)
title(['Lag-Lead Compensated Root Locus with ' , num2str(z), ...
' Damping Ratio Line and Lag Pole at ',num2str(-pc)])
[K,p]=rlocfind(GGleadGlag);  %Allows input by selecting point on graphic
'Closed-loop poles = '
p
i=input('Give pole number that is operating point   ');
'Summary of estimated specifications for lag-lead compensated system'
operatingpoint=p(i)
```

```
gain=K
estimated_settling_time=4/abs(real(p(i)))
estimated_peak_time=pi/abs(imag(p(i)))
estimated_percent_overshoot=pos
estimated_damping_ratio=z
estimated_natural_frequency=sqrt(real(p(i))^2+imag(p(i))^2)
Kpleadlag=dcgain(K*GGleadGlag)
T=feedback(K*GGleadGlag,1);
'Press any key to continue and obtain the step response'
pause
step(T)
whitebg('w')
title(['Step Response for Lag-Lead Compensated System with  ',
num2str(z),...
' Damping Ratio and Lag Pole at ',num2str(-pc)],'color','black')
```

Computer response:

```
ans =

G(s)


Zero/pole/gain:
          1
-----------------------
(s+8) (s+6) (s+4) (s+2)

Select a point in the graphics window

selected_point =

  -1.5036 + 2.6553i


ans =

Closed-loop poles =


p =

  -8.4807 + 2.6674i
  -8.4807 - 2.6674i
  -1.5193 + 2.6674i
  -1.5193 - 2.6674i

Give pole number that is operating point   3

ans =
```

```
Summary of estimated specifications for uncompensated system


operatingpoint =

  -1.5193 + 2.6674i


gain =

  360.8014


estimated_settling_time =

    2.6328


estimated_peak_time =

    1.1778


estimated_percent_overshoot =

   16.3034


estimated_damping_ratio =

    0.5000


estimated_natural_frequency =

    3.0698


Kpo =

    0.9396


ans =
```

```
Press any key to continue and obtain the step response


ans =

Press any key to go to Lead compensation


ans =

Compensated system


ans =

Lead Zero at -b

Enter a Test Lead Compensator Pole, (s+a). a =     10

ans =

G(s)Glead(s)


Zero/pole/gain:
            (s+5)
------------------------------
(s+10) (s+8) (s+6) (s+4) (s+2)

Are you done? (y=0,n=1)  1
Enter a Test Lead Compensator Pole, (s+a). a =     15

ans =

G(s)Glead(s)


Zero/pole/gain:
            (s+5)
------------------------------
(s+15) (s+8) (s+6) (s+4) (s+2)

Are you done? (y=0,n=1)  0
Select a point in the graphics window
```

```
selected_point =

  -1.9076 + 3.2453i


ans =

Closed-loop poles =


p =

 -13.0497 + 1.9313i
 -13.0497 - 1.9313i
  -5.0654
  -1.9176 + 3.2514i
  -1.9176 - 3.2514i

Give pole number that is operating point   4

ans =

Summary of estimated specifications for lead-compensated system


operatingpoint =

  -1.9176 + 3.2514i


gain =

  1.3601e+003


estimated_settling_time =

    2.0860


estimated_peak_time =

    0.9662
```

```
estimated_percent_overshoot =

   16.3034


estimated_damping_ratio =

    0.5000


estimated_natural_frequency =

    3.7747


Kplead =

    1.1806


ans =

Press any key to continue and obtain the step response


ans =

Press any key to continue and design lag compensation


ans =

Improvement in steady-state error with lead compensator is


error_improvement =

    1.1243


additional_error_improvement =

   26.6842
```

```
Kpnn =

   57.1876


pc =

    0.0010


zc =

    0.0484


ans =

G(s)Glead(s)Glag(s)


Zero/pole/gain:
           (s+5) (s+0.04844)
----------------------------------------
(s+15) (s+8) (s+6) (s+4) (s+2) (s+0.001)

Select a point in the graphics window

selected_point =

  -1.8306 + 3.2919i


ans =

Closed-loop poles =


p =

 -13.0938 + 2.0650i
 -13.0938 - 2.0650i
  -5.0623
  -1.8617 + 3.3112i
  -1.8617 - 3.3112i
  -0.0277
```

```
Give pole number that is operating point   4

ans =

Summary of estimated specifications for lag-lead compensated system


operatingpoint =

  -1.8617 + 3.3112i


gain =

  1.4428e+003


estimated_settling_time =

    2.1486


estimated_peak_time =

    0.9488


estimated_percent_overshoot =

   16.3034


estimated_damping_ratio =

    0.5000


estimated_natural_frequency =

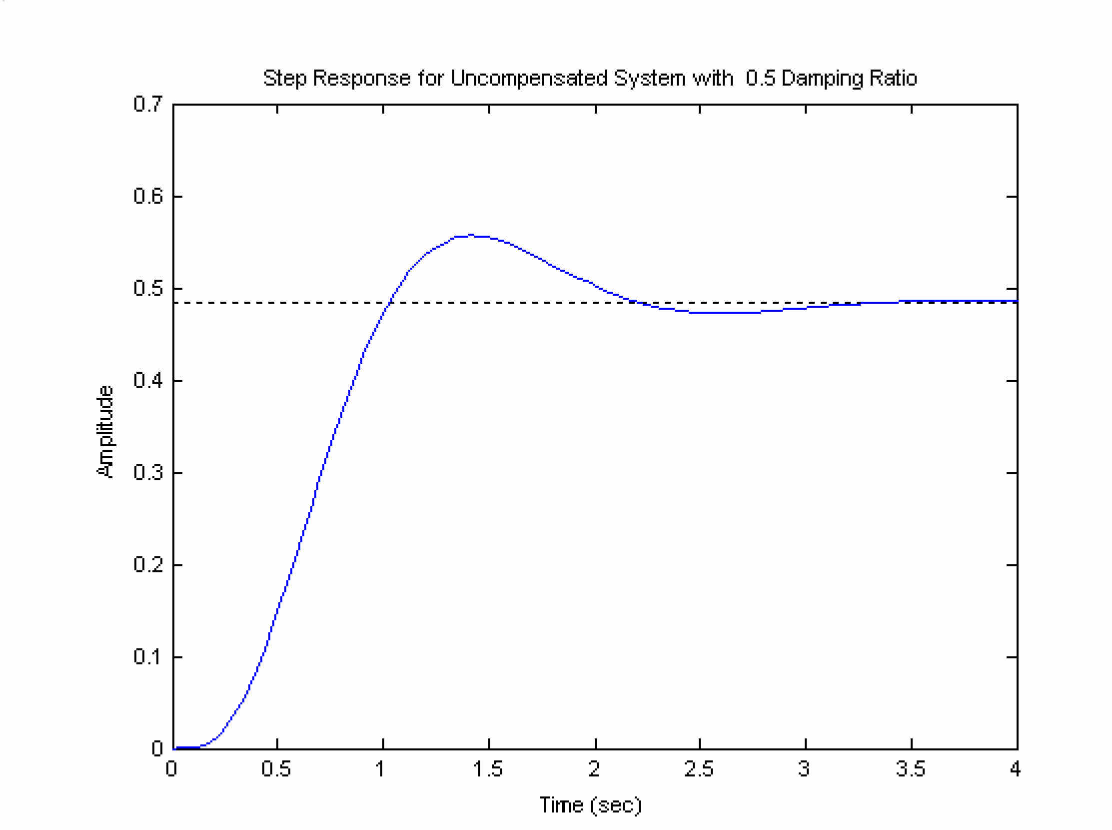
    3.7987

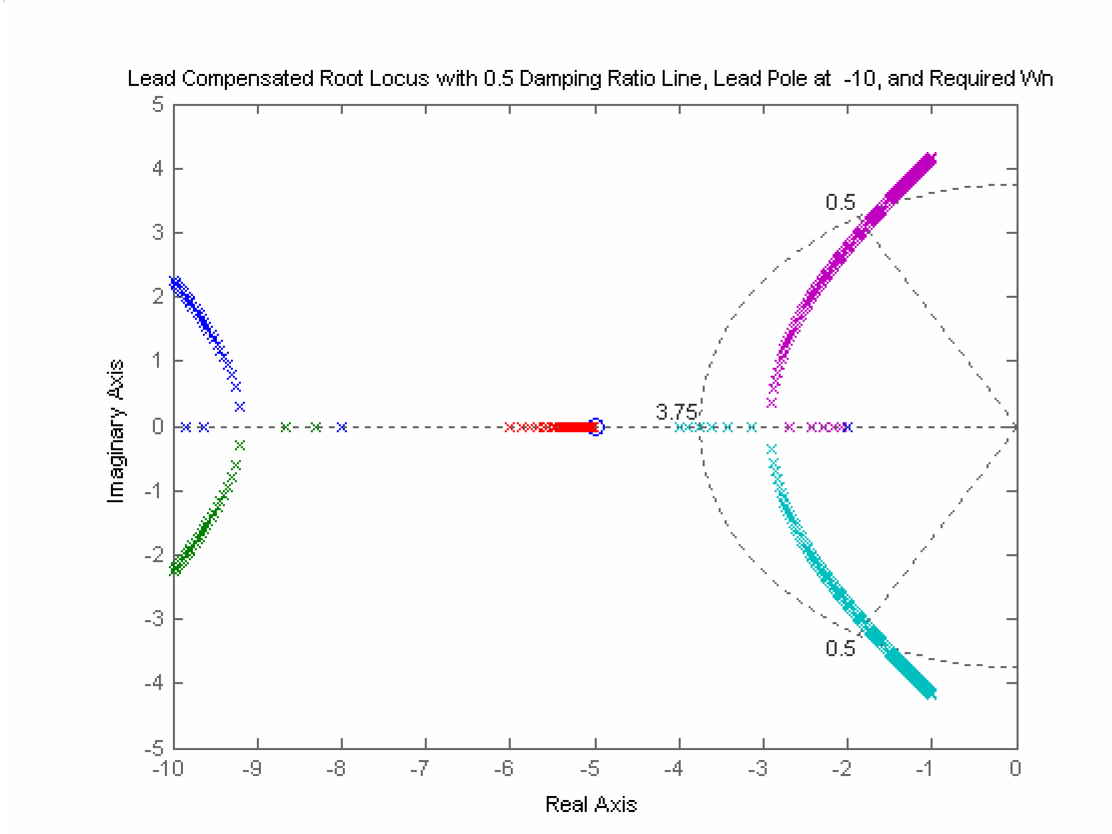

Kpleadlag =
```

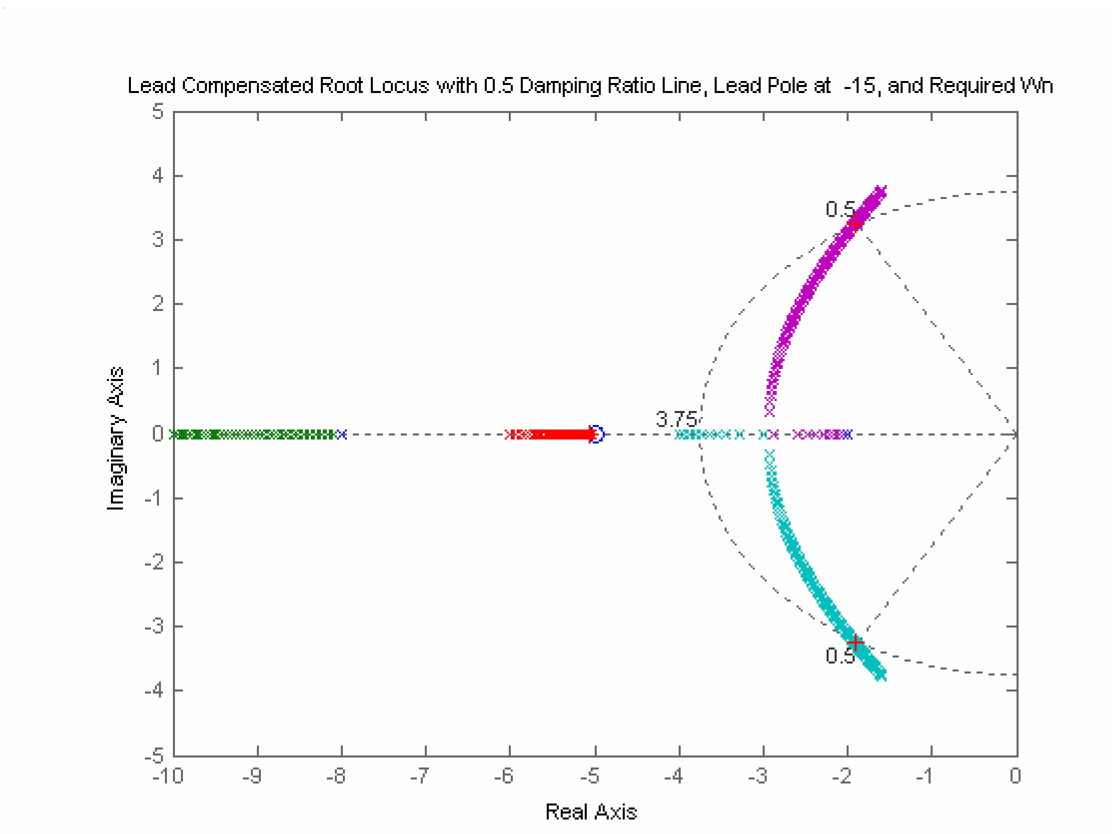
```
   60.6673

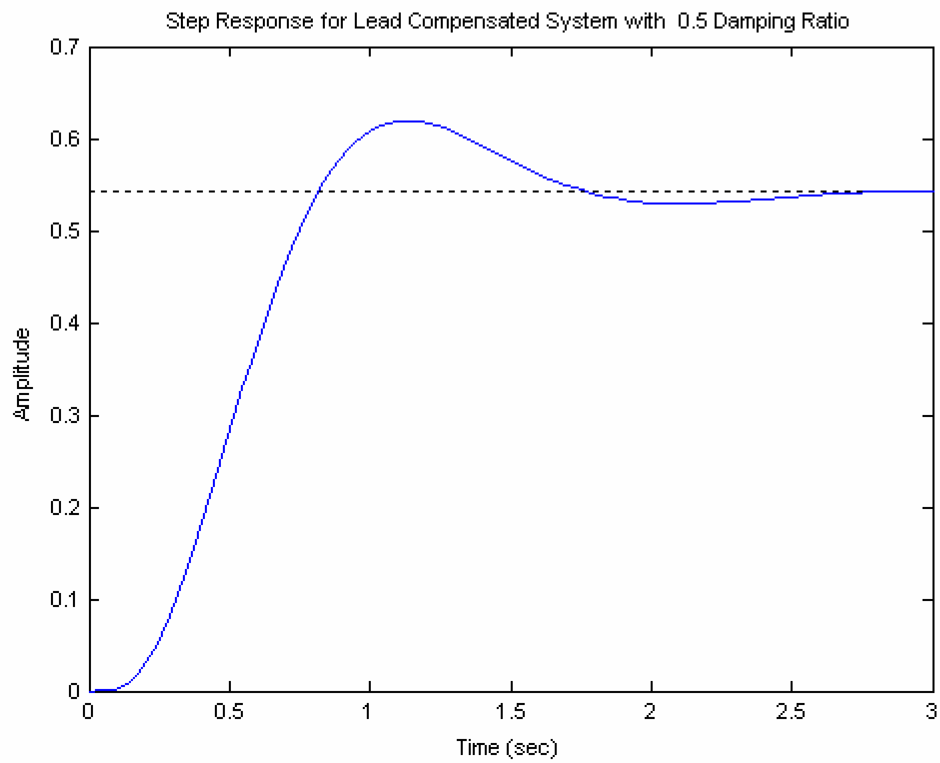

ans =

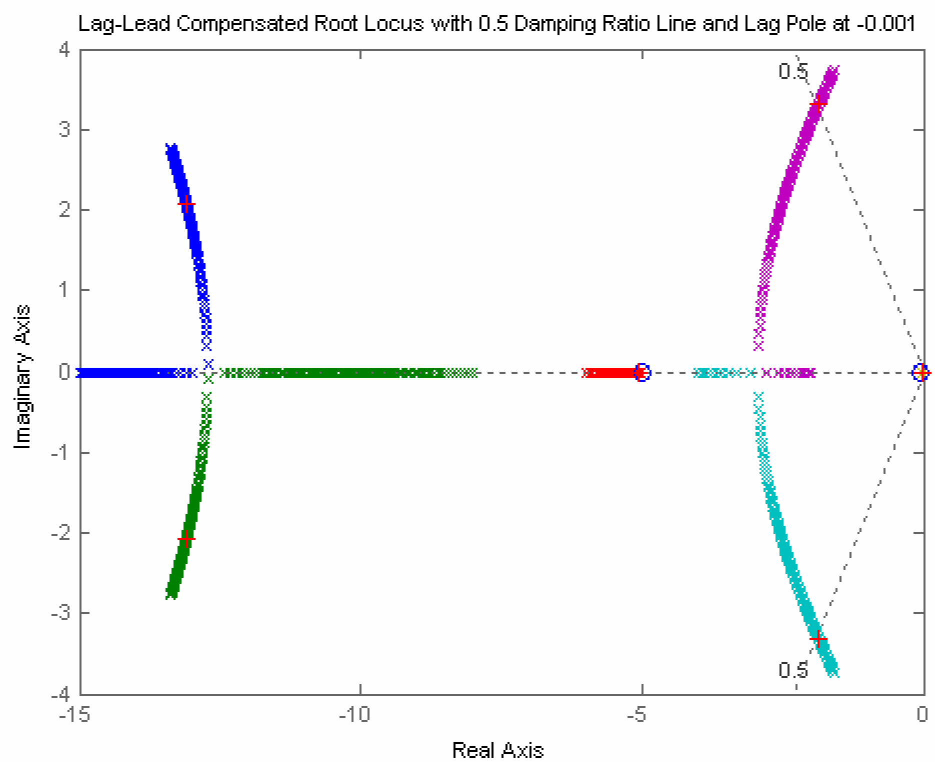
Press any key to continue and obtain the step response
```

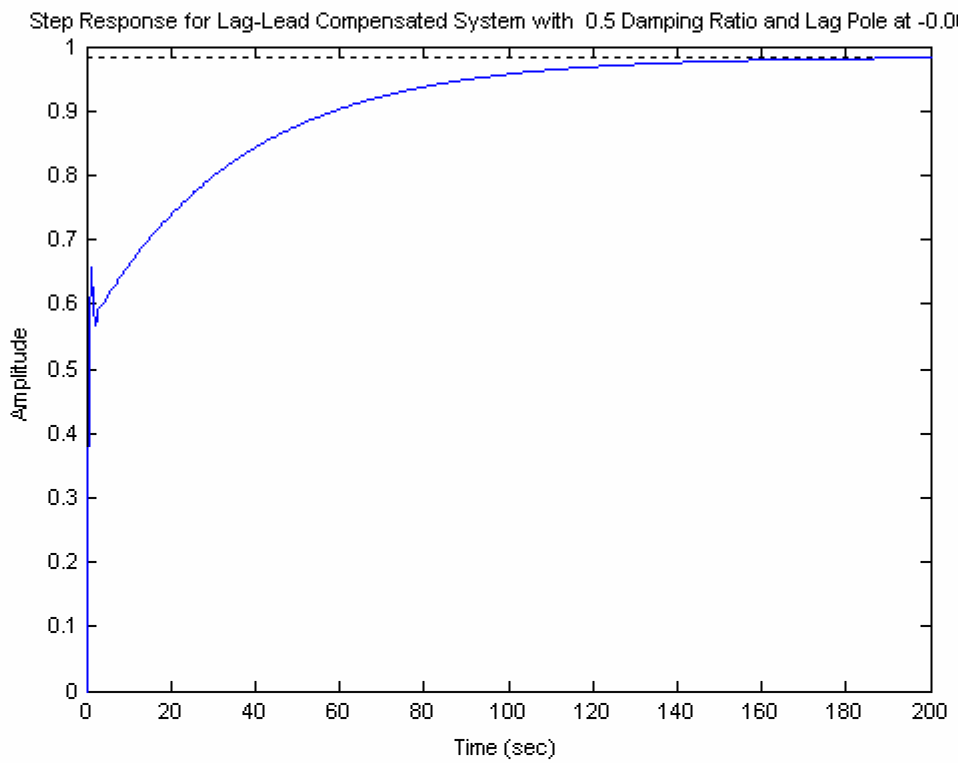



Lead Compensated Root Locus with 0.5 Damping Ratio Line, Lead Pole at -10, and Required Wn
Imaginary Axis
Real Axis
0.5
3.75
0.5
5
4
3
2
1
0
-1
-2
-3
-4
-5
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0


Lead Compensated Root Locus with 0.5 Damping Ratio Line, Lead Pole at -15, and Required Wn
Imaginary Axis
Real Axis
0.5
3.75
0.5
5
4
3
2
1
0
-1
-2
-3
-4
-5
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0

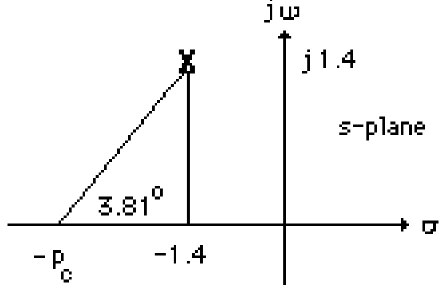

Step Response for Lead Compensated System with 0.5 Damping Ratio
Amplitude
Time (sec)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3


Lag-Lead Compensated Root Locus with 0.5 Damping Ratio Line and Lag Pole at -0.001
Imaginary Axis
Real Axis
4
3
2
1
0
-1
-2
-3
-4
-15
-10
-5
0
0.5
0.5



21.

a. For the settling time to be 2.86 seconds with 4.32% overshoot, the real part of the compensated dominant poles must be $\frac{4}{T_s} = \frac{4}{2.86} = 1.4$. Hence the compensated dominant poles are $-1.4 \pm j1.4$. Assume the compensator zero to be at -1 canceling the system pole at -1. The summation of angles to the design point at $-1.4 \pm j1.4$ is -176.19^o. Thus the contribution of the compensator pole is $176.19^o - 180^o = 3.81^o$. Using the geometry below, $\frac{1.4}{p_c - 1.4} = \tan 3.81^o$, or $p_c = 22.42$.



Adding the compensator pole and using $-1.4 \pm j1.4$ as the test point, K = 88.68.

b. Uncompensated: Search the 135^o line (4.32% overshoot) and find the uncompensated dominant pole at $-0.419 + j0.419$ with K = 1.11. Thus $K_v = \frac{1.11}{3} = 0.37$. Hence, $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.419} = 9.55$

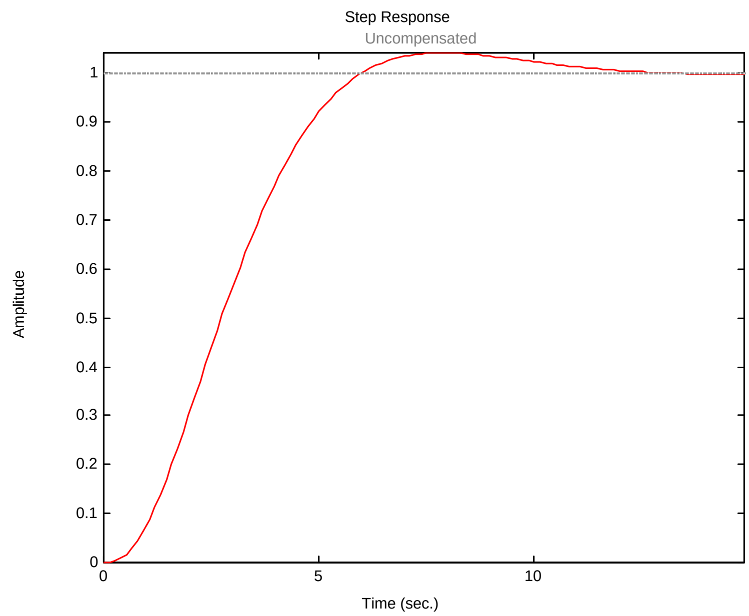
seconds and %OS = 4.32%. Compensated: $K_v = \frac{88.68}{22.42 \times 3} = 1.32$ (Note: steady-state error improvement is greater than 2). $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1.4} = 2.86$ seconds and %OS = 4.32%.

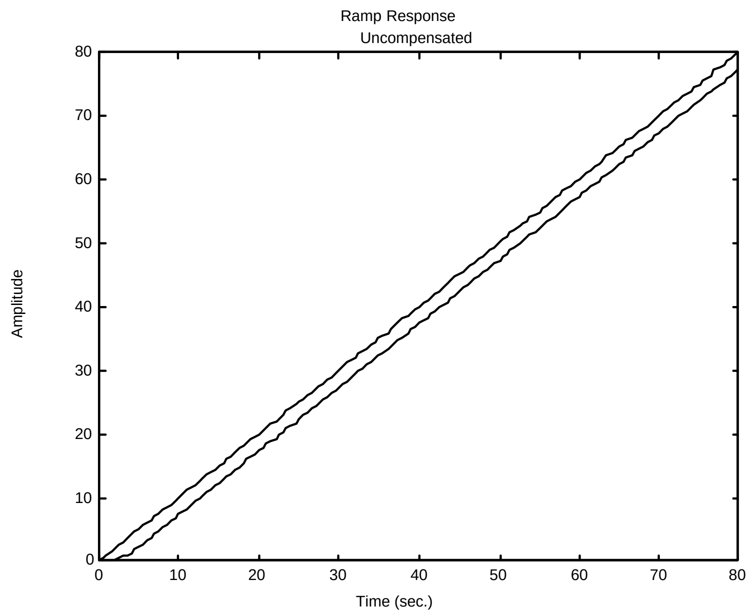
c. Uncompensated: K = 1.11; Compensated: K = 88.68.

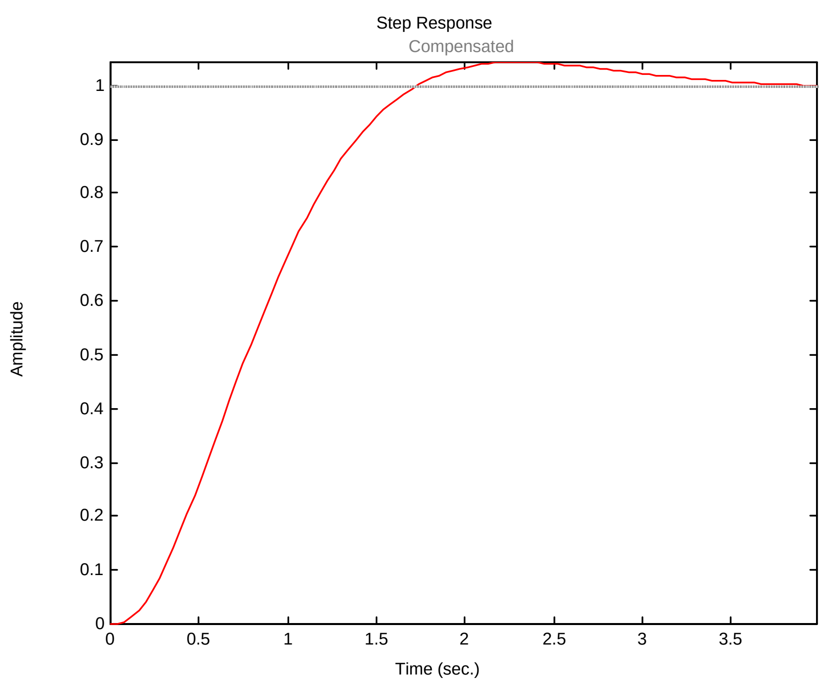
d. Uncompensated: Searching the real axis segments for K = 1.11 yields a higher-order pole at -3.16 which is more than five times the real part of the uncompensated dominant poles. Thus the second-order approximation for the uncompensated system is valid.

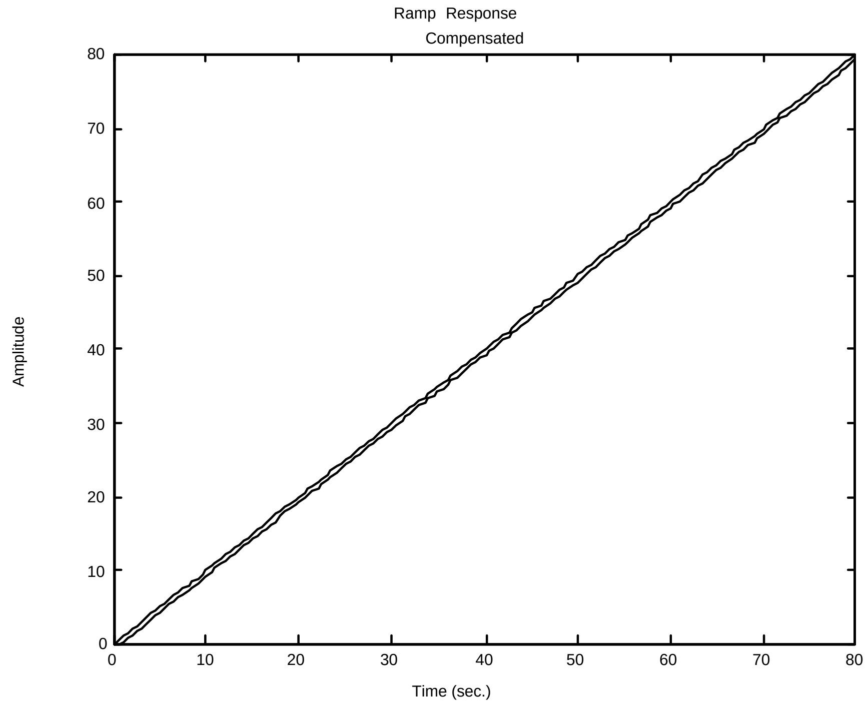
Compensated: Searching the real axis segments for K = 88.68 yields a higher-order pole at -22.62 which is more than five times the real part of the compensated dominant poles' real part. Thus the second order approximation is valid.

e.




Ramp Response
Uncompensated
Amplitude
Time (sec.)
0
10
20
30
40
50
60
70
80


Step Response
Compensated
Amplitude
Time (sec.)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3
3.5

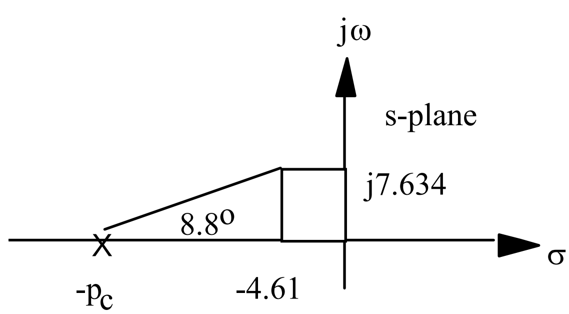


22.

a. Searching the 30% overshoot line (ζ = 0.358; 110.97°) for 180° yields -1.464 + j3.818 with a gain, K = 218.6.

b. $T_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.818} = 0.823$ second. $K_v = \frac{218.6}{(5)(11)} = 3.975$.

c. Lead design: From the requirements, the percent overshoot is 15% and the peak time is 0.4115 second. Thus, $\zeta = \frac{-\ln(\%/100)}{\sqrt{\pi^2+\ln^2(\%/100)}} = 0.517$; $\omega_d = \frac{\pi}{T_p} = 7.634 = \omega_n\sqrt{1-\zeta^2}$. Hence, $\omega_n = 8.919$. The design point is located at $-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -4.61 + j7.634$. Assume a lead compensator zero at -5. Summing the angles of the uncompensated system's poles as well as the compensator zero at -5 yields –171.2°. Therefore, the compensator pole must contribute (171.2° - 180°) = -8.8°. Using the geometry below,



$\frac{7.634}{p_c - 4.61}$ = tan (8.8°) . Hence, p_c = 53.92. The compensated open-loop transfer function is

$\frac{K}{s(s+11)(s+53.92)}$. Evaluating the gain for this function at the point, -4.61 + j7.634 yields

K = 4430.

Lag design: The uncompensated $K_v = \frac{218.6}{(5)(11)} = 3.975$. The required K_v is 30*3.975 = 119.25.

The lead compensated K_v = $\frac{4430}{(11)(53.92)}$ = 7.469. Thus, we need an improvement over the lead compensated system of 119.25/7.469 = 15.97. Thus, use a lag compensator

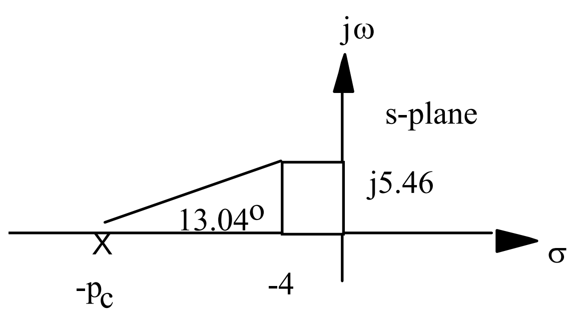
$G_{lag}(s) = \frac{s+0.01597}{s+0.001}$. The final open-loop function is $\frac{4430(s+0.01597)}{s(s+11)(s+53.92)(s+0.001)}$.

23.

a. Searching along the 10% overshoot line (ζ = 0.591) the operating point is found to be –1.85 + j2.53 with K = 21.27. A third pole is at –10.29. Thus, the estimated performance before compensation is: 10% overshoot, $T_s = \frac{4}{1.85} = 2.16$ seconds, and $K_p = \frac{21.27}{(8)(10)} = 0.266$.

b. Lead design: Place compensator zero at –3. The desired operating point is found from the desired specifications. $\zeta\omega_n = \frac{4}{T_s} = \frac{4}{1} = 4$ and $\omega_n = \frac{4}{\zeta} = \frac{4}{0.591} = 6.768$. Thus,

$\text{Im} = \omega_n\sqrt{1-\zeta^2} = 6.768\sqrt{1-0.591^2} = 5.46$. Hence the design point is –4 +j5.46. The angular contribution of the system poles and compensator zero at the design point is –166.96°. Thus, the compensator pole must contribute –180° + 166.96° = -13.04°. Using the geometry below,

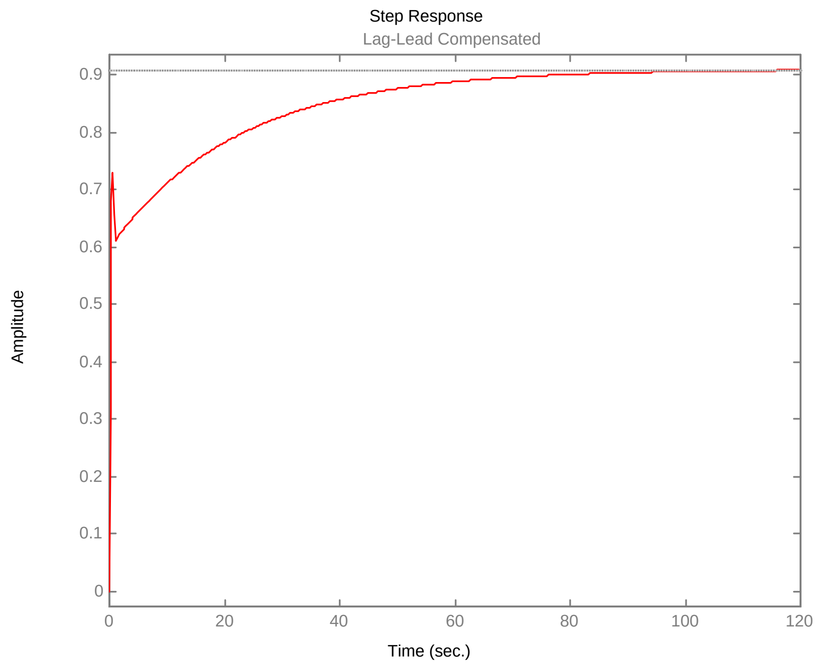


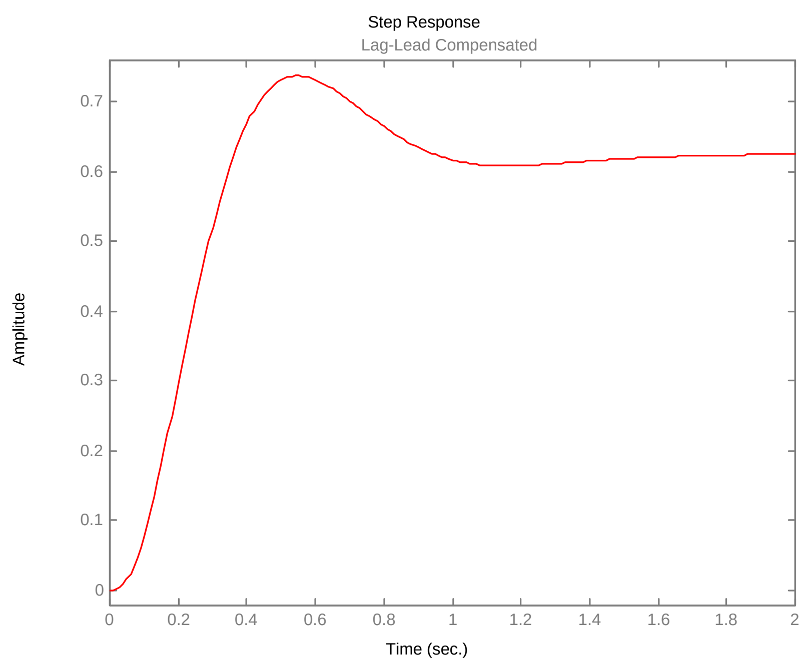
$\frac{5.46}{p_c - 4}$ = tan (13.04°) . Hence, p_c = 27.57. The compensated open-loop transfer function is

$\frac{K(s+3)}{(s^2+4s+8)(s+10)(s+27.57)}$. Evaluating the gain for this function at the point -4 + j5.46 yields K = 1092 with higher-order poles at –4.055 and –29.52.

Lag design: For the lead-compensated system, $K_p = 1.485$. Thus, we need an improvement of $\frac{10}{1.485} = 6.734$ times. Hence, $G_{lag}(s) = \frac{(s+0.06734)}{(s+0.01)}$. Finally, the equivalent forward-path transfer function is $G_e(s) = \frac{1092(s+3)(s+0.06734)}{(s^2+4s+8)(s+10)(s+27.57)(s+0.01)}$.

c.





24.

a. Uncompensated: Search the 135° line (4.32% overshoot) for 180° and find the dominant pole at –3 + j3 with K = 10.

Lag Compensated: Search the 135° line (4.32% overshoot) for 180° and find the dominant pole at -2.88 + j2.88 with K = 9.95.

b. Uncompensated: $K_p = \frac{10}{2 \text{ x } 4} = 1.25$

Lag compensated: $K_p = \frac{9.95 \text{ x } 0.5}{2 \text{ x } 4 \text{ x } 0.1} = 6.22$

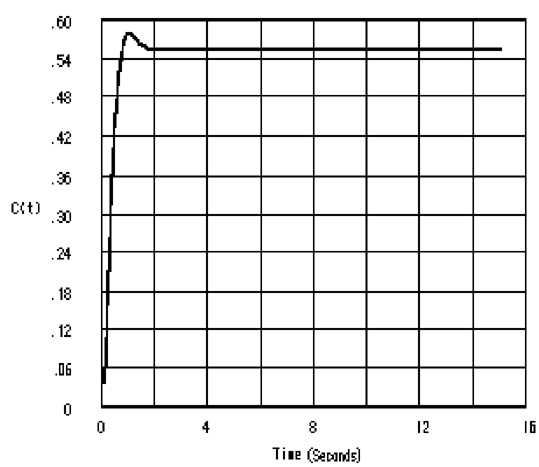
c. %OS = 4.32% both cases;

Uncompensated $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{3}$ 1.33 seconds; Compensated $T_s = \frac{4}{2.88} = 1.39$ seconds

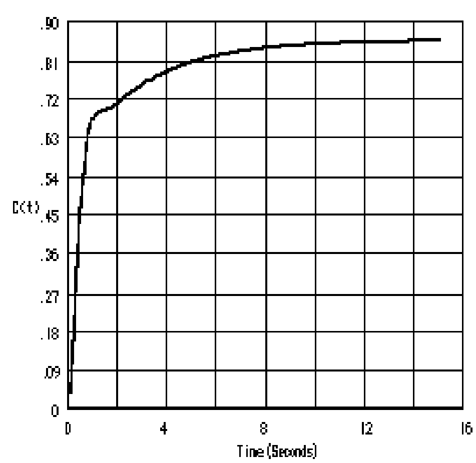
d. Uncompensated: Exact second-order system; approximation OK

Compensated: Search real axis segments of the root locus and find a higher-order pole at -0.3. System should be simulated to see if there is effective pole/zero cancellation with zero at - 0.5.

e.



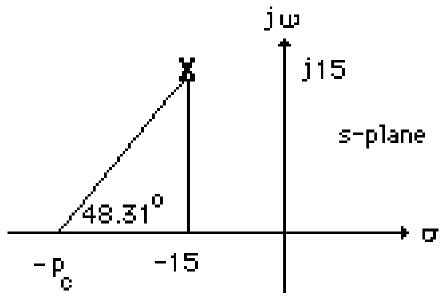
Uncompensated



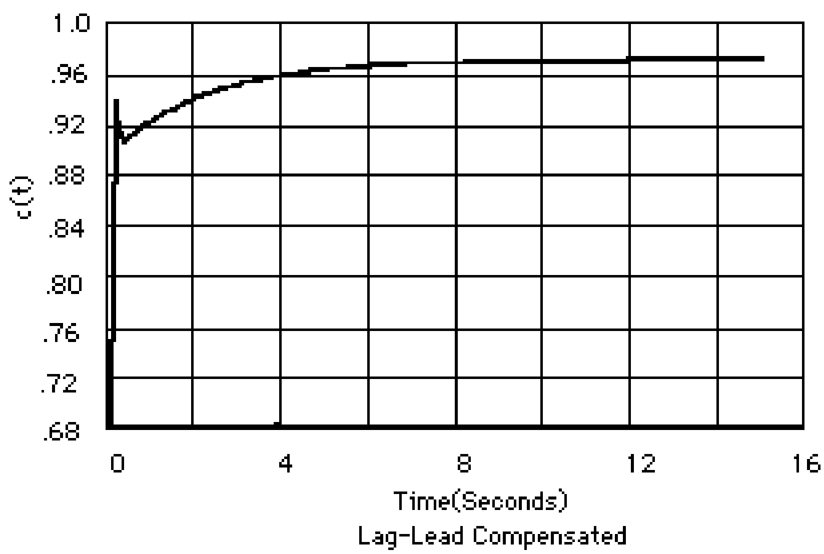
Lag Compensated

The compensated system's response takes a while to approach the final value.

f. We will design a lead compensator to speed up the system by a factor of 5. The lead-compensated dominant poles will thus be placed at $-15 \pm j15$. Assume a compensator zero at - 4 that cancels the open-loop pole at - 4. Using the system's poles and the compensator's zero, the sum of angles to the design point, $-15 \pm j15$ is 131.69^o. Thus, the angular contribution of the compensator pole must be $131.69^o - 180^o = -48.31^o$. Using the geometry below, $p_c = 28.36$.



Using the compensated open-loop transfer function, $G_e(s) = \frac{K(s+0.5)(s+4)}{(s+2)(s+4)(s+0.1)(s+28.36)}$ and using the design point $-15 \pm j15$, K = 404.1.The time response of the lag-lead compensated system is shown below.

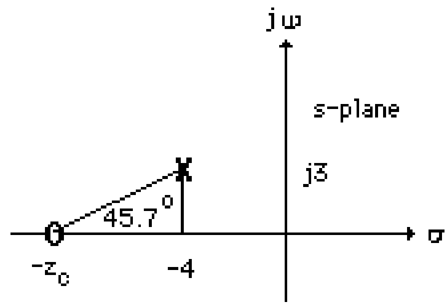


25.

Since $T_p = 1.047$, the imaginary part of the compensated closed-loop poles will be $\frac{\pi}{1.047} = 3$.

Since $\frac{Im}{Re} = \tan(\cos^{-1}\zeta)$, the magnitude of the real part will be $\frac{Im}{\tan(\cos^{-1}\zeta)} = 4$. Hence, the design point is $-4 + j3$. Assume an PI controller, $G_c(s) = \frac{s+0.1}{s}$, to reduce the steady-state error to zero.

Using the system's poles and the pole and zero of the ideal integral compensator, the summation of angles to the design point is -225.7°. Hence, the ideal derivative compensator must contribute 225.7°-180° = 45.7° . Using the geometry below, z_c = 6.93.



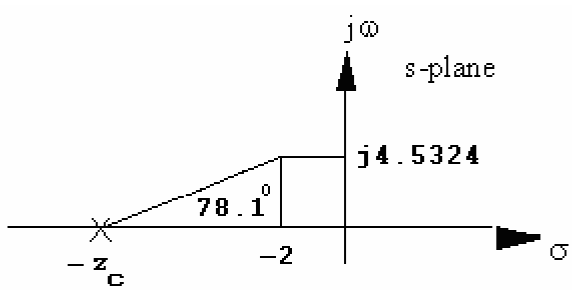
The PID controller is thus $\frac{(s+6.93)(s+0.1)}{s}$. Using all poles and zeros of the system and PID controller, the gain at the design point is K = 3.08. Searching the real axis segment, a higher-order pole is found at - 0.085. A simulation of the system shows the requirements are met.

26.

a. The desired operating point is found from the desired specifications. $\zeta\omega_n = \frac{4}{T_s} = \frac{4}{2} = 2$ and

$\omega_n = \frac{2}{\zeta} = \frac{2}{0.4037} = 4.954$. Thus, $\text{Im} = \omega_n\sqrt{1-\zeta^2} = 4.954\sqrt{1-0.4037^2} = 4.5324$. Hence the design point is –2 +j4.5324. Now, add a pole at the origin to increase system type and drive error to zero for step inputs.

Now design a PD controller. The angular contribution to the design point of the system poles and pole at the origin is 101.9^0. Thus, the compensator zero must contribute $180^0 - 101.9^0 = 78.1^0$. Using the geometry below,



$\frac{4.5324}{z_c - 2} = \tan(78.1^0)$. Hence, z_c = 2.955. The compensated open-loop transfer function with PD compensation is $\frac{K(s+2.955)}{s(s+4)(s+6)(s+10)}$. Adding the compensator zero to the system and

evaluating the gain for this at the point $-2 + j4.5324$ yields K = 294.51 with a higher-order pole at -2.66 and -13.34.

PI design: Use $G_{PI}(s) = \frac{(s+0.01)}{s}$. Hence, the equivalent open-loop transfer function is

$$G_e(s) = \frac{K(s+2.955)(s+0.01)}{s^2(s+4)(s+6)(s+10)}$$ with K = 294.75.

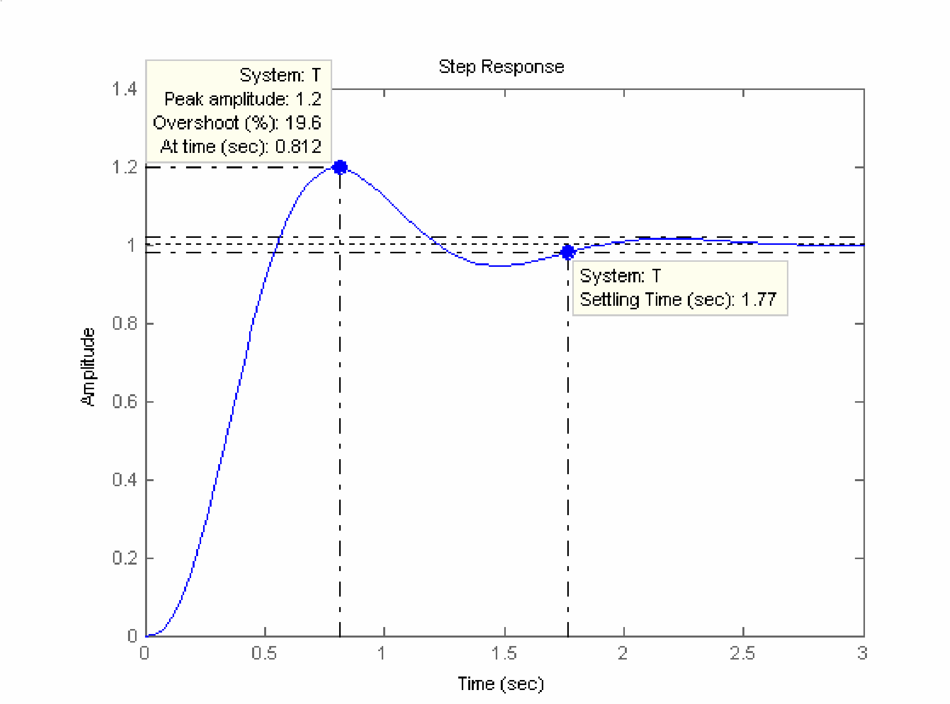
b.

Program (Step Response):

```
numg=[-2.995 -0.01];
deng=[0  0 -4 -6 -10];
K=294.75;
G=zpk(numg,deng,K)
T=feedback(G,1);
step(T)
```

Computer response:

```
Zero/pole/gain:
294.75 (s+2.995) (s+0.01)
-------------------------
 s^2 (s+4) (s+6) (s+10)
```

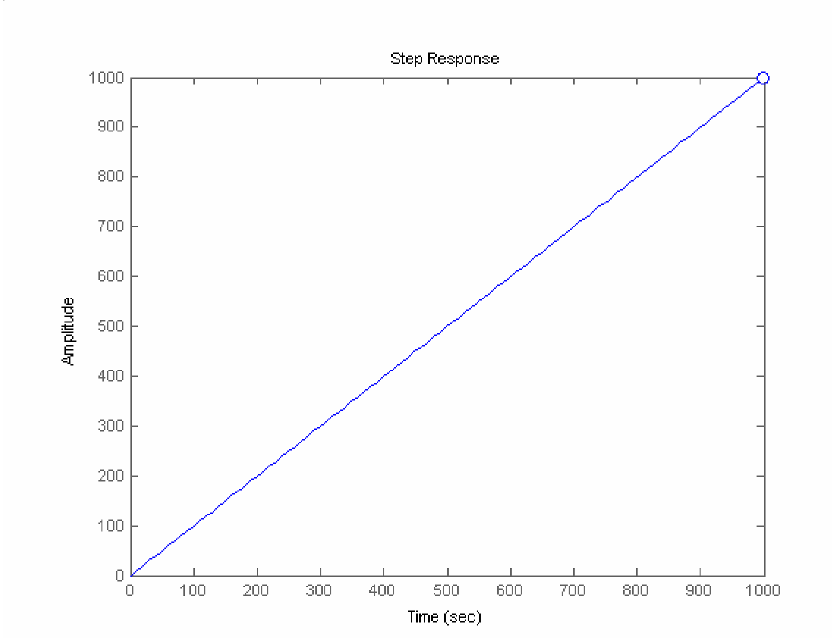


Program (Ramp Response):

```
numg=[-2.995 -0.01];
deng=[0 0 -4 -6 -10];
K=294.75;
G=zpk(numg,deng,K)
T=feedback(G,1);
Ta=tf([1],[1 0]);
step(T*Ta)
```

Computer response:

```
Zero/pole/gain:
294.75 (s+2.995) (s+0.01)
-------------------------
 s^2 (s+4) (s+6) (s+10)
```



27.

Program:

```
numg=[]
deng=[-4 -6 -10]
'G(s)'
G=zpk(numg,deng,1)
pos=input('Type desired percent overshoot ');
z=-log(pos/100)/sqrt(pi^2+[log(pos/100)]^2);
Ts=input('Type desired settling time ');
zci=input(...
'Type desired position of integral controller zero (absolute value) ');
wn=4/(Ts*z);
desired_pole=(-z*wn)+(wn*sqrt(1-z^2)*i)
angle_at_desired_pole=(180/pi)*angle(evalfr(G,desired_pole))
PD_angle=180-angle_at_desired_pole;
zcpd=((imag(desired_pole)/tan(PD_angle*pi/180))-real(desired_pole));
'PD Compensator'
numcpd=[1 zcpd];
dencpd=[0 1];
'Gcpd(s)'
Gcpd=tf(numcpd,dencpd)
Gcpi=zpk([-zci],[0],1)
Ge=G*Gcpd*Gcpi
rlocus(Ge)
sgrid(z,0)
title(['PID Compensated Root Locus with ' ,...
      num2str(pos), '% Damping Ratio Line'])
[K,p]=rlocfind(Ge);
'Closed-loop poles = '
p
f=input('Give pole number that is operating point   ');

'Summary of estimated specifications for selected point'
```

```
'on PID compensated root locus'

operatingpoint=p(f)
gain=K
estimated_settling_time=4/abs(real(p(f)))

estimated_peak_time=pi/abs(imag(p(f)))

estimated_percent_overshoot=pos

estimated_damping_ratio=z

estimated_natural_frequency=sqrt(real(p(f))^2+imag(p(f))^2)
T=feedback(K*Ge,1);
step(T)
title(['Step Response for PID Compensated System with  ' ,...
      num2str(pos),'% Damping Ratio Line'])
pause
one_over_s=tf(1,[1 0]);
Tr=T*one_over_s;
t=0:0.01:10;
step(one_over_s,Tr)
title('Ramp Response for PID Compensated System')
```

Computer response:

```
numg =

     []


deng =

     0    -4    -6   -10


ans =

G(s)


Zero/pole/gain:
         1
--------------------
s (s+4) (s+6) (s+10)

Type desired percent overshoot 25
Type desired settling time 2
Type desired position of integral controller zero (absolute value) 0.01

desired_pole =

  -2.0000 + 4.5324i


angle_at_desired_pole =

  101.8963


ans =

PD Compensator


ans =
```

```
Gcpd(s)

Transfer function:
s + 2.955

Zero/pole/gain:
(s+0.01)
--------
   s

Zero/pole/gain:
  (s+2.955) (s+0.01)
----------------------
s^2 (s+4) (s+6) (s+10)

Select a point in the graphics window

selected_point =

  -1.9931 + 4.5383i


ans =

Closed-loop poles =


p =

 -13.3485
  -1.9920 + 4.5377i
  -1.9920 - 4.5377i
  -2.6575
  -0.0100

Give pole number that is operating point   2

ans =

Summary of estimated specifications for selected point


ans =

on PID compensated root locus


operatingpoint =

  -1.9920 + 4.5377i


gain =

  295.6542


estimated_settling_time =

    2.0081


estimated_peak_time =
```

```
    0.6923

estimated_percent_overshoot =

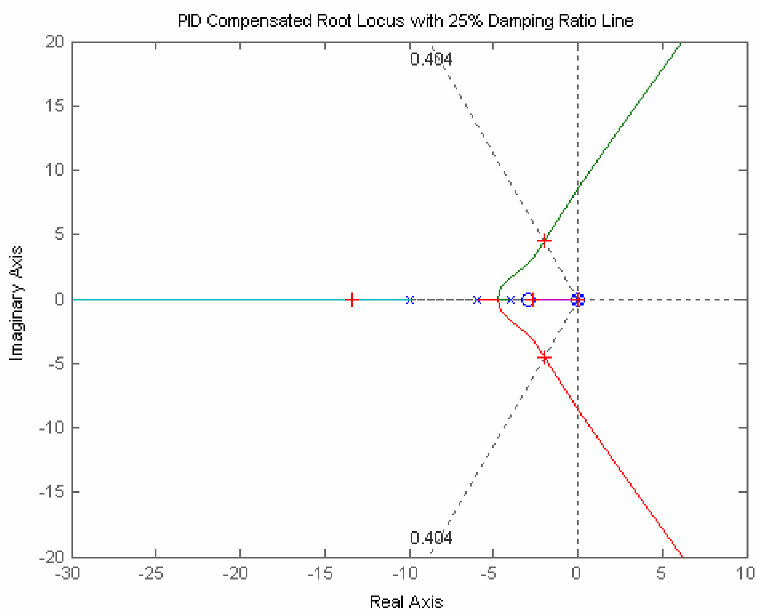
    25

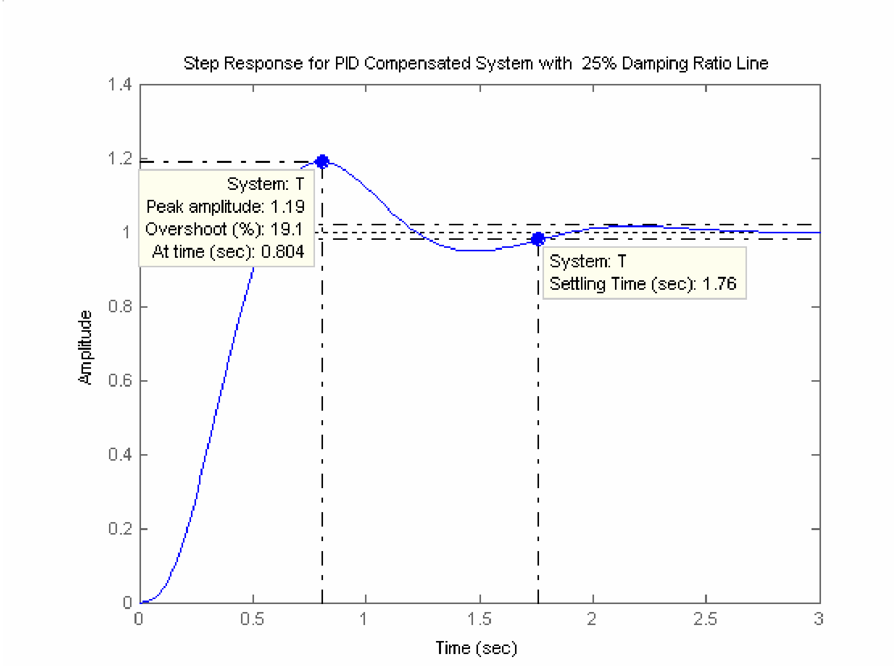
estimated_damping_ratio =

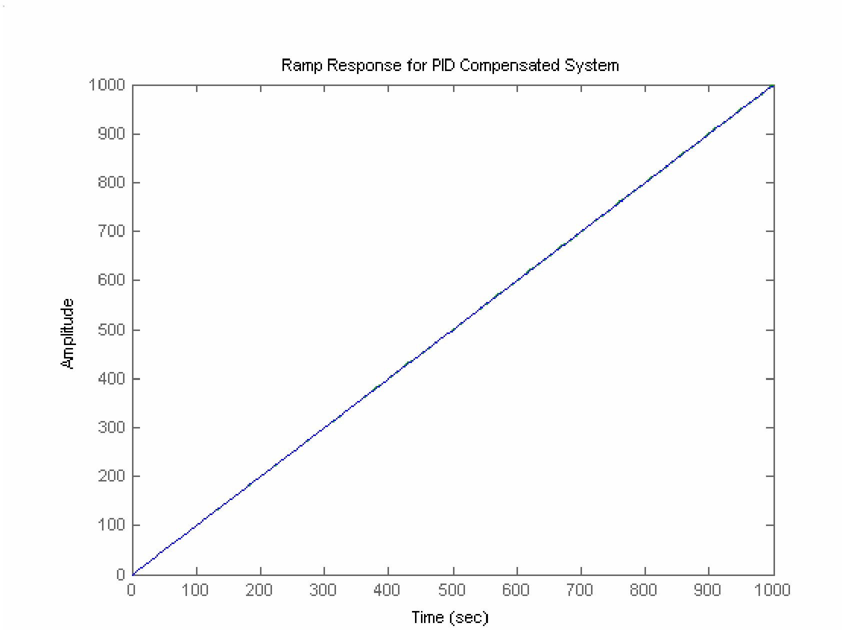
    0.4037

estimated_natural_frequency =

    4.9557
```







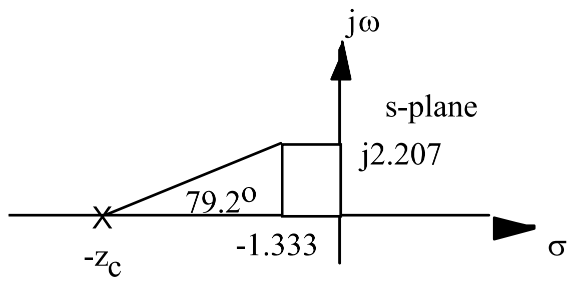
28.

Open-loop poles are at -2, - 0.134, and -1.87. An open-loop zero is at -3. Searching the 121.13° line ($\zeta = 0.517$), find the closed-loop dominant poles at -0.747 + j1.237 with K = 1.58. Searching the real axis segments locates a higher-order pole at -2.51. Since the open-loop zero is a zero of H(s), it is not a closed-loop zero. Thus, there are no closed-loop zeros.

29.

a. The damping ratio for 15% overshoot is 0.517. The desired operating point is found from the desired specifications. $\zeta\omega_n = \frac{4}{T_s} = \frac{4}{3} = 1.333$ and $\omega_n = \frac{1.333}{\zeta} = \frac{1.333}{0.517} = 2.578$. Thus,

$\text{Im} = \omega_n\sqrt{1-\zeta^2} = 2.578\sqrt{1-0.517^2} = 2.207$. Hence the design point is –1.333 + j2.207. The angular contribution of the system poles and compensator zero at the design point is 100.8^0. Thus, the compensator zero must contribute $180^0 – 100.8^0 = 79.2^0$. Using the geometry below,



$\frac{2.207}{z_c - 1.333}$ = tan (79.2°) . Hence, z_c = 1.754. The compensated open-loop transfer function with PD compensation is $\frac{K(s+1.754)}{s(s+2)(s+4)(s+6)}$. Evaluating the gain for this function at the point –1.333 + j2.207 yields K = 47.28 with higher-order poles at –1.617 and –7.718. Following Figure 9.49(c) in the text, $\frac{1}{K_f} = 1.754$. Therefore, $K_f = 0.5701$. Also, using the notation of Figure 9.49(c), $K_1K_f = 47.28$, from which $K_1 = 82.93$.

b.
Program:
```
K1=82.93;
numg=K1;
deng=poly([0 -2 -4 -6]);
'G(s)'
G=tf(numg,deng);
Gzpk=zpk(G)
Kf=0.5701
numh=Kf*[1 1.754];
denh=1
'H(s)'
H=tf(numh,denh);
Hzpk=zpk(H)
'T(s)'
T=feedback(G,H);
T=minreal(T)
step(T)
title('Step Response for Feedback Compensated System')
```

Computer response:
```
ans =

G(s)

Zero/pole/gain:
       82.93
-------------------
s (s+6) (s+4) (s+2)
```

```
Kf =

    0.5701

denh =

     1

ans =

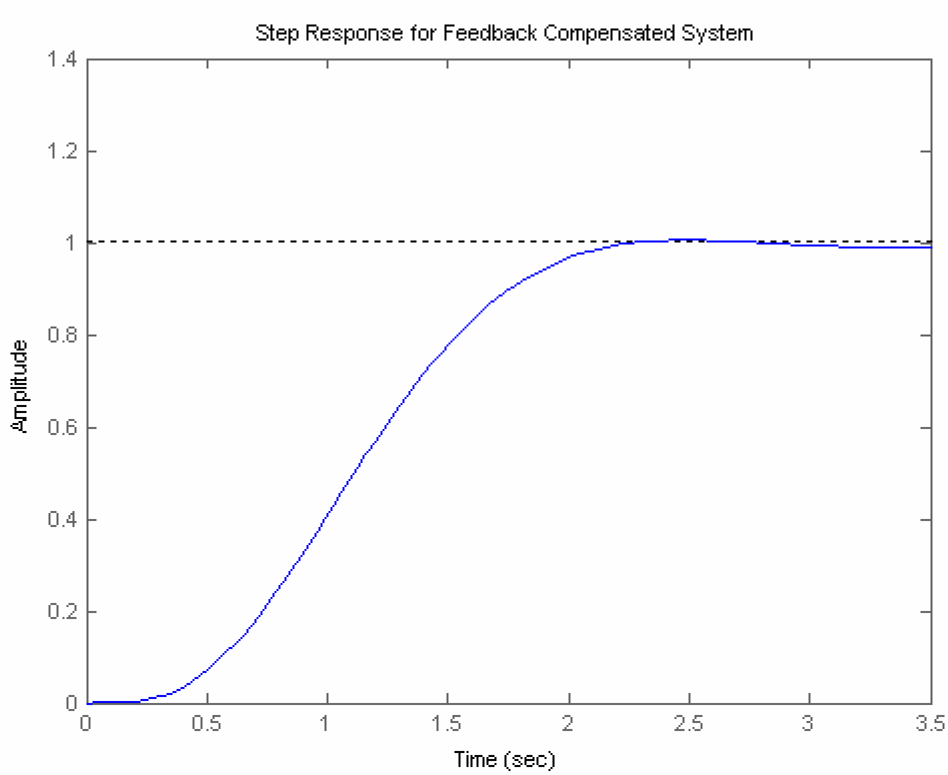
H(s)

Zero/pole/gain:
0.5701 (s+1.754)

ans =

T(s)

Transfer function:
                 82.93
---------------------------------------
s^4 + 12 s^3 + 44 s^2 + 95.28 s + 82.93
```

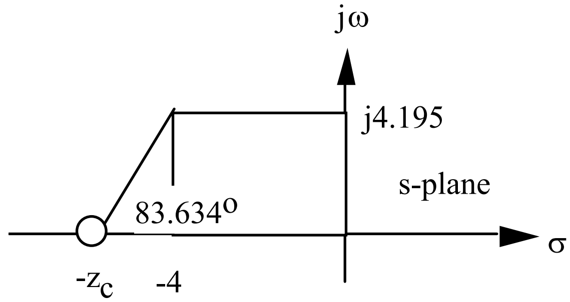


30.

a. $\sigma_d = \zeta\omega_n = 4/T_s = 4/1 = 4$. 5% overshoot -> $\zeta = 0.69$. Since $\zeta\omega_n = 4$, $\omega_n = 5.8$.

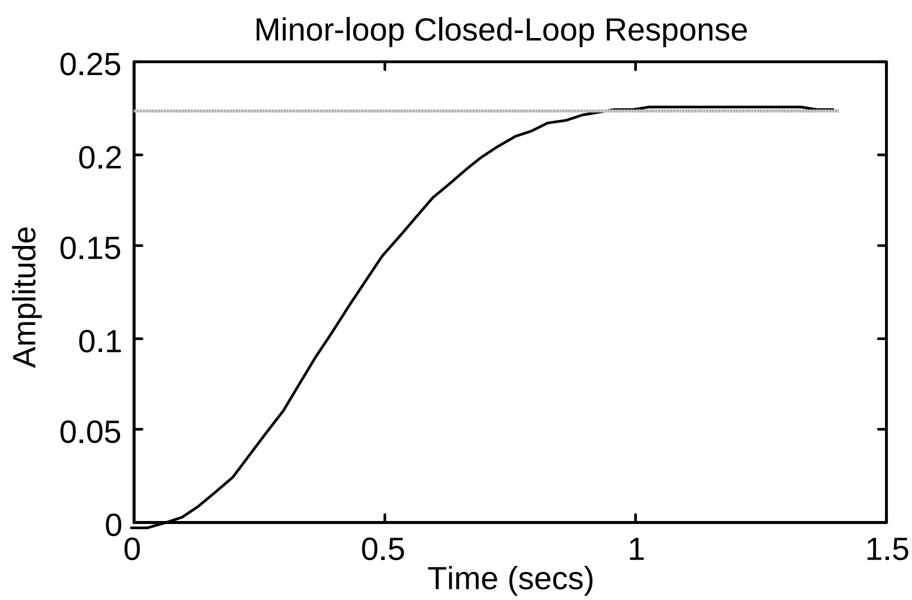
$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.195$. Thus, the design point is $-1 + j4.195$. The sum of angles from the minor-loop's open-loop poles to the design point is -263.634^o. Thus, the minor-loop's open-loop zero must

contribute 83.634^o to yield 180^o at the design point. Hence, $\frac{4.195}{z_c - 4} = \tan 83.634^o$, or $z_c = a = 4.468$ from the geometry below.



Adding the zero and calculating the gain at the design point yields $K_1 = 38.33$. Therefore, the minor-loop open-loop transfer function is $K_1G(s)H(s) = \frac{38.33(s+4.468)}{s(s+4)(s+9)}$. The equivalent minor-loop closed-loop transfer function is $G_{ml}(s) = \frac{K_1G(s)}{1+K_1G(s)H(s)} = \frac{38.33}{s^3+13s^2+74.33s+171.258}$. A simulation of the step response of the minor loop is shown below.

Computer response:



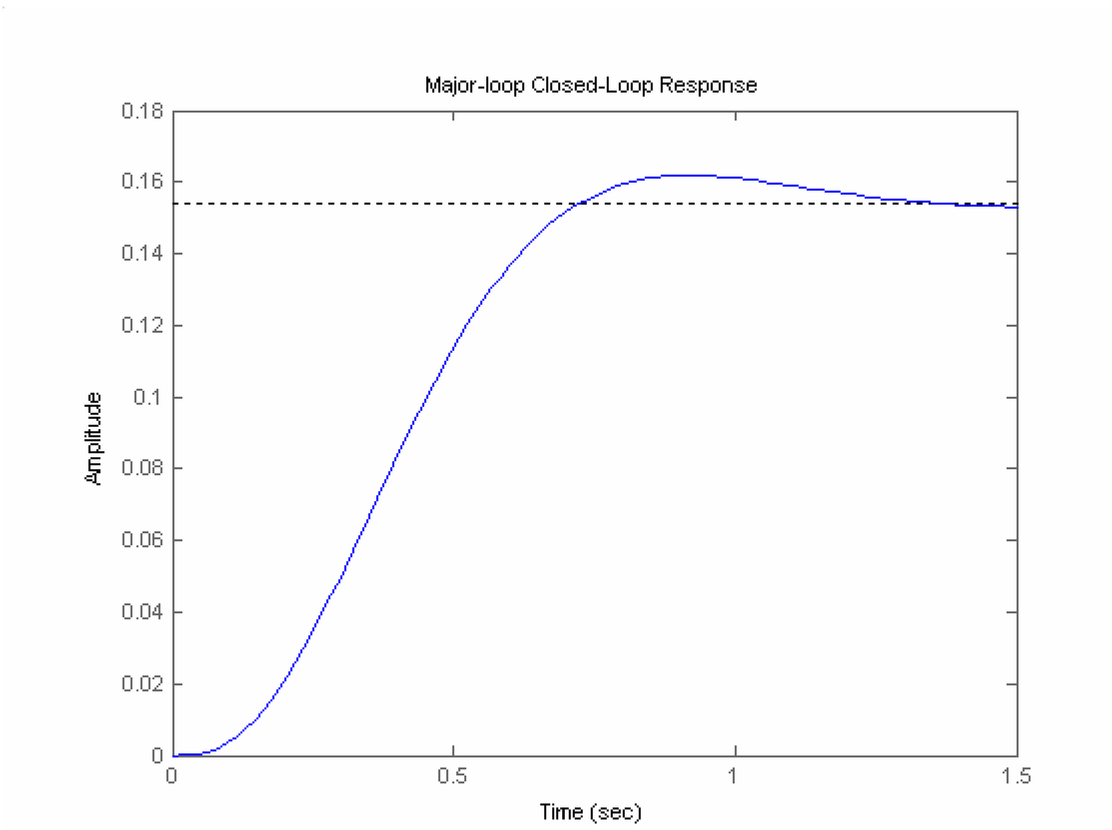
b. The major-loop open-loop transfer function is $G_e(s) = \frac{38.33K}{s^3 + 13s^2 + 74.33s + 171.258}$.

Drawing the root locus using $G_e(s)$ and searching along the 10% overshoot line ($\zeta = 0.591$) for 180^o yields the point $-3.349 + j4.572$ with a gain $38.33K = 31.131$, or $K = 0.812$.

c.
Program:
```
numg=31.131;
deng=[1 13 74.33 171.258];
'G(s)'
G=tf(numg,deng)
T=feedback(G,1);
step(T)
title('Major-loop Closed-Loop Response')
```

Computer response:
```
G(s)


Transfer function:
            31.13
------------------------------
s^3 + 13 s^2 + 74.33 s + 171.3
```



d. Adding the PI compensator, $G_e(s) = \dfrac{31.131(s+0.1)}{s(s^3+13s^2+74.33s+171.258)}$.

Program:
```
numge=31.131*[1 0.1];
denge=[1 13 74.33 171.258 0];
'Ge(s)'
Ge=tf(numge,denge)
T=feedback(Ge,1);
t=0:0.1:10;
step(T,t)
title('Major-loop Closed-Loop Response with PI Compensator')
pause
step(T)
```

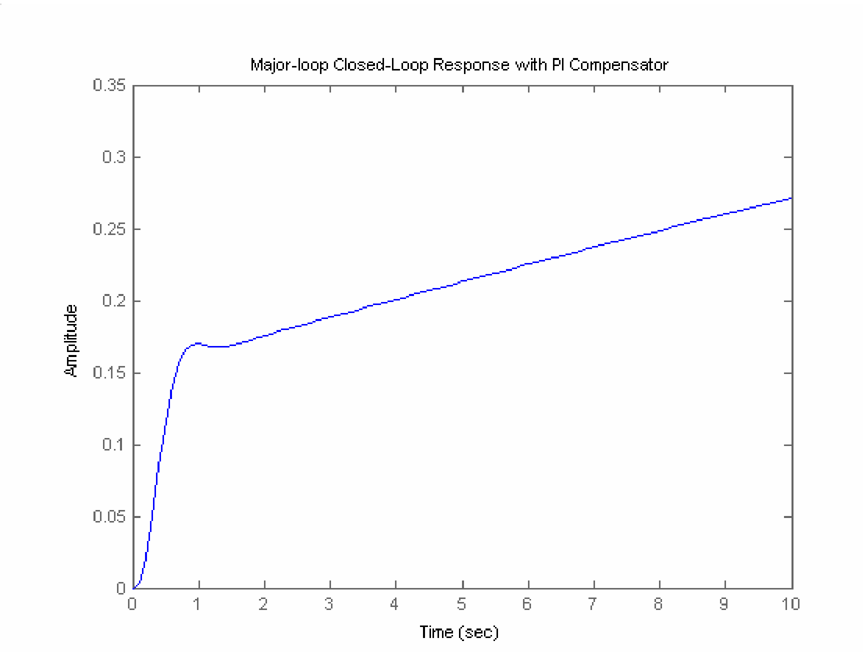
```
title('Major-loop Closed-Loop Response with PI Compensator')
```

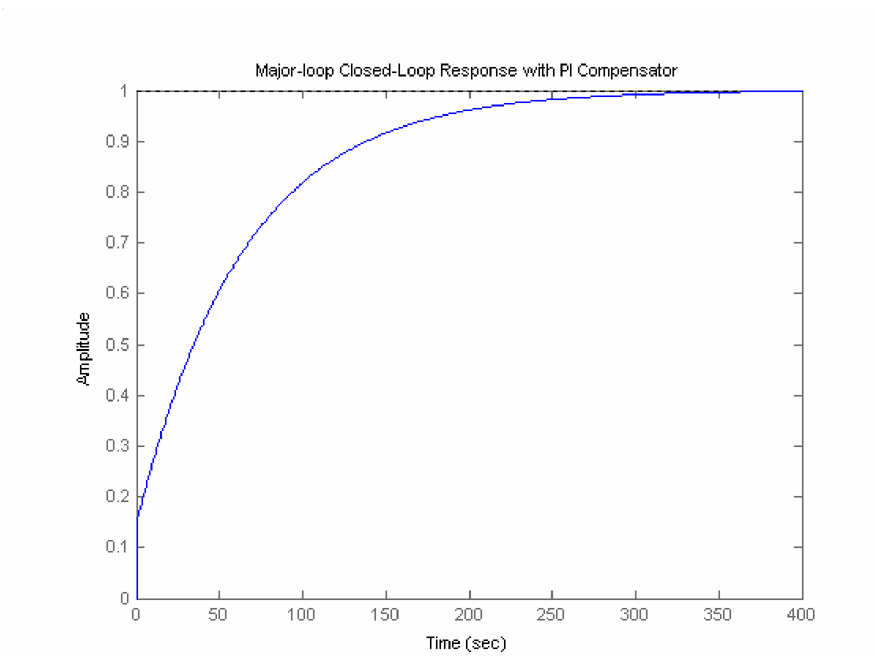
Computer response:

```
ans =

Ge(s)

Transfer function:
         31.13 s + 3.113
---------------------------------
s^4 + 13 s^3 + 74.33 s^2 + 171.3 s
```





31.

a. PI controller: Using Table 9.10, $\frac{R_2}{R_1}\frac{s+\frac{1}{R_2C}}{s} = \frac{s+0.01}{s}$, $R_2C = 100$. Let C = 25 µF. Therefore, $R_2 = 4$ MΩ. For unity gain, $R_1 = 4$ MΩ. Compensate elsewhere in the loop for the compensator negative sign.

b. **PD controller:** Using Table 9.10, $R_2C(s+\frac{1}{R_1C}) = s+2$. Hence, $R_1C = 0.5$. Let C = 1 µF. Therefore, $R_1 = 500$ KΩ. For unity gain, $R_2C = 1$, or $R_2 = 1$ MΩ. Compensate elsewhere in the loop for the compensator negative sign.

32.

a. Lag compensator: See Table 9.11. $\frac{s+\frac{1}{R_2C}}{s+\frac{1}{(R_1+R_2)C}} = \frac{s+0.1}{s+0.01}$. Thus, $R_2C = 10$, and $(R_1 + R_2)C = 100$. Letting C = 10 µ F, we find $R_2 = 1$ MΩ. Also $R_1C = 100 - R_2C = 90$, which yields $R_1 = 9$ MΩ. The loop gain also must be multiplied by $\frac{R_1+R_2}{R_2}$.

b. Lead compensator: See Table 9.11. $\frac{s+\frac{1}{R_1C}}{s+\frac{1}{R_1C}+\frac{1}{R_2C}} = \frac{s+2}{s+5}$. Thus, $R_1C = 0.5$, and $\frac{1}{R_1C}+\frac{1}{R_2C} = 5$. Letting C = 1 µF, $R_2 = 333$ KΩ, and $R_1 = 500$ KΩ.

c. Lag-lead compensation: See Table 9.11.

$$\frac{(s+\frac{1}{R_1C_1})(s+\frac{1}{R_2C_2})}{s^2+(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1})s+\frac{1}{R_1R_2C_1C_2}} = \frac{(s+0.1)(s+1)}{s^2+10.01s+0.1}$$

. Thus, $R_1C_1 = 1$, and $R_2C_2 = 10$. Also, $\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1} = 1 + 0.1 + \frac{1}{R_2C_1} = 10.01$, or $R_2C_1 = 0.112$. Letting C_1 = 10 µF, we find $R_1 = 10$ MΩ , $R_2 = 1.12$ MΩ, and $C_2 = 8.9$ µF.

33.

a. Lag compensator: See Table 9.10 and Figure 9.58. $\frac{s+0.1}{s+0.01} = \frac{C_1}{C_2}\frac{(s+\frac{1}{R_1C_1})}{(s+\frac{1}{R_2C_2})}$. Therefore,

$R_1C_1 = 10$; $R_2C_2 = 100$. Letting $C_1 = C_2 = 20$ µF, we find $R_1 = 500$ KΩ and $R_2 = 5$ MΩ. Compensate elsewhere in the loop for the compensator negative sign.

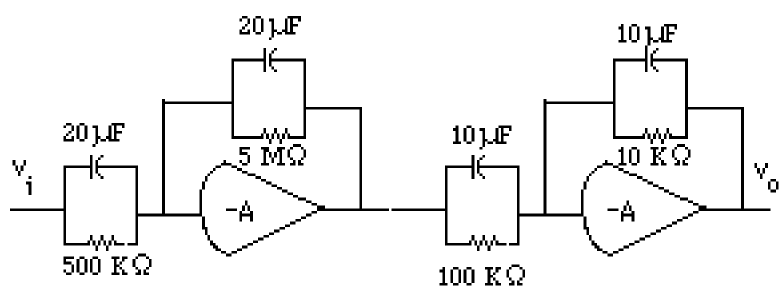
b. Lead compensator: See Table 9.10 and Figure 9.58. $\frac{s+2}{s+5} = \frac{C_1}{C_2}\frac{(s+\frac{1}{R_1C_1})}{(s+\frac{1}{R_2C_2})}$. Therefore,

$R_1C_1 = 0.5$ and $R_2C_2 = 0.2$. Letting $C_1 = C_2 = 20$ μF, we find $R_1 = 25$ KΩ and $R_2 = 10$ MΩ. Compensate elsewhere in the loop for the compensator negative sign.

c. Lag-lead compensator: See Table 9.10 and Figure 9.58. For lag portion, use (a). For lead:

$\frac{s+1}{s+10} = \frac{C_1}{C_2}\frac{(s+\frac{1}{R_1C_1})}{(s+\frac{1}{R_2C_2})}$. Therefore, $R_1C_1 = 1$ and $R_2C_2 = 0.1$. Letting $C_1 = C_2 = 10$ μF, we find

$R_1 = 100$ KΩ and $R_2 = 10$ KΩ. The following circuit can be used to implement the design.



SOLUTIONS TO DESIGN PROBLEMS

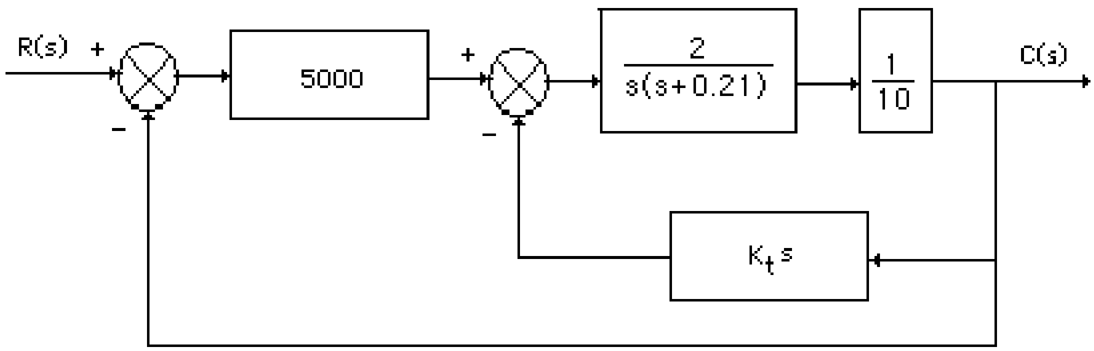
34.

a. $$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_aJ}}{s(s+\frac{1}{J}(D+\frac{K_tK_b}{R_a}))}$$

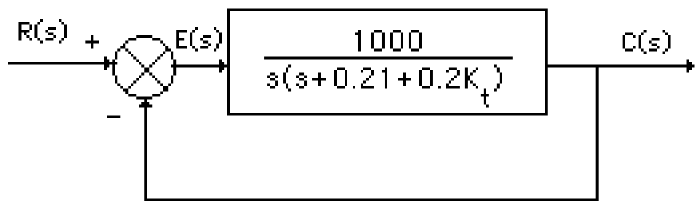
$$K_b = \frac{E_a}{\omega} = \frac{5}{\frac{60000}{2\pi} \times \frac{1}{60} \times 2\pi} = 0.005;\ J_{eq} = 5\left(\frac{4}{10} \times \frac{1}{4}\right)^2 = 0.05;\ D_{eq} = 1\left(\frac{1}{10}\right)^2 = 0.01;$$

$$\frac{K_t}{R_a} = \frac{T_s}{E_a} = \frac{0.5}{5} = 0.1.\ \text{Therefore, } \frac{\theta_m(s)}{E_a(s)} = \frac{2}{s(s+0.21)}.$$

b. The block diagram of the system is shown below.



Forming an equivalent unity feedback system,



Now, $T(s) = \dfrac{1000}{s^2 + (0.21 + 0.2K_t)s + 1000}$. Thus, $\omega_n = \sqrt{1000}$; $2\zeta\omega_n = 0.21 + 0.2K_t$. Since $\zeta = 0.5$,

$K_t = 157.06$.

c. Uncompensated: $K_t = 0$; $T(s) = \dfrac{1000}{s^2 + 0.21s + 1000}$; $\omega_n = 31.62$ rad/s; $\zeta = 3.32 \times 10^{-3}$;

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 98.96\%$; $T_s = \dfrac{4}{\zeta\omega_n} = 38.09$ seconds;

$T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 9.93 \times 10^{-2}$ second; $K_v = \dfrac{1000}{0.21} = 4761.9$.

Compensated: $K_t = 157.06$; $T(s) = \dfrac{1000}{s^2 + 31.62s + 1000}$; $\omega_n = 31.62$ rad/s; $\zeta = 0.5$;

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 16.3\%$; $T_s = \dfrac{4}{\zeta\omega_n} = 0.253$ second; $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.115$ second;

$K_v = \dfrac{1000}{31.62} = 31.63$.

35.

a. $T(s) = \dfrac{25}{s^2 + s + 25}$; Therefore, $\omega_n = 5$; $2\zeta\omega_n = 1$; $\zeta = 0.1$;

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 73\%$; $T_s = \dfrac{4}{\zeta\omega_n} = 8$ seconds.

b. From Figure P9.6(b), $T(s) = \frac{25K_1}{s^2 + (1 + 25K_f)s + 25K_1}$. Thus,

$\omega_n = \sqrt{25K_1}$; $2\zeta\omega_n = 1 + 25K_f$. For 25% overshoot, $\zeta = 0.404$. For $T_s = 0.2 = \frac{4}{\zeta\omega_n}$, $\zeta\omega_n = 20$.

Therefore $1 + 25K_f = 2\zeta\omega_n = 40$, or $K_f = 1.56$. Also, $\omega_n = \frac{20}{\zeta} = 49.5$.

Hence $K_1 = \frac{\omega_n^2}{25} = \frac{49.5^2}{25} = 98.01$.

c. Uncompensated: $G(s) = \frac{25}{s(s+1)}$; Therefore, $K_v = 25$, and $e(\infty) = \frac{1}{K_v} = 0.04$.

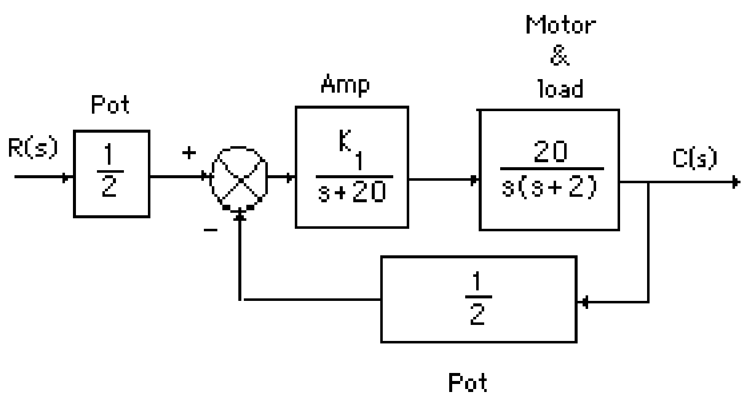
Compensated: $G(s) = \frac{25K_1}{s(s + 1 + 25K_f)}$; Therefore, $K_v = \frac{25 \text{ x } 98.01}{1+25 \text{ x } 1.56} = 61.26$, and

$e(\infty) = \frac{1}{K_v} = 0.0163$.

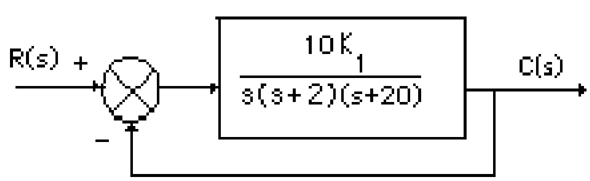
36.

a. The transfer functions of the subsystems are as follows:

Pot: $G_p(s) = \frac{5\pi}{10\pi} = \frac{1}{2}$; Amplifier: $G_a(s) = \frac{K_1}{s+20}$; Motor and load: Since the time to rise to 63% of the final value is 0.5 second, the pole is at -2. Thus, the motor transfer function is of the form, $G_m(s) = \frac{K}{s(s+2)}$. But, from the problem statement, $\frac{K}{2} = \frac{100}{10}$, or K = 20. The block diagram of the system is shown below.



Block diagram



Equivalent system

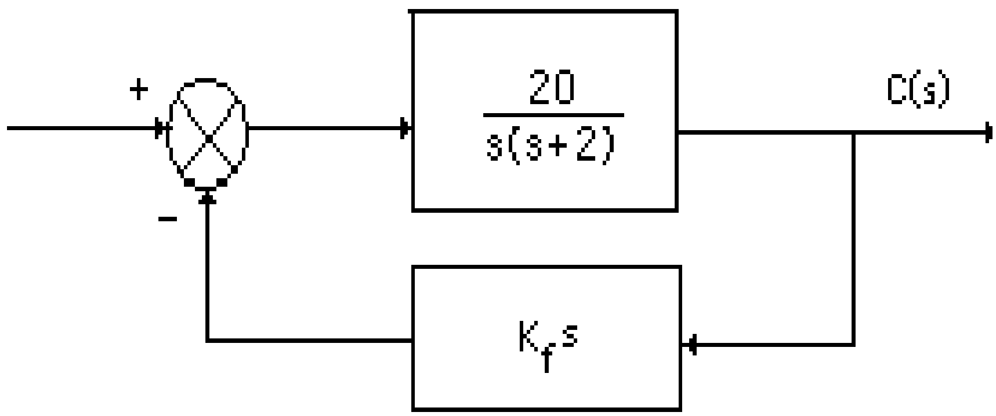
Using the equivalent system, search along the 117.126° line (20% overshoot) and find the dominant second-order pole at - 0.89 + j1.74 with $K = 10K_1 = 77.4$. Hence, $K_1 = 7.74$.

b. $K_v = \frac{77.4}{2 \times 20} = 1.935$. Therefore, $e(\infty) = \frac{1}{K_v} = 0.517$.

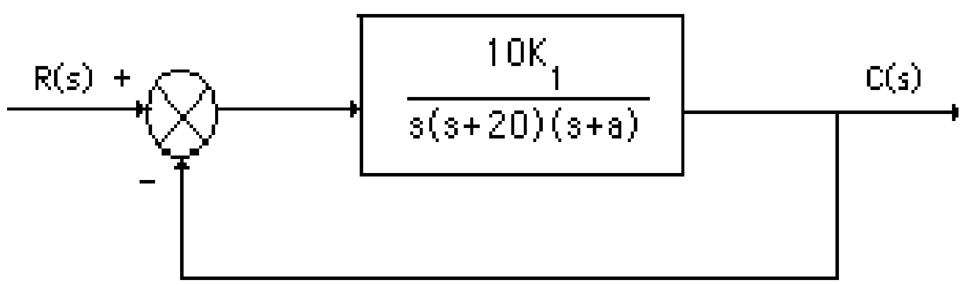
c. %OS = 20%; $\zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} = 0.456$; $\omega_n = \sqrt{0.89^2 + 1.74^2} = 1.95$ rad/s;

$T_s = \frac{4}{\zeta\omega_n} = 4.49$ seconds; $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.81$ seconds.

d. The block diagram of the minor loop is shown below.

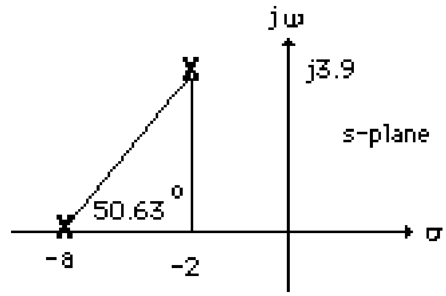


The transfer function of the minor loop is $G_{ML}(s) = \frac{20}{s(s+2+20K_f)}$. Hence, the block diagram of the equivalent system is



where $a = 2 + 20K_f$. The design point is now found. Since %OS = 20%, $\zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} =$ 0.456. Also, since $T_s = \frac{4}{\zeta\omega_n} = 2$ seconds, $\omega_n = 4.386$ rad/s. Hence, the design point is $-2 + j3.9$.

Using just the open-loop poles at the origin and at -20, the summation of angles to the design point is -129.37^o. The pole at -a must then be contributing $129.37^o - 180^o = -50.63^o$. Using the geometry below, a = 5.2, or $K_f = 0.16$.



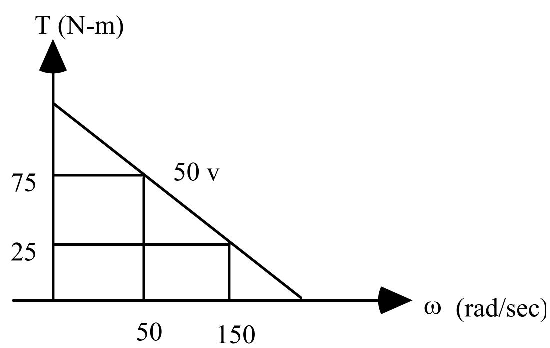
Adding the pole at -5.2 and using the design point, we find $10K_1 = 407.23$, or $K_1 = 40.723$. Summarizing the compensated transient characteristics: $\zeta = 0.456$; $\omega_n = 4.386$; %OS = 20%; $T_s = \frac{4}{\zeta\omega_n} = 2$ seconds; $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.81$ seconds; $K_v = \frac{407.23}{20 \times 5.2} = 3.92$.

37.

Block diagram

Preamplifier/Power amplifier: $\frac{K_1}{(s+40)}$; Pots: $\frac{20\pi \text{ volts}}{5(2\pi) \text{ rad.}} = 2$.

Torque-speed curve:



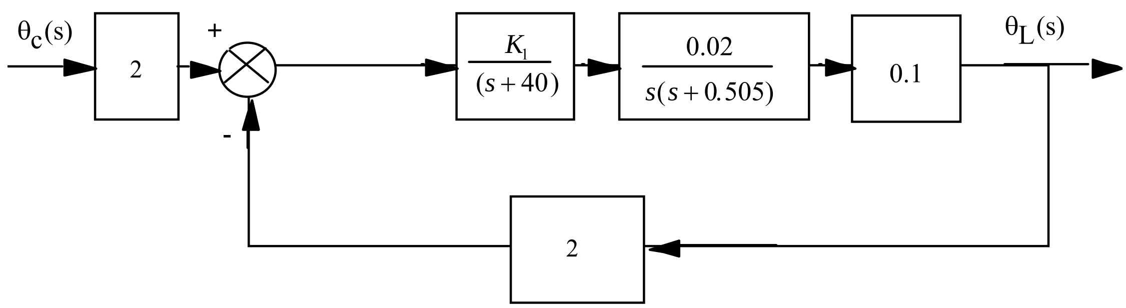
where $1432.35 \frac{\text{rev}}{\text{min}} \times \frac{1}{60} \frac{\text{min}}{\text{sec}} \times 2\pi \frac{\text{rad}}{\text{rev}} = 150$ rad/sec; $477.45 \frac{\text{rev}}{\text{min}} \times \frac{1}{60} \frac{\text{min}}{\text{sec}} \times 2\pi \frac{\text{rad}}{\text{rev}} = 50$ rad/sec.

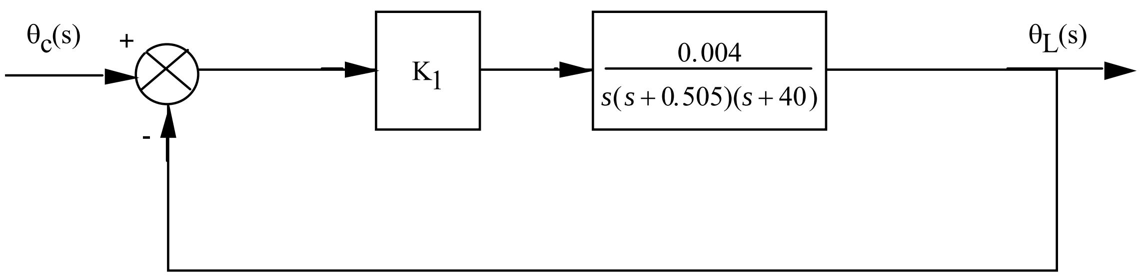
The slope of the line is $-\frac{50}{100} = -0.5$. Thus, its equation is y = -0.5x + b. Substituting one of the points, find b = 100. Thus $T_{stall} = 100$, and $\omega_{no\ load} = 200$. $\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} = \frac{100}{50} = 2$; $K_b = \frac{e_a}{\omega_{no\ load}} = \frac{50}{200} = 0.25$.

Motor: $\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_aJ)}{s(s+\frac{1}{J}(D+\frac{K_tK_b}{R_a}))} = \frac{0.02}{s(s+0.505)}$, where J = 100, D = 50.

Gears: 0.1

Drawing block diagram:

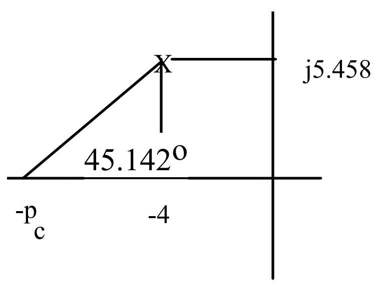




b. Compensator design - Lead

10% overshoot and T_s = 1 sec yield a design point of - 4 + j5.458. Sum of angles of uncompensated system poles to this point is -257.491°. If we place the lead compensator zero over the uncompensated system pole at -0.505, the angle at the design point is -134.858°. Thus, the lead compensator pole must contribute 134.858° - 180° = -45.142°. Using the geometry below

$\frac{5.458}{p_c - 4}$ = tan(45.142°), or p_c = 9.431.



Using the uncompensated poles and the lead compensator, the gain at the design point is $0.004K_1$ = 1897.125.

Compensator design - Lag

With lead compensation, $K_v = \frac{1897.125}{(40)(9.431)} = 5.0295.029$. Since we want $K_v = 1000$, $\frac{z_{lag}}{p_{lag}} = \frac{1000}{5.029} = 198.85$. Use $p_{lag} = 0.001$. Hence $z_{lag} = 0.1988$. The lag compensated

$$G_e(s) = \frac{1897.125(s+0.1988)}{s(s+40)(s+9.431)(s+0.001)}.$$

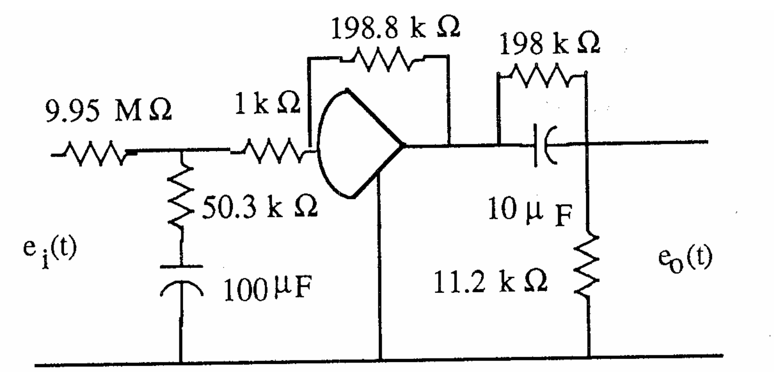
c. Compensator schematic

lag: $\frac{1}{R_2C} = 0.1988$. Let C = 100 μF. Then $R_2 = 50.3$ kΩ. Now, $\frac{1}{(R_1+R_2)C} = 0.001$.

Thus, $R_1 = 9.95$ MΩ. Buffer gain = reciprocal of lag compensator's $\frac{R_2}{R_1 + R_2}$. Hence buffer gain = $\frac{R_1 + R_2}{R_2} = 198.8$.

lead: $\frac{1}{R_1C} = 0.505$. Let C = 10 μF. Then $R_1 = 198$ kΩ. Now, $\frac{1}{R_1C} + \frac{1}{R_2C} = 9.431$.

Thus, $R_2 = 11.2$ k□Ω.



d.

Program:

```
numg= 1897.125*[1 0.1988];
deng=poly([0 -40 -9.431 -.001]);
'G(s)'
G=tf(numg,deng);
Gzpk=zpk(G)
rlocus(G)
pos=10
z=-log(pos/100)/sqrt(pi^2+[log(pos/100)]^2)
sgrid(z,0)
title(['Root Locus with ' , num2str(pos), ' Percent Overshoot Line'])
[K,p]=rlocfind(G)  %Allows input by selecting point on graphic
pause
T=feedback(K*G,1);
step(T)
title(['Step Response for Design of ' , num2str(pos), ' Percent'])
```

Computer response:

```
ans =

G(s)


Zero/pole/gain:
    1897.125 (s+0.1988)
----------------------------
s (s+40) (s+9.431) (s+0.001)
pos =

    10


z =

    0.5912

Select a point in the graphics window

selected_point =

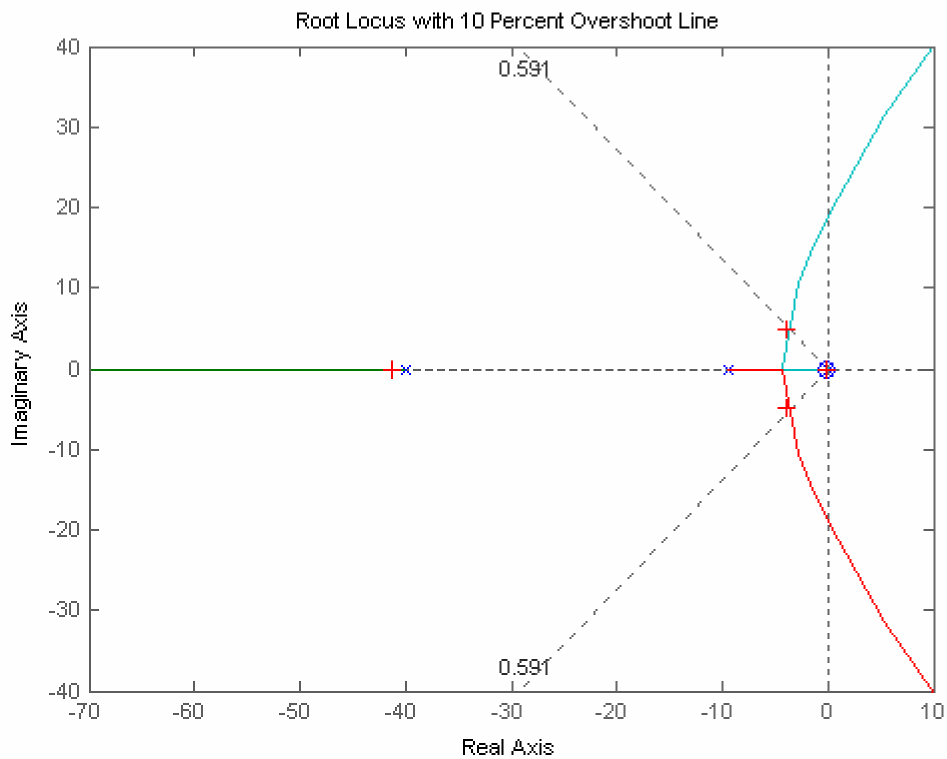
  -3.3649 + 4.8447i

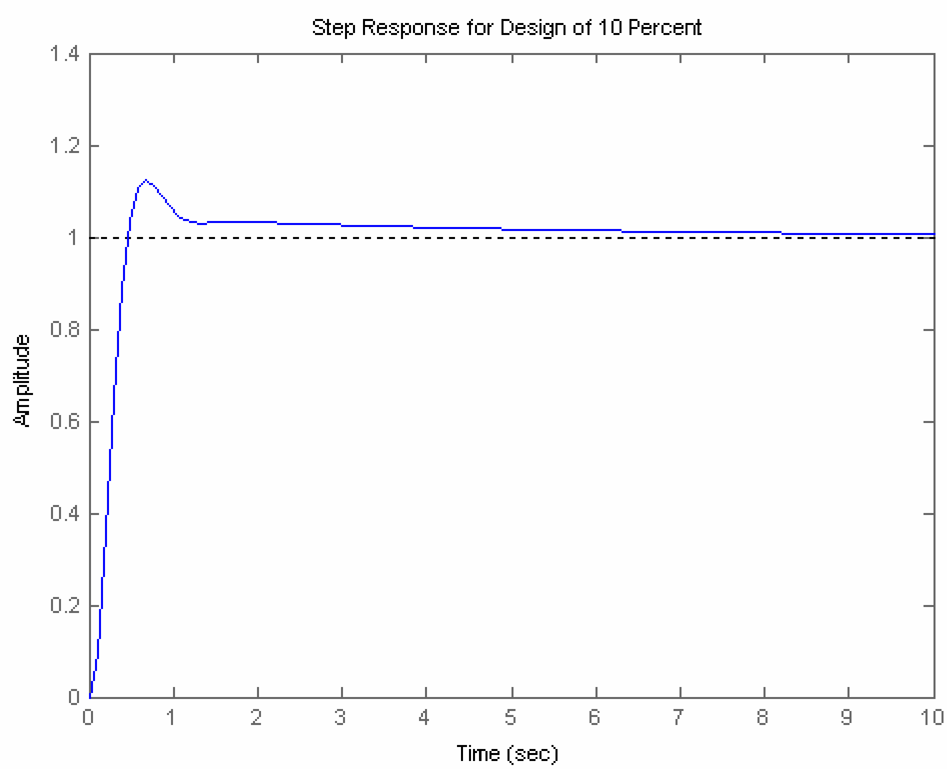

K =

    0.9090


p =

 -41.3037
  -3.9602 + 4.9225i
  -3.9602 - 4.9225i
  -0.2080
```


Root Locus with 10 Percent Overshoot Line
0.591
0.591
Imaginary Axis
Real Axis
40
30
20
10
0
-10
-20
-30
-40
-70
-60
-50
-40
-30
-20
-10
0
10


Step Response for Design of 10 Percent
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
9
10

38.

Consider only the minor loop. Searching along the 143.13° line (ζ = 0.8), locate the minor-loop dominant poles at -3.36 ± j2.52 with K_f = 8.53. Searching the real axis segments for K_f = 8.53 locates a higher-order pole at - 0.28. Using the minor-loop poles as the open-loop poles for the entire system, search along the 120° line (ζ = 0.5) and find the dominant second-order poles at -1.39 + j2.41 with K = 27.79. Searching the real axis segment locates a higher-order pole at - 4.2.

39.

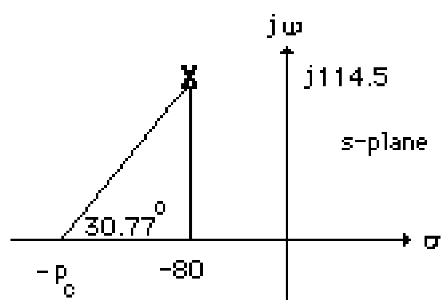
Consider only the minor loop. Searching along the 143.13° line (ζ = 0.8), locate the minor-loop dominant poles at -7.74 ± j5.8 with K_f = 36.71. Searching the real axis segments for K_f = 36.71 locates a higher-order pole at - 0.535. Using the minor-loop poles at -7.74 ± j5.8 and - 0.535 as the open-loop poles (the open-loop zero at the origin is not a closed-loop zero) for the entire system, search along the 135° line (ζ = 0.707; 4.32% overshoot) and find the dominant second-order poles at - 4.38 + j4 .38 with K = 227.91. Searching the real axis segment locates a higher-order pole at -7.26. Uncompensated system performance: Setting K_f = 0 and searching along the 135° line (4.32% overshoot) yields -2.39 + j2.39 as the point on the root locus with K = 78.05. Searching the real axis segments of the root locus for K = 78.05 locates a higher-order pole at -11.2. The following table compares the predicted uncompensated characteristics with the predicted compensated characteristics.

Uncompensated	**Compensated**
$G(s) = \dfrac{78.05}{(s+1)(s+5)(s+10)}$	$G(s) = \dfrac{227.91}{s^3+16s^2+101.71s+50}$
Dominant poles: -2.39 + j2.39	Dominant poles: - 4.38 + j4 .38
$\zeta = 0.707$	$\zeta = 0.707$
$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 4.32\%$	$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 4.32\%$
$\omega_n = \sqrt{2.39^2+2.39^2} = 3.38$ rad/s	$\omega_n = \sqrt{4.38^2+4.38^2} = 6.19$ rad/s
$T_s = \dfrac{4}{\zeta\omega_n} = 1.67$ seconds	$T_s = \dfrac{4}{\zeta\omega_n} = 0.91$ second
$T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.31$ seconds	$T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.72$ second
$K_p = \dfrac{78.05}{1 \times 5 \times 10} = 1.56$	$K_p = \dfrac{227.91}{50} = 4.56$
Higher-order pole: -11.22	Higher-order pole: -7.26
Second-order approximation OK	Higher-order pole not 5x further from imaginary axis than dominant poles. Simulate to be sure of the performance.

40.

In Problem 46, Chapter 8 , the dominant poles, - 40 ± j57.25, yielded T_s = 0.1 second and 11.14% overshoot. The unity feedback system consisted of a gain adjusted forward transfer function of

$G(s) = \frac{20000K}{s(s+100)(s+500)(s+800)}$, where K = 102,300. To reduce the settling time by a factor of 2 to 0.05 seconds and keep the percent overshoot the same, we double the coordinates of the dominant poles to $-80 \pm j114.5$. Assume a lead compensator with a zero at -100 that cancels the plant's pole at -100. The summation of angles of the remaining plant poles to the design point is 149.23°. Thus, the angular contribution of the compensator pole must be 149.23° - 180° = 30.77°. Using the geometry below, $\frac{114.5}{p_c - 80} = \tan 30.77°$, or $p_c = 272.3$.



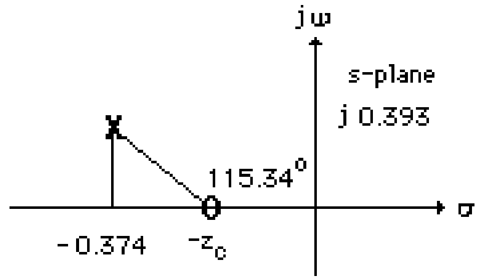
Adding this pole to the poles at the origin, -500, and -800 yields K = 9.92 x 10^9 at the design point, $-80 \pm j114.5$. Any higher-order poles will have a real part greater than 5 times that of the dominant pair. Thus, the second-order approximation is OK.

41.

Uncompensated: $G(s)H(s) = \frac{0.35K}{(s+0.4)(s+0.5)(s+0.163)(s+1.537)}$. Searching the 133.639° line (%OS = 5%), find the dominant poles at $- 0.187 \pm j0.196$ with gain, 0.35K = 2.88 x 10^{-2}. Hence, the estimated values are: %OS = 5%; $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.187} = 21.39$ seconds; $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{0.196} =$ 16.03 seconds; $K_p = 0.575$.

PD compensated: Design for 8 seconds peak time and 5% overshoot.

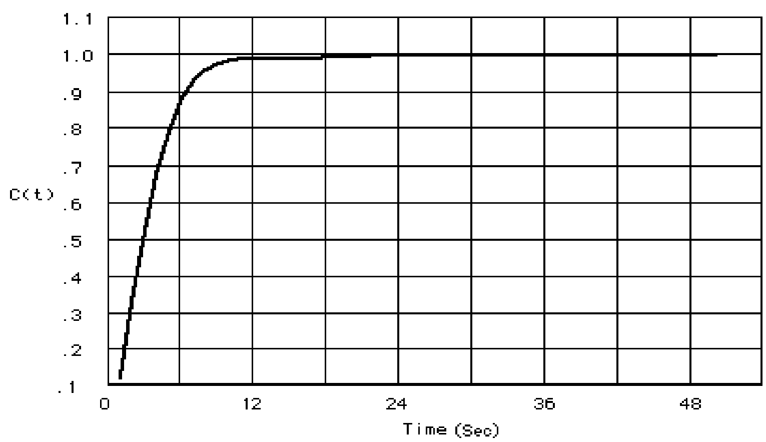
$$\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.69.$$ Since $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 8$ seconds and $\omega_n\sqrt{1-\zeta^2} = 0.393$, $\omega_n = 0.5426$. Hence, $\zeta\omega_n = 0.374$. Thus, the design point is $- 0.374 + j0.393$. The summation of angles from the system's poles to the design point is -295.34°. Thus, the angular contribution of the controller zero must be 295.34°-180° = 115.34°. Using the geometry below,



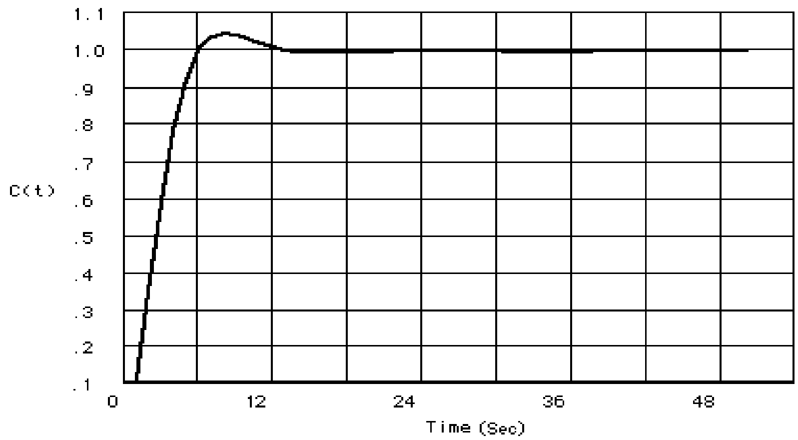
$\frac{0.393}{0.374 - z_c}$ = tan (180° - 115.34°), from which z_c = 0.19. Adding this zero to the system's poles and using the design point, - 0.374 + j0.393, the gain, 0.35K = 0.205.

PID compensated: Assume the integral controller, $G_c(s) = \frac{s+0.01}{s}$. The total open-loop transfer function is $G_{PID}(s)H(s) = \frac{0.35K(s+0.19)(s+0.01)}{s(s+0.4)(s+0.5)(s+0.163)(s+1.537)}$.

Check: The PID compensated system yields a very slow rise time due to the lag zero at 0.01. The rise time can be sped up by moving the zero further from the imaginary axis with resultant changes in the transient response. The plots below show the step response with the PI zero at - 0.24.



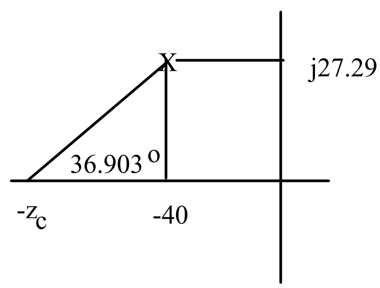
The response compares favorably with a two-pole system step response that yields 5% overshoot and a peak time of 8 seconds as shown below.



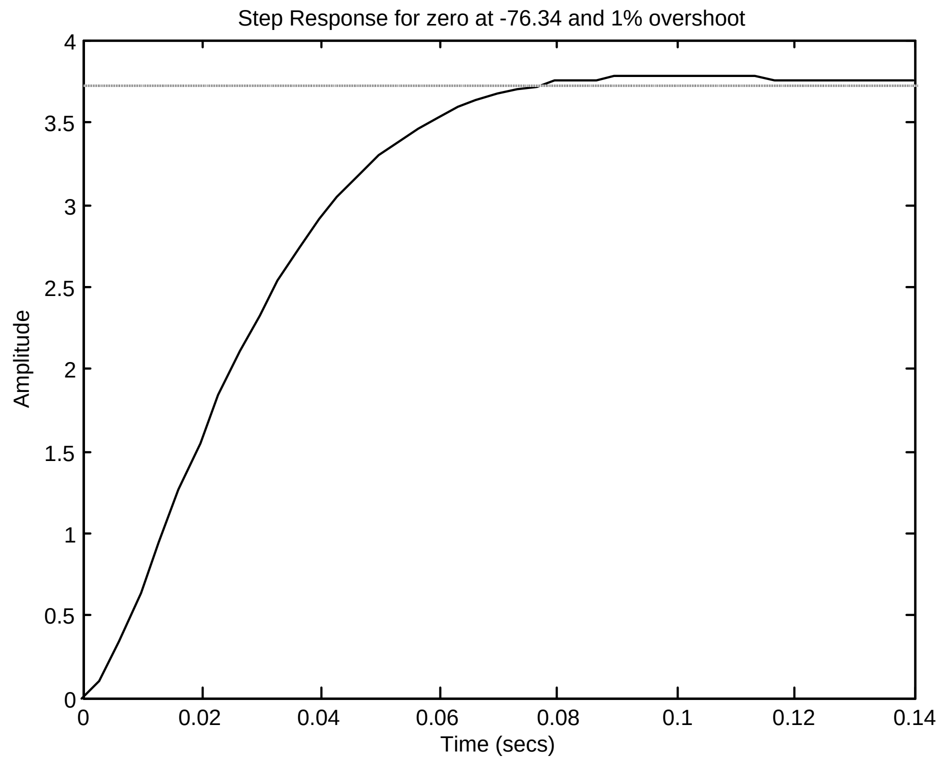
42.

a. PD compensator design: Pushing the gain, 10, to the right past the summing junction, the system can be represented as an equivalent unity feedback system with $G_e(s) = \frac{10^6}{(s^2 - 4551)(s+286)}$.

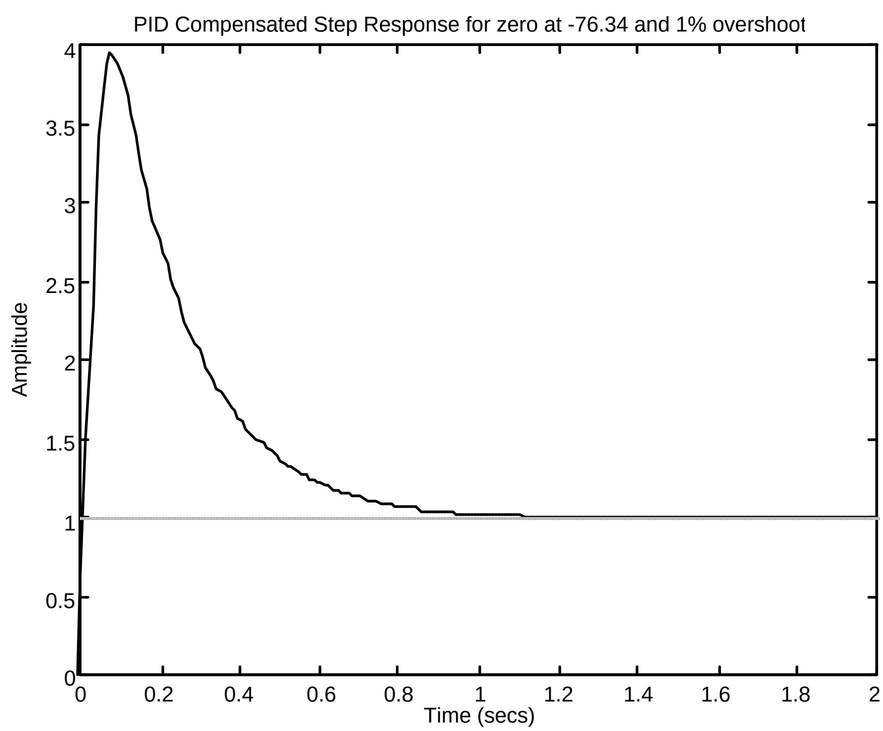
This system is unstable at any gain. For 1% overshoot and $T_s = 0.1$, the design point is –40 + j27.29. The summation of angles from the poles of $G_e(s)$ to this point is -216.903°. Therefore, the compensator zero must contribute 216.903° - 180° = 36.903°. Using the following geometry:



$\frac{27.29}{z_c - 40}$ = tan (36.903). Thus, z_c = 76.34. Adding this zero to the poles of $G_e(s)$, the gain at the design point is 10^6K = 23377. The PD compensated response is shown below.



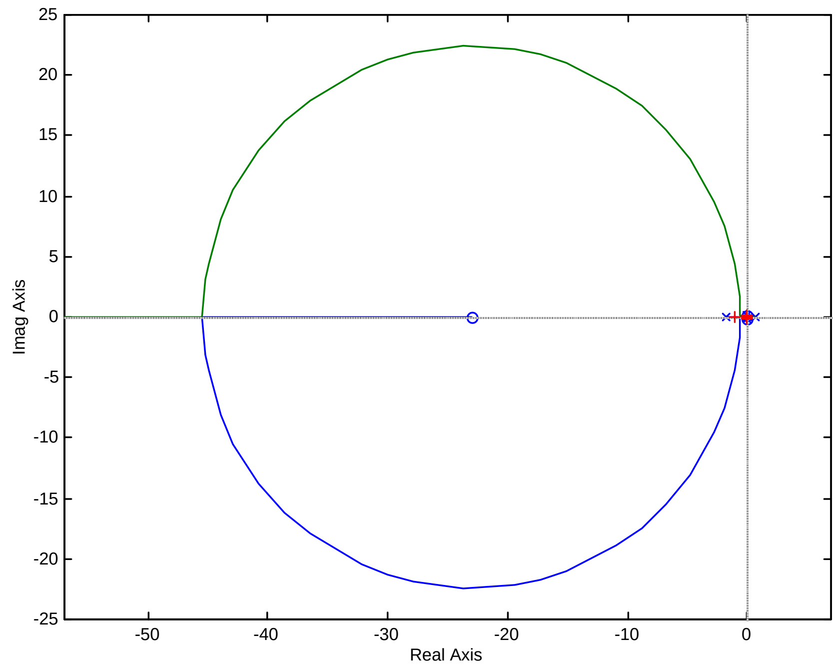
b. PI compensator design: To reduce the steady-state error to zero, we add a PI controller of the form $\frac{s+1}{s}$. The PID compensated step response is shown below.



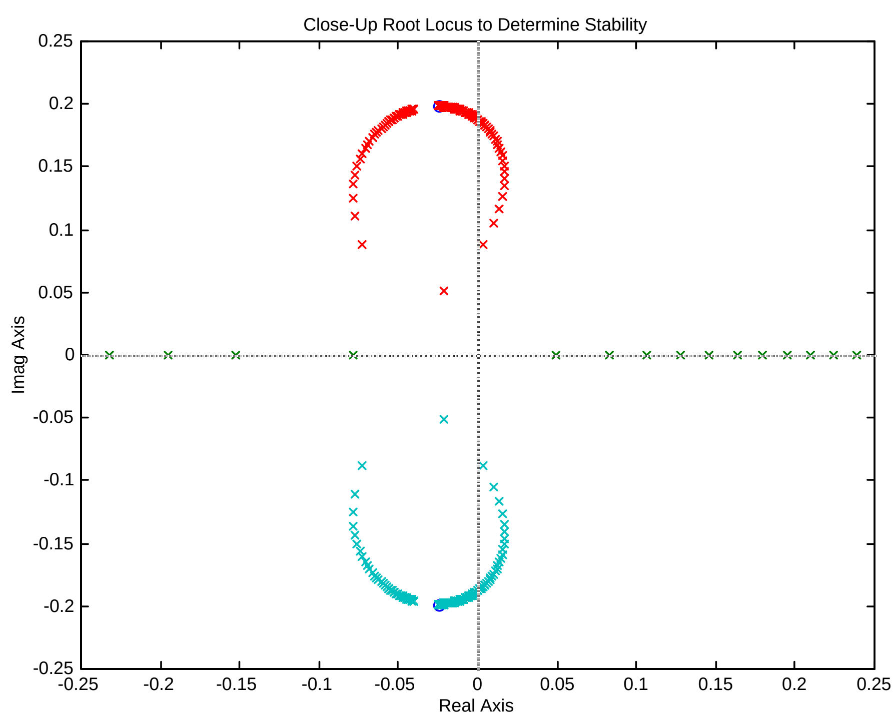
We can see the 1% overshoot at about 0.1 second as in the PD compensated system above. But the system now corrects to zero error.

43.

a. Root locus sketch yields;



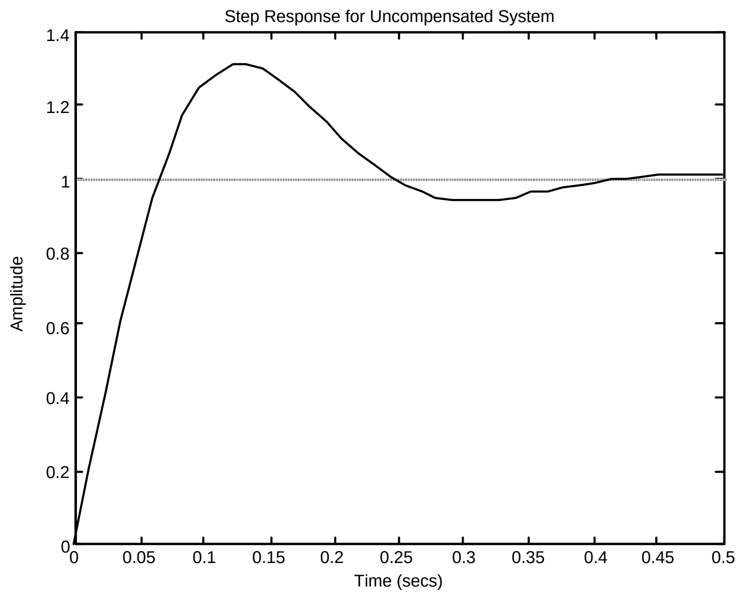
Root locus sketch near imaginary axis yields;



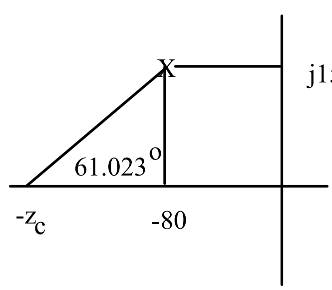
Searching imaginary axis for 180° yields: j0.083 at a gain of 0.072K = 0.0528 and j0.188 at a gain of 0.072K = 0.081. Also, the gain at the origin is 0.0517. Thus, the system is stable for 0.0517 < 0.072K < 0.0528; 0.072K > 0.081. Equivalently, for 0.7181 < K < 0.7333; 0.072K > 1.125.

b. See (a)

c. Uncompensated system: Searching the 20% overshoot line, we find the operating point at -8.987 + j17.4542 = 19.71∠117.126° at 0.072K = 16.94 for the uncompensated system. Simulating the response at this gain yields,

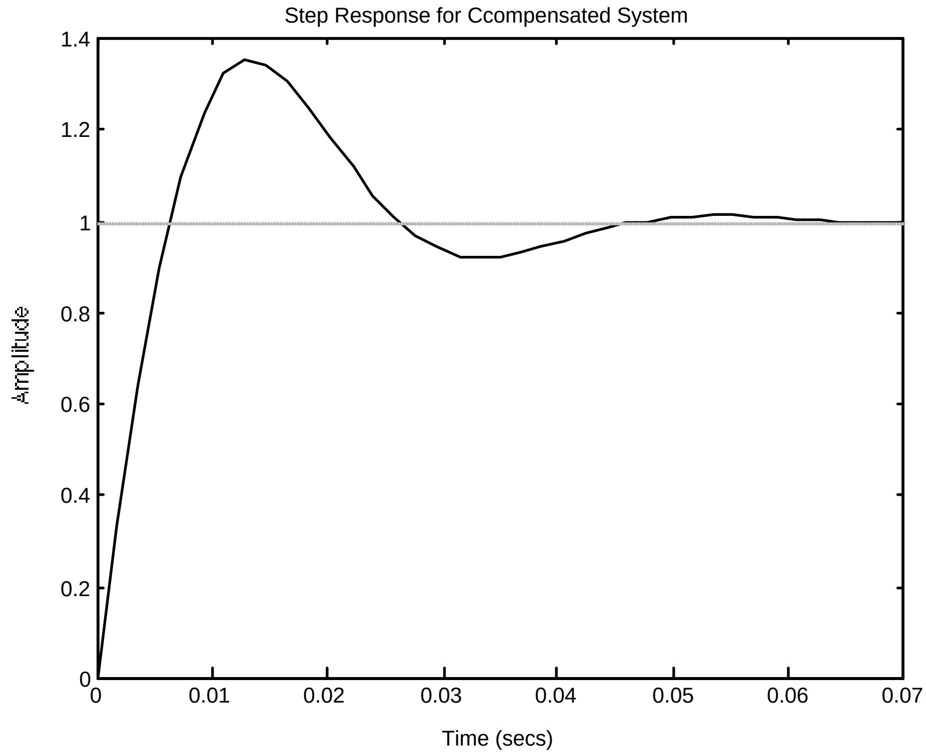


For 20% overshoot and T_s = 0.05 s, a design point of –80 + j156.159 is required. The sum of angles to the design point is -123.897°. To meet the requirements at the design point, a zero would have to contribute +303.897°, which is too high for a single zero. Let us first add the pole at the origin to drive the steady-state error to zero to reduce the angle required from the zero. Summing angles with this pole at the origin yields -241.023. Thus a zero contributing 61.023° is required. Using the geometry below with $\frac{156.159}{z_c - 80}$ = tan (61.023), z_c = 166.478.

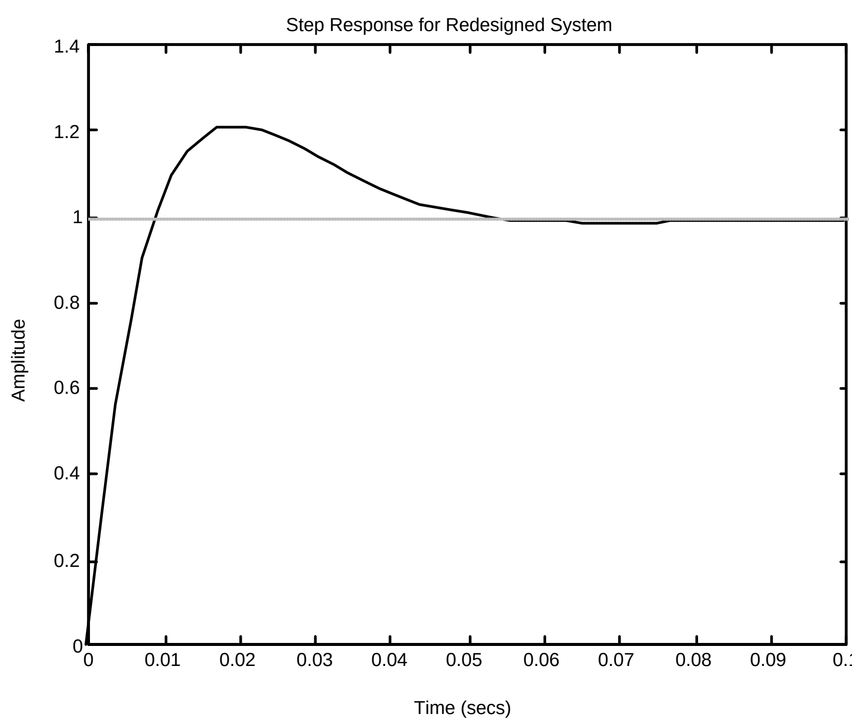


The gain at the design point is 0.072K = 181.55.

d.



The settling time requirement has been met, but the percent overshoot has not. Repeating the design for 1% overshoot and a T_S = 0.05 s yields a design point of –80 + j54.575. The compensator zero is found to be at -47.855 at a gain 0.072K = 180.107.

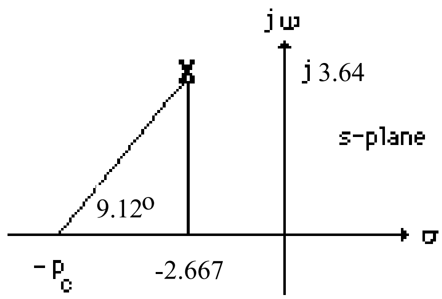


44.

$$\zeta\omega_n = \frac{4}{T_s} = 2.667;\ \ \zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.591.$$ Thus, $\omega_n = 4.512$ rad/s.

$\text{Im} = \omega_n\sqrt{1-\zeta^2} = 4.512\sqrt{1-0.591^2} = 3.64$. Thus, and the operating point is $-2.667 \pm j3.64$.

Summation of angles, assuming the compensating zero is at –5 (to cancel the open-loop pole at –5, is –170.88°. Therefore, the compensator pole must contribute 180° – 170.88° = -9.12°. Using the geometry shown below,

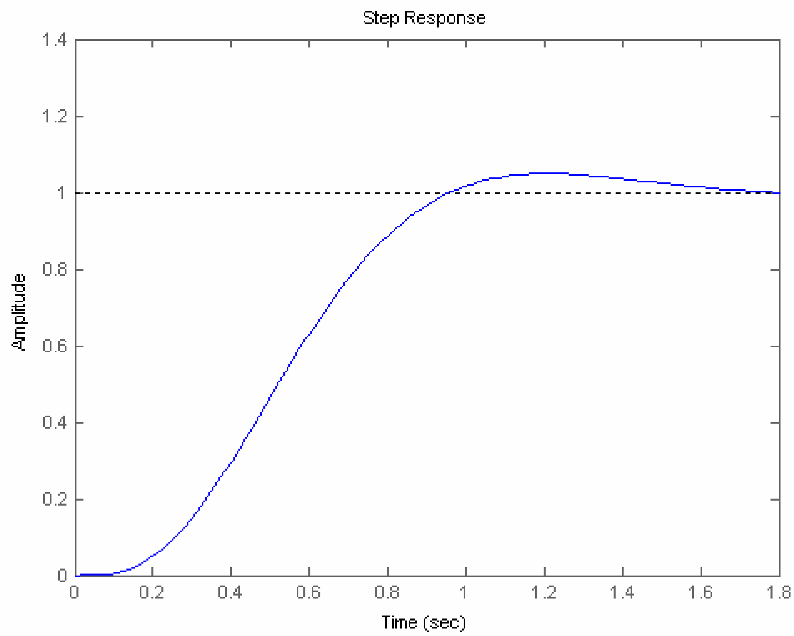


$\frac{3.64}{p_c - 2.667} = \tan 9.12°$. Thus, $p_c = 25.34$. Adding the compensator pole and using $-2.667 \pm j3.64$ as the test point, 50K = 2504, or K = 50.08. Thus the compensated open-loop transfer function is

$$G_e(s) = \frac{2504(s+5)}{s(s+5)(s^2+10s+50)(s+25.34)}$$. Higher-order pole are at –25.12, -5, and-4.898. The pole at –5 is cancelled by the closed-loop zero at -5. The pole at –4.898 is not far enough away from the dominant second-order pair. Thus, the system should be simulated to determine if the response meets the requirements.

Program:
```
syms s
numg=2504;
deng=expand(s*(s^2+10*s+50)*(s+25.34));
deng=sym2poly(deng);
G=tf(numg,deng);
Gzpk=zpk(G)
T=feedback(G,1);
step(T)
```

Computer response:
```
Zero/pole/gain:
             2504
----------------------------
s (s+25.34) (s^2 + 10s + 50)
```



45.

a. From Chapter 8,

$$G_e(s) = \frac{0.6488K\,(s+53.85)}{(s^2 + 8.119s + 376.3)\,(s^2 + 15.47s + 9283)}$$

Cascading the notch filter,

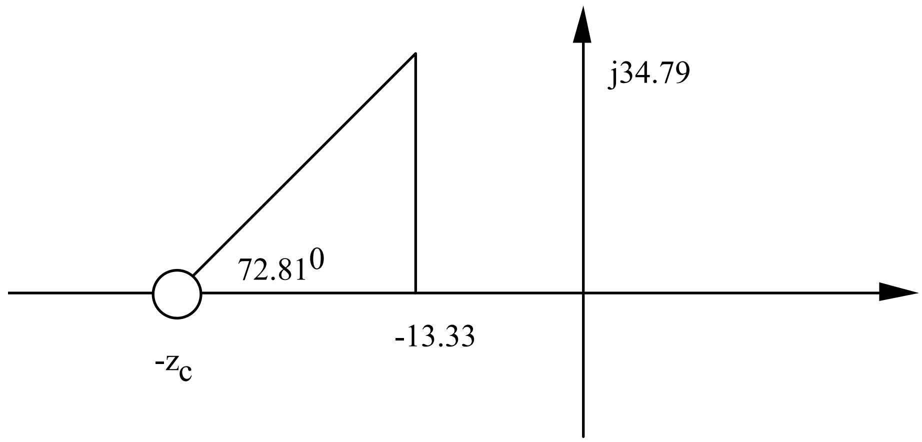
$$G_{et}(s) = \frac{0.6488K\,(s+53.85)(s^2 + 16s + 9200)}{(s^2 + 8.119s + 376.3)\,(s^2 + 15.47s + 9283)(s+60)^2}$$

Arbitrarily design for %OS = 30% ($\zeta = 0.358$) and $T_s = 0.3$ s. This places desired poles at

$-13.33 \pm j34.79$. At the design point, the sum of the angles without the PD controller is 107.19^0.

Thus,

$$\frac{34.79}{z_c - 13.33} = \tan 72.81$$



From which, $z_c = 24.09$. Putting this into the forward path,

$$G_{et}(s) = \frac{0.6488K\,(s+53.85)(s^2 + 16s + 9200)(s+24.09)}{(s^2 + 8.119s + 376.3)\,(s^2 + 15.47s + 9283)(s+60)^2}$$

Using root locus, the gain 0.6488K = 1637, or K = 2523.

b. Add a PI controller

$$G_{PI}(s) = \frac{(s+0.1)}{s}$$

Thus,

$$G_{et}(s) = \frac{0.6488K\,(s+53.85)(s^2 + 16s + 9200)(s+24.09)(s+0.1)}{s\,(s^2 + 8.119s + 376.3)\,(s^2 + 15.47s + 9283)(s+60)^2}$$

Using root locus, the gain 0.6488K = 1740, or K = 2682.

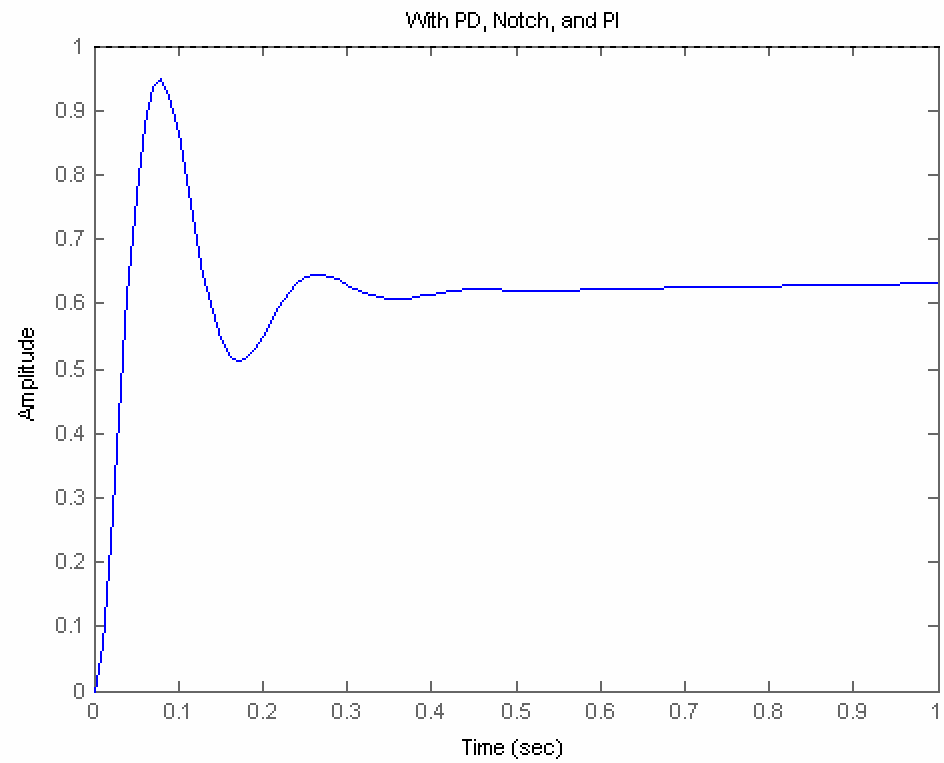
c.

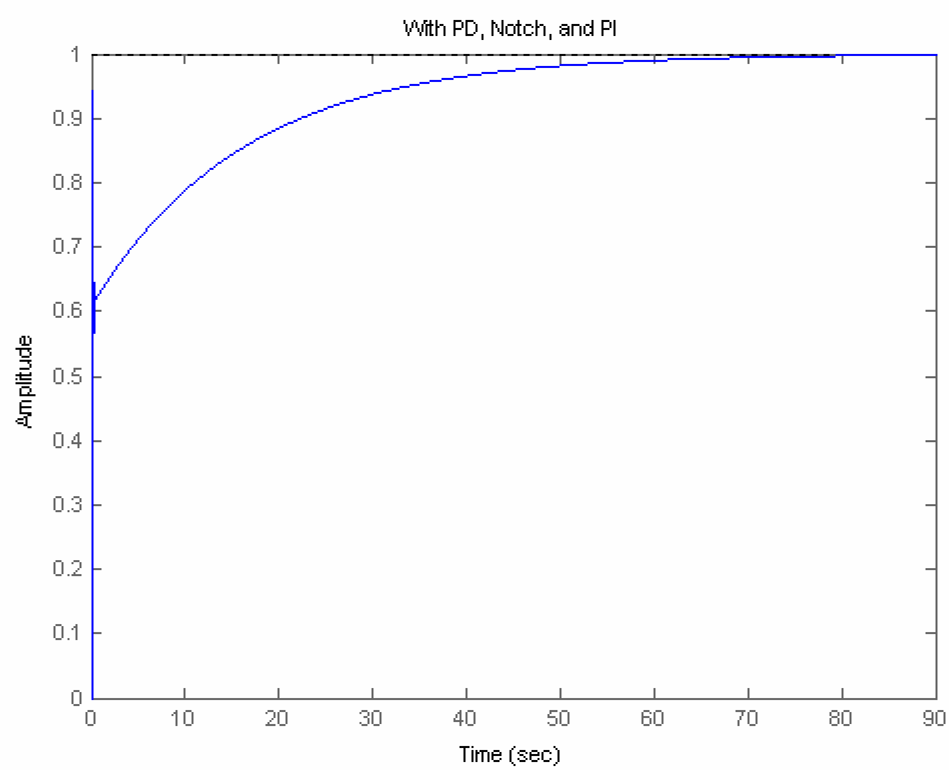
Program:

```
syms s
numg=1637*(s+53.85)*(s^2+16*s+9200)*(s+24.09)*(s+0.1);
deng=s*(s^2+15.47*s+9283)*(s^2+8.119*s+376.3)*(s+60)^2;
numg=sym2poly(numg);
deng=sym2poly(deng);
G=tf(numg,deng);
Gzpk=zpk(G)
T=feedback(G,1);
step(T,0:0.01:1)
title(['With PD, Notch, and PI'])
pause
step(T)
title(['With PD, Notch, and PI'])
```

Computer response:

```
Zero/pole/gain:
  1637 (s+53.85) (s+24.09) (s+0.1) (s^2 + 16s + 9200)
-------------------------------------------------------
s (s+60)^2 (s^2 + 8.119s + 376.3) (s^2 + 15.47s + 9283)
```


With PD, Notch, and PI
Amplitude
Time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


With PD, Notch, and PI
Amplitude
Time (sec)
0
10
20
30
40
50
60
70
80
90
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

T E N

Frequency Response Techniques

SOLUTION TO CASE STUDY CHALLENGE

Antenna Control: Stability Design and Transient Performance

First find the forward transfer function, G(s).

Pot:

$$K_1 = \frac{10}{\pi} = 3.18$$

Preamp:

$$K$$

Power amp:

$$G_1(s) = \frac{100}{s(s+100)}$$

Motor and load:

$$J = 0.05 + 5\left(\frac{1}{5}\right)^2 = 0.25\ ;\ D = 0.01 + 3\left(\frac{1}{5}\right)^2 = 0.13;\ \frac{K_t}{Ra} = \frac{1}{5}\ ;\ K_b = 1.$$

Therefore,

$$G_m(s) = \frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a J}}{s\left(s+\frac{1}{J}\left(D + \frac{K_t K_b}{R_a}\right)\right)} = \frac{0.8}{s(s+1.32)}\ .$$

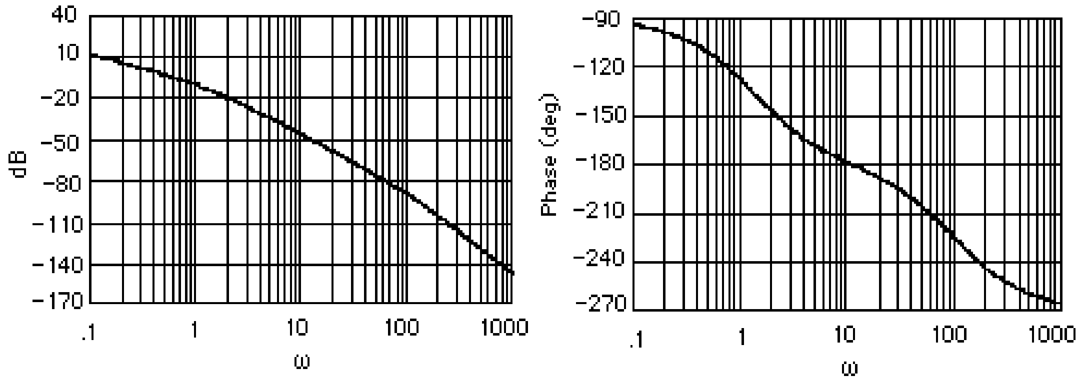
Gears:

$$K_2 = \frac{50}{250} = \frac{1}{5}$$

Therefore,

$$G(s) = K_1 K G_1(s) G_m(s) K_2 = \frac{50.88K}{s(s+1.32)(s+100)}$$

Plotting the Bode plots for K = 1,



a. Phase is 180^{o} at $\omega = 11.5$ rad/s. At this frequency the gain is - 48.41 dB, or K = 263.36. Therefore, for stability, $0 < K < 263.36$.

b. If K = 3, the magnitude curve will be 9.54 dB higher and go through zero dB at $\omega = 0.94$ rad/s. At this frequency, the phase response is -125.99^{o}. Thus, the phase margin is $180^{o} - 125.99^{o} = 54.01^{o}$. Using Eq. (10.73), $\zeta = 0.528$. Eq. (4.38) yields %OS = 14.18%.

c.

Program:
```
numga=50.88;
denga=poly([0 -1.32 -100]);
'Ga(s)'
Ga=tf(numga,denga);
Gazpk=zpk(Ga)
'(a)'
bode(Ga)
title('Bode Plot at Gain of 50.88')
pause
[Gm,Pm,Wcp,Wcg]=margin(Ga);
'Gain for Stability'
Gm
pause
'(b)'
numgb=50.88*3;
dengb=denga;
'Gb(s)'
Gb=tf(numgb,dengb);
Gbzpk=zpk(Gb)
bode(Gb)
title('Bode Plot at Gain of 3*50.88')
[Gm,Pm,Wcp,Wcg]=margin(Gb);
'Phase Margin'
Pm
for z=0:.01:1
Pme=atan(2*z/(sqrt(-2*z^2+sqrt(1+4*z^4))))*(180/pi);
if Pm-Pme<=0;
break
end
end
z
percent=exp(-z*pi/sqrt(1-z^2))*100
```

Computer response:
```
ans =

Ga(s)
```

```
Zero/pole/gain:
      50.88
------------------
s (s+100) (s+1.32)

ans =

(a)

ans =

Gain for Stability

Gm =

  262.8585

ans =

(b)

ans =

Gb(s)

Zero/pole/gain:
      152.64
------------------
s (s+100) (s+1.32)

ans =

Phase Margin

Pm =

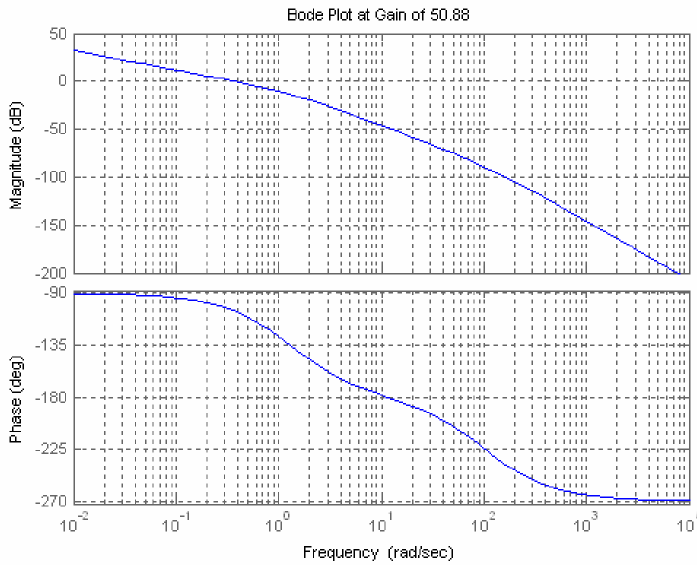
   53.9644

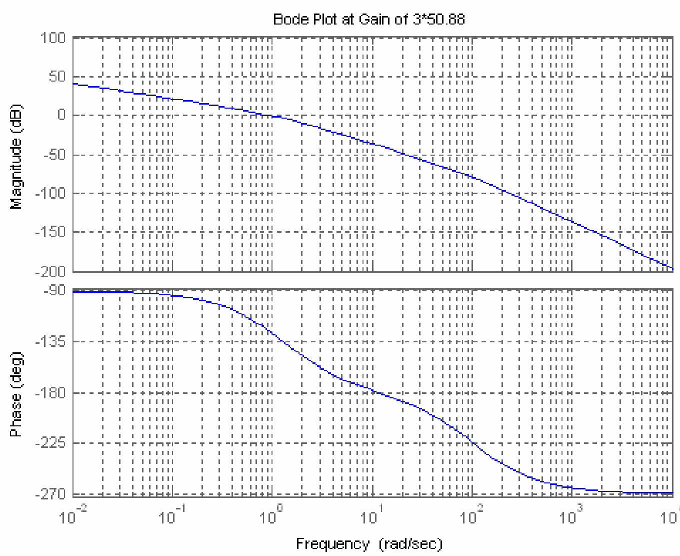
z =

    0.5300

percent =

   14.0366
```





ANSWERS TO REVIEW QUESTIONS

1. a. Transfer functions can be modeled easily from physical data; **b.** Steady-state error requirements can be considered easily along with the design for transient response; **c.** Settles ambiguities when sketching root locus; (d) Valuable tool for analysis and design of nonlinear systems.

2. A sinusoidal input is applied to a system. The sinusoidal output's magnitude and phase angle is measured in the steady-state. The ratio of the output magnitude divided by the input magnitude is the magnitude response at the applied frequency. The difference between the output phase angle and the input phase angle is

the phase response at the applied frequency. If the magnitude and phase response are plotted over a range of different frequencies, the result would be the frequency response for the system.

3. Separate magnitude and phase curves; polar plot

4. If the transfer function of the system is G(s), let s=jω. The resulting complex number's magnitude is the magnitude response, while the resulting complex number's angle is the phase response.

5. Bode plots are asymptotic approximations to the frequency response displayed as separate magnitude and phase plots, where the magnitude and frequency are plotted in dB.

6. Negative 6 dB/octave which is the same as 20 dB/decade

7. Negative 24 dB/octave or 80 dB/decade

8. Negative 12 dB/octave or 40 dB/decade

9. Zero degrees until 0.2; a negative slope of 45^o/decade from a frequency of 0.2 until 20; a constant -90^o phase from a frequency of 20 until ∞

10. Second-order systems require a correction near the natural frequency due to the peaking of the curve for different values of damping ratio. Without the correction the accuracy is in question.

11. Each pole yields a maximum difference of 3.01 dB at the break frequency. Thus for a pole of multiplicity three, the difference would be 3x3.01 or 9.03 dB at the break frequency, - 4.

12. Z = P - N, where Z = # of closed-loop poles in the right-half plane, P = # of open-loop poles in the right-half plane, and N = # of counter-clockwise encirclements of -1 made by the mapping.

13. Whether a system is stable or not since the Nyquist criterion tells us how many rhp the system has

14. A Nyquist diagram, typically, is a mapping, through a function, of a semicircle that encloses the right half plane.

15. Part of the Nyquist diagram is a polar frequency response plot since the mapping includes the positive jω axis.

16. The contour must bypass them with a small semicircle.

17. We need only map the positive imaginary axis and then determine that the gain is less than unity when the phase angle is 180^o.

18. We need only map the positive imaginary axis and then determine that the gain is greater than unity when the phase angle is 180^o.

19. The amount of additional open-loop gain, expressed in dB and measured at 180^o of phase shift, required to make a closed-loop system unstable.

20. The phase margin is the amount of additional open-loop phase shift, Φ_M, required at unity gain to make the closed-loop system unstable.

21. Transient response can be obtained from (1) the closed-loop frequency response peak, (2) phase margin

22. a. Find T(jω)=G(jω)/[1+G(jω)H(jω)] and plot in polar form or separate magnitude and phase plots. **b.** Superimpose G(jω)H(jω) over the M and N circles and plot. **c.** Superimpose G(jω)H(jω) over the Nichols chart and plot.

23. For Type zero: K_p = low frequency gain; For Type 1: K_v = frequency value at the intersection of the initial slope with the frequency axis; For Type 2: K_a = square root of the frequency value at the intersection of the initial slope with the frequency axis.

24. No change at all

25. A straight line of negative slope, ωT, where T is the time delay

26. When the magnitude response is flat and the phase response is flat at 0^o.

SOLUTIONS TO PROBLEMS

1.

a.

$$G(s) = \frac{1}{s(s+2)(s+4)}; \quad G(j\omega) = \frac{1}{-6\omega^2 + i(8\omega - \omega^3)}$$

$$M(\omega) = \frac{1}{\sqrt{(8\omega-\omega^3)^2 + (6\omega^2)^2}}; \quad \Phi(\omega) = -\left(\pi + \arctan\left[\frac{8-\omega^2}{-6\omega}\right]\right)$$

b.

$$G(s) = \frac{(s+5)}{(s+2)(s+4)}; \quad G(j\omega) = \frac{(\omega^2+40) - i(\omega^2+22)\omega}{\omega^4 + 20\omega^2 + 64}$$

$$M(\omega) = \frac{\sqrt{(\omega^2+40)^2 + \omega^2(\omega^2+22)^2}}{\omega^4 + 20\omega^2 + 64}; \quad \Phi(\omega) = \arctan\left(\frac{-[\omega^2+22]\omega}{\omega^2+40}\right)$$

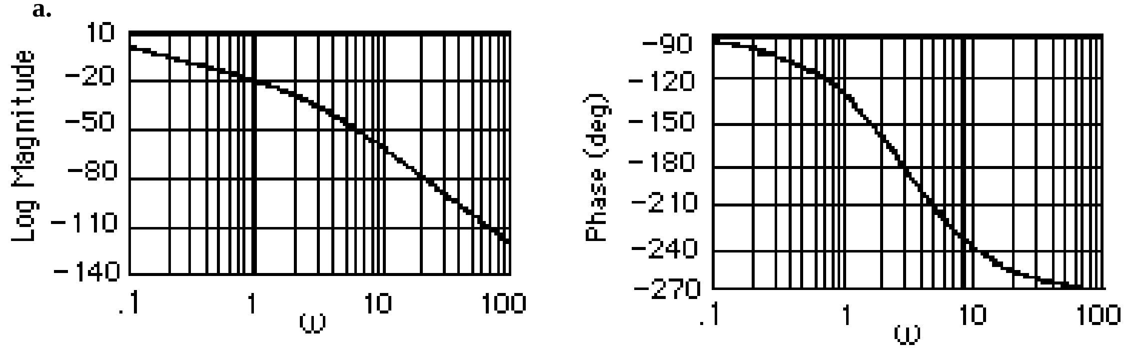
c.

$$G(s) = \frac{(s+3)(s+5)}{s(s+4)(s+2)}; \quad G(j\omega) = \frac{-2\omega(\omega^2+13) - i(\omega^4+25\omega^2+120)}{\omega^5 + 20\omega^3 + 64\omega}$$

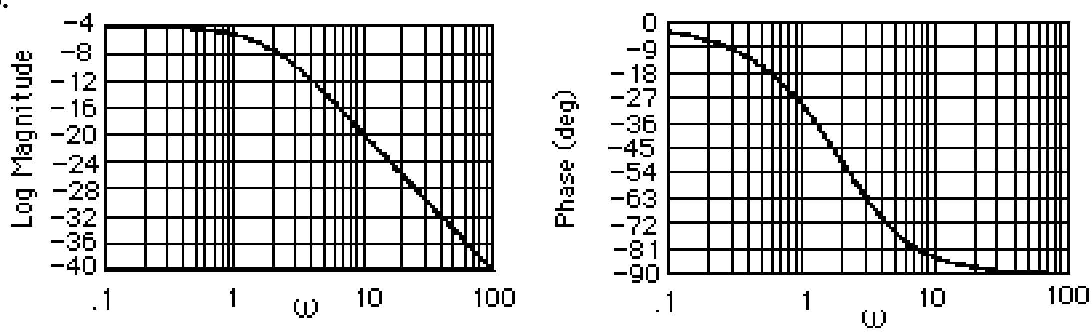
$$M(\omega) = \frac{\sqrt{(2\omega[\omega^2+13])^2 + (\omega^4+25\omega^2+120)^2}}{\omega^5 + 20\omega^3 + 64\omega}; \quad \Phi(\omega) = \pi + \arctan\left(\frac{\omega^4+25\omega^2+120}{2\omega[\omega^2+13]}\right)$$

2.

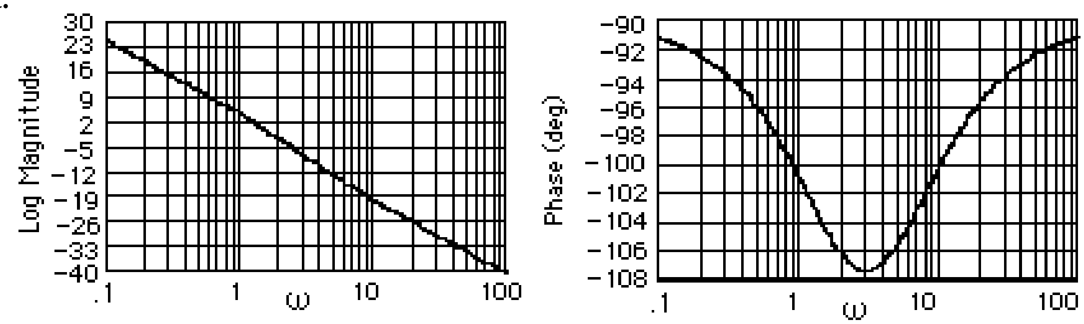
a.



b.

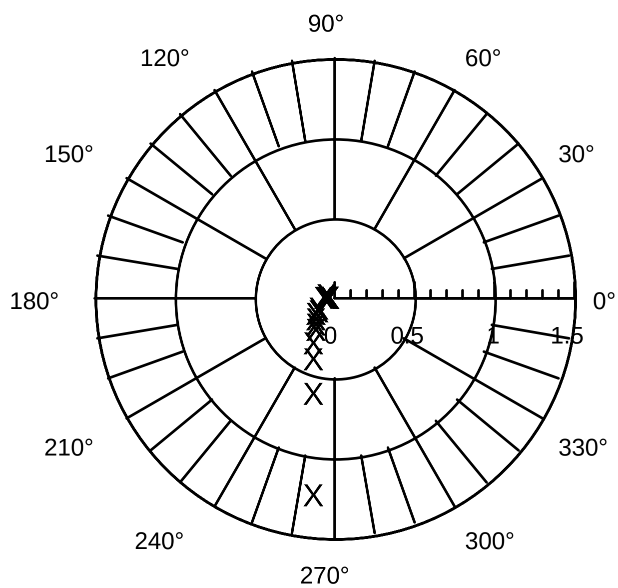

Log Magnitude
-4
-8
-12
-16
-20
-24
-28
-32
-36
-40
.1
1
ω
10
100
Phase (deg)
0
-9
-18
-27
-36
-45
-54
-63
-72
-81
-90
.1
1
ω
10
100

c.

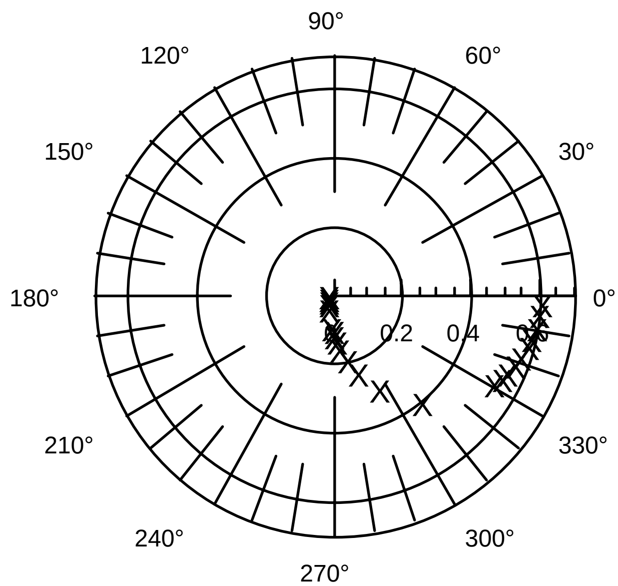

Log Magnitude
30
23
16
9
2
-5
-12
-19
-26
-33
-40
.1
1
ω
10
100
Phase (deg)
-90
-92
-94
-96
-98
-100
-102
-104
-106
-108
.1
1
ω
10
100

3.

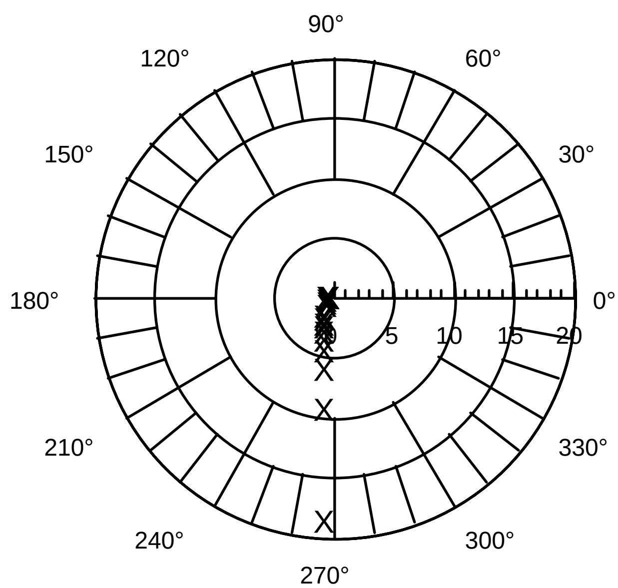
a.


90°
120°
60°
150°
30°
180°
0°
0
0.5
1
1.5
210°
330°
240°
300°
270°

b.

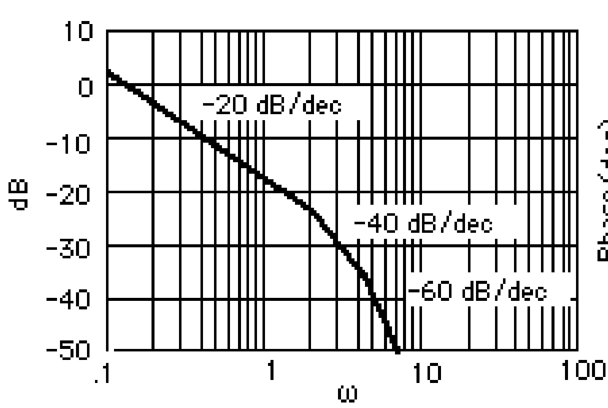

90°
120°
60°
150°
30°
180°
0°
0.2
0.4
210°
330°
240°
300°
270°

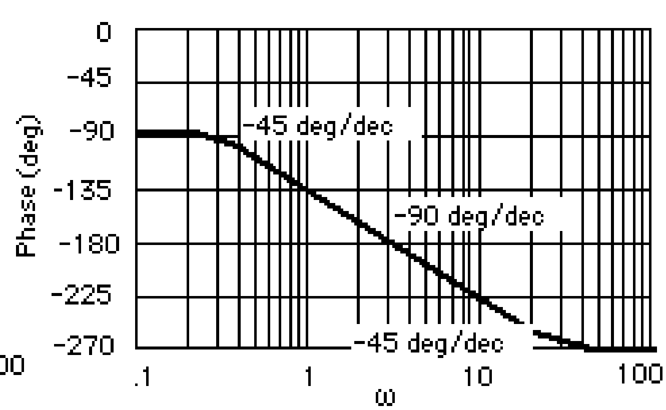
c.


90°
120°
60°
150°
30°
180°
0°
0
5
10
15
20
210°
330°
240°
300°
270°

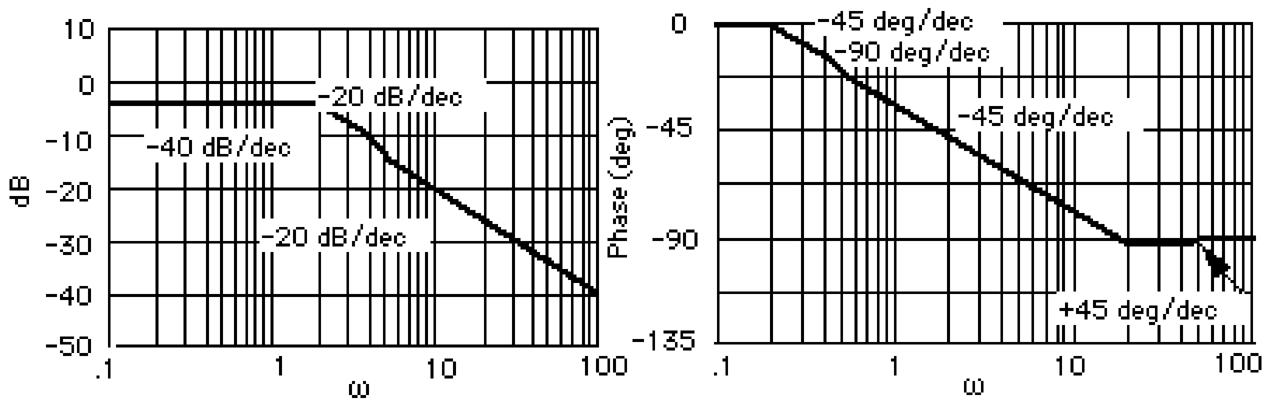
4.

a.

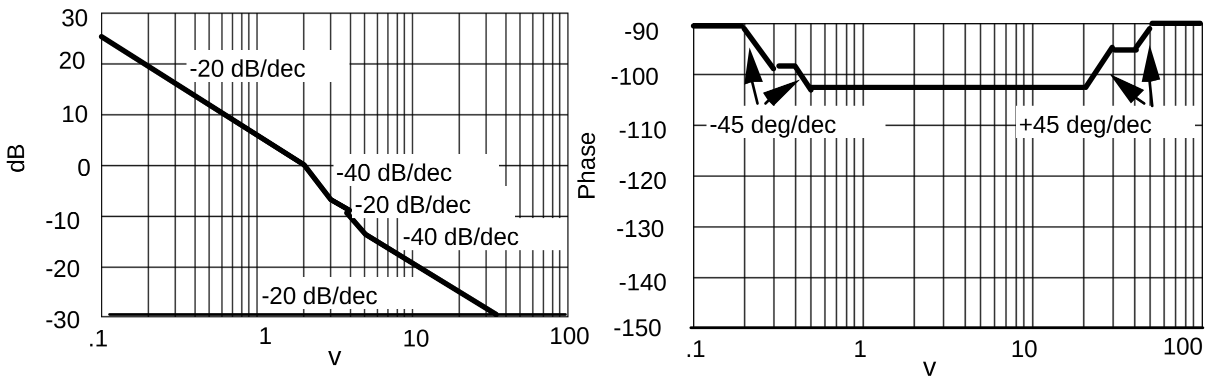

10
0
-10
-20
-30
-40
-50
dB
-20 dB/dec
-40 dB/dec
-60 dB/dec
.1
1
10
100
ω


0
-45
-90
-135
-180
-225
-270
Phase (deg)
-45 deg/dec
-90 deg/dec
-45 deg/dec
.1
1
10
100
ω

b.

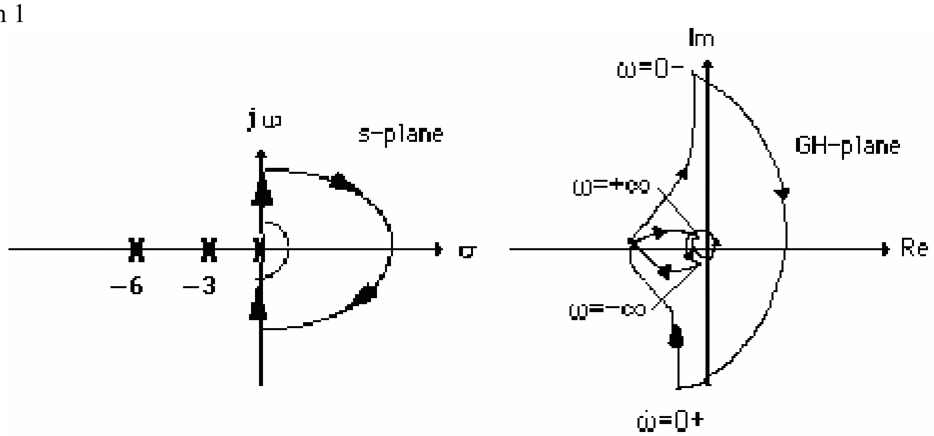


c.

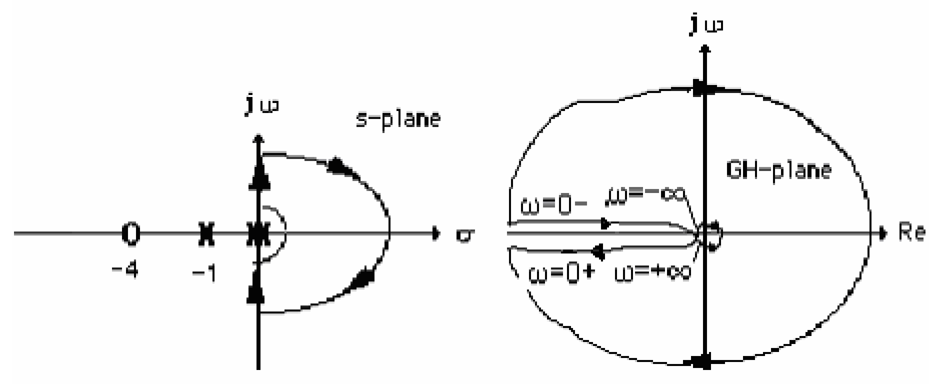


5.

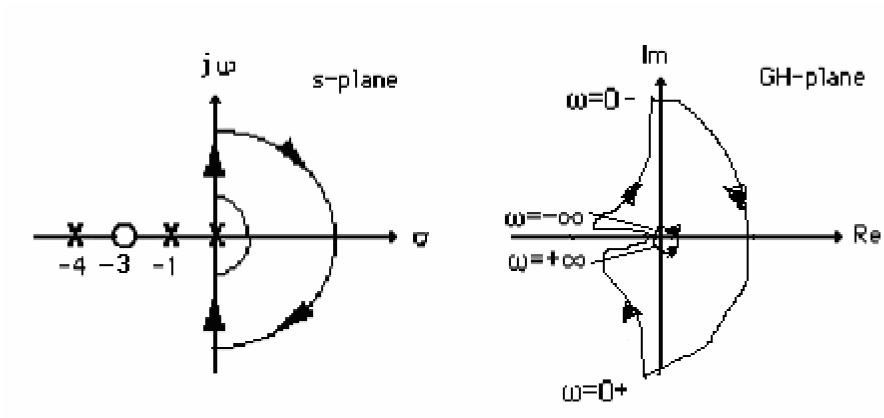
a. System 1



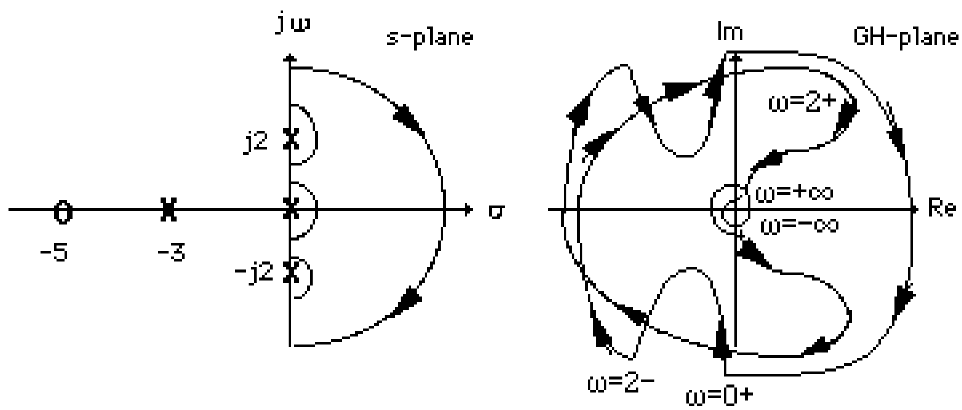
b. System 2



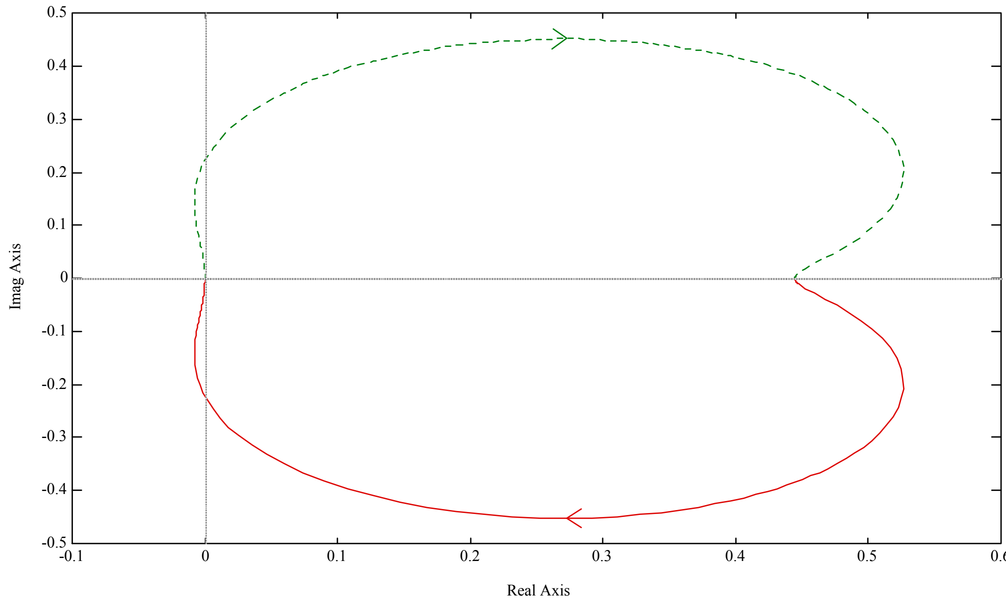
c. System 3



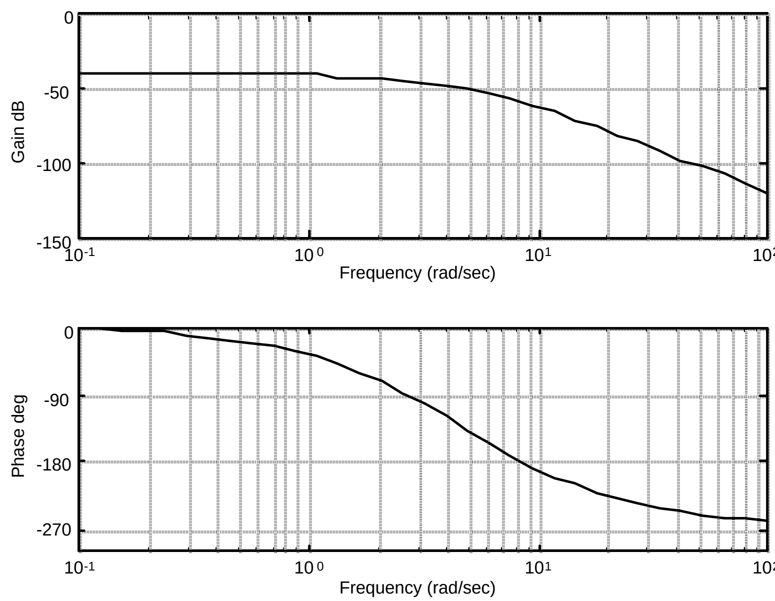
d.



6.



7.



8.

Program:

```
numg=[1 5];
deng=conv([1 6 100],[1 4 25]);
G=tf(numg,deng);
'G(s)'
Gzpk=zpk(G)
nyquist(G)
axis([-3e-3,4e-3,-5e-3,5e-3])
w=0:0.1:100;
[re,im]=nyquist(G,w);
for i=1:1:length(w)
```

```
M(i)=abs(re(i)+j*im(i));
A(i)=atan2(im(i),re(i))*(180/pi);
if 180-abs(A(i))<=1;
re(i);
im(i);
K=1/abs(re(i));
fprintf('\nw = %g',w(i))
fprintf(', Re = %g',re(i))
fprintf(', Im = %g',im(i))
fprintf(', M = %g',M(i))
fprintf(', Angle = %g',A(i))
fprintf(', K = %g',K)
Gm=20*log10(1/M(i));
fprintf(', Gm = %g',Gm)
break
end
end
```

Computer response:

```
ans =

G(s)


Zero/pole/gain:
              (s+5)
----------------------------------
(s^2  + 4s + 25) (s^2  + 6s + 100)

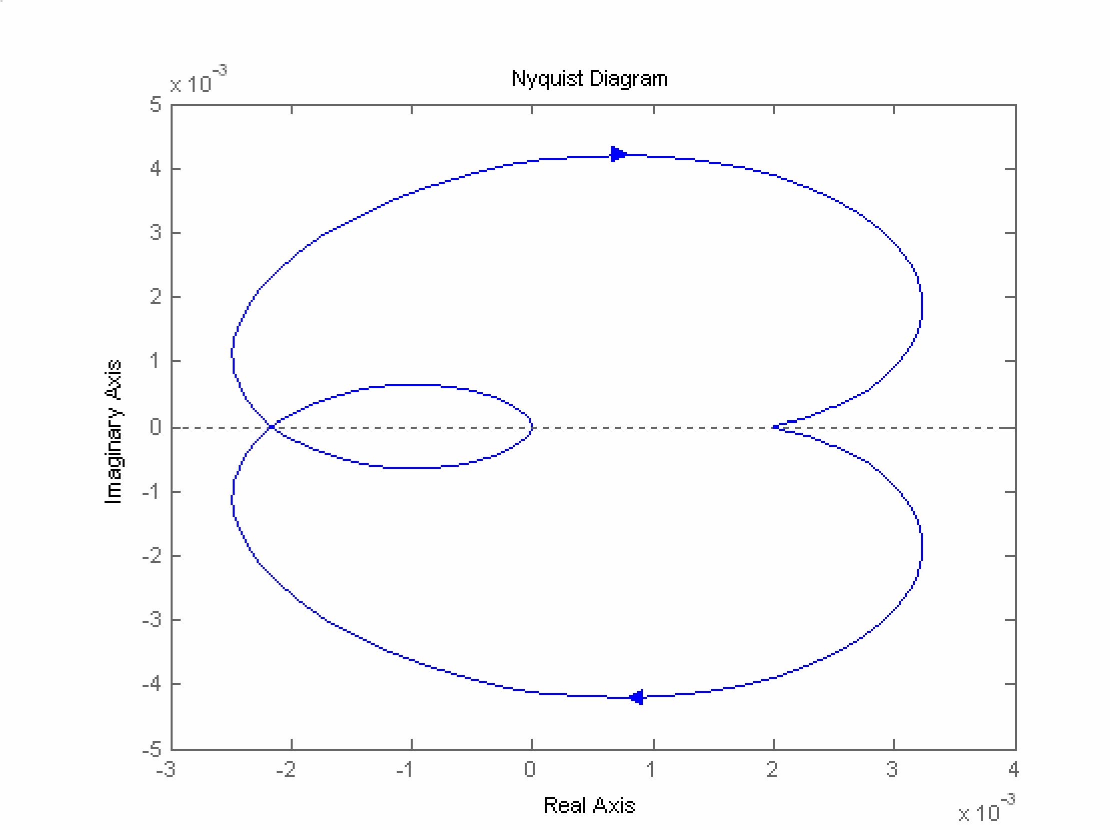

w = 10.1, Re = -0.00213722, Im = 2.07242e-005, M = 0.00213732, Angle =
179.444, K = 467.898, Gm = 53.4026

ans =

G(s)


Zero/pole/gain:
              (s+5)
----------------------------------
(s^2  + 4s + 25) (s^2  + 6s + 100)


w = 10.1, Re = -0.00213722, Im = 2.07242e-005, M = 0.00213732, Angle =
179.444, K = 467.898, Gm = 53.4026
```



9.

a. Since the real-axis crossing is at -0.3086, P = 0, N = 0. Therefore Z = P - N = 0. System is stable.

Derivation of real-axis crossing:

$$G(j\omega) = \left.\frac{50}{s(s+3)(s+6)}\right|_{s=j\omega} = \frac{50\left[-9\omega^2 - j\omega\left(18-\omega^2\right)\right]}{81\omega^4 + \left(18\omega - \omega^3\right)}.$$

Thus, the imaginary part = 0 at $\omega = \sqrt{18}$. Substituting this frequency into $G(j\omega)$, the real part is evaluated to be -0.3086.

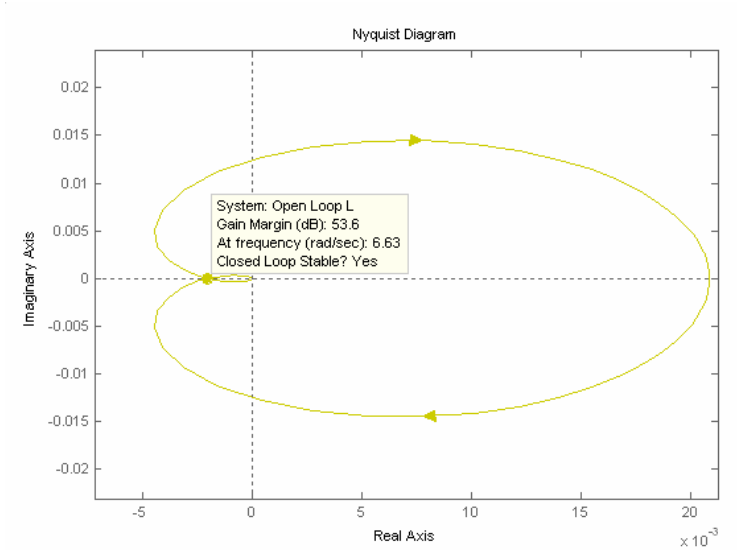
b. P = 0, N = -2. Therefore Z = P - N = 2. System is unstable.

c. P = 0, N = 0. Therefore Z = P - N = 0. System is stable

d. P = 0, N = -2. Therefore Z = P - N = 2. System is unstable.

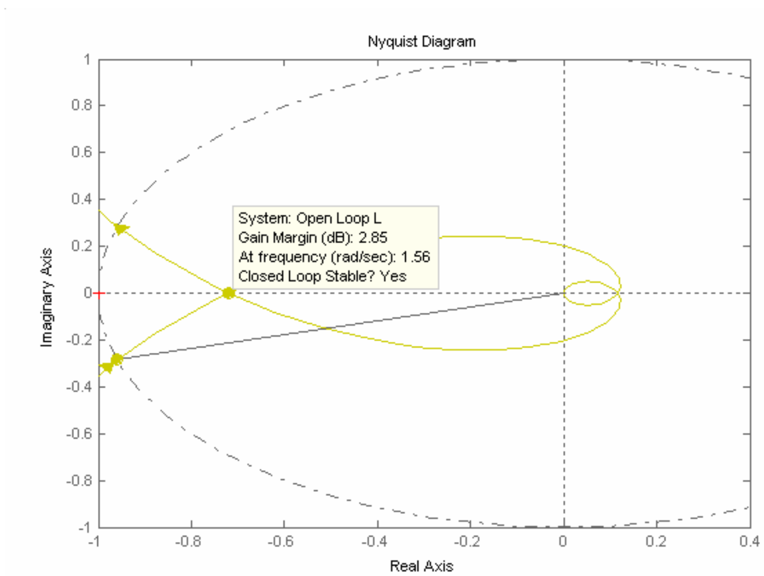
10.

System 1: For K = 1,



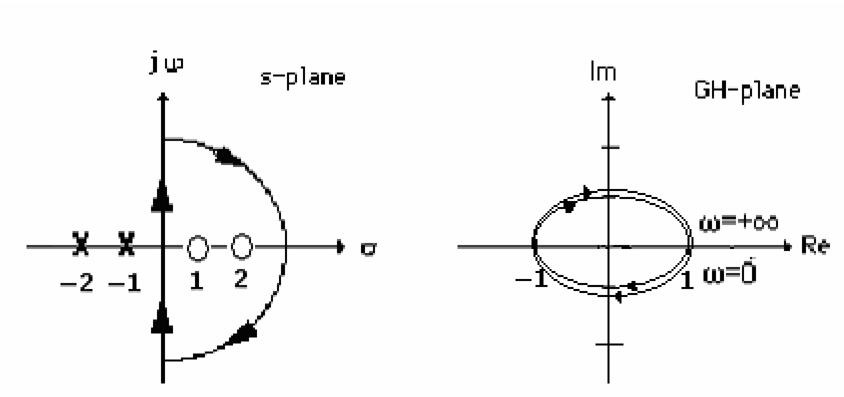
The Nyquist diagram intersects the real axis at -0.0021. Thus K can be increased to 478.63 before there are encirclements of -1. There are no poles encircles by the contour. Thus P = 0. Hence, Z = P - N, Z = 0 + 0 if K <478.63; Z = 0 –(-2) if K > 478.63. Therefore stability if 0 < K < 478.63.

System 2: For K = 1,



The Nyquist diagram intersects the real axis at -0.720. Thus K can be increased to 1.39 before there are encirclements of -1. There are no poles encircles by the contour. Thus P = 0. Hence, Z = P - N, Z = 0 + 0 if K <1.39; Z = 0 – (-2) if K > 1.39. Therefore stability if 0 < K < 1.39.

System 3: For K = 1,

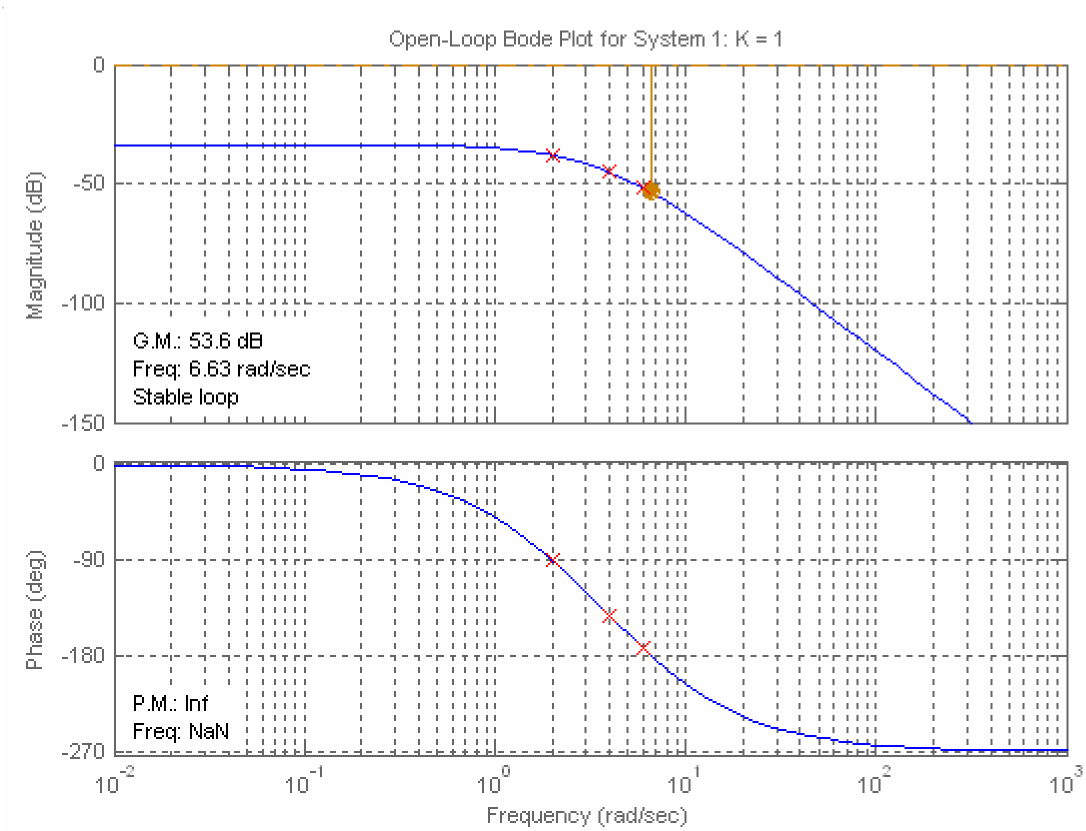


Stable if 0<K<1.

11.

Note: All results for this problem are based upon a non-asymptotic frequency response.

System 1: Plotting Bode plots for K = 1 yields the following Bode plot,



K = 1000:

For K = 1, phase response is 180° at ω = 6.63 rad/s. Magnitude response is -53.6 dB at this frequency.

For K = 1000, magnitude curve is raised by 60 dB yielding + 6.4 dB at 6.63 rad/s. Thus, the gain margin is

- 6.4 dB.

Phase margin: Raising the magnitude curve by 60 dB yields 0 dB at 9.07 rad/s, where the phase curve is 200.3°. Hence, the phase margin is 180°-200.3° = - 20.3°.

K = 100:

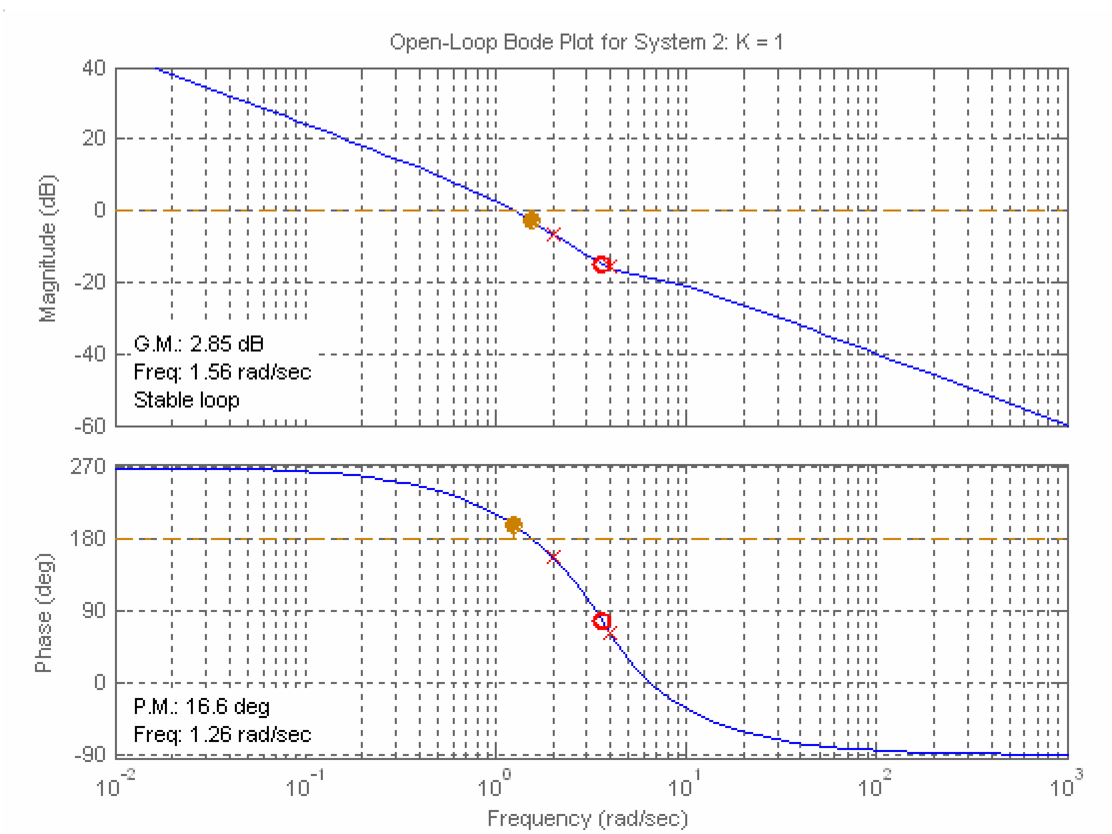
For K = 1, phase response is 180° at ω = 6.63 rad/s. Magnitude response is -53.6 dB at this frequency. For K = 100, magnitude curve is raised by 40 dB yielding – 13.6 dB at 6.63 rad/s. Thus, the gain margin is 13.6 dB.

Phase margin: Raising the magnitude curve by 40 dB yields 0 dB at 2.54 rad/s, where the phase curve is 107.3°. Hence, the phase margin is 180°-107.3° = 72.7°.

K = 0.1:

For K = 1, phase response is 180° at ω = 6.63 rad/s. Magnitude response is -53.6 dB at this frequency. For K = 0.1, magnitude curve is lowered by 20 dB yielding – 73.6 dB at 6.63 rad/s. Thus, the gain margin is 73.6 dB..

System 2: Plotting Bode plots for K = 1 yields



K = 1000:

For K = 1, phase response is 180° at ω = 1.56 rad/s. Magnitude response is -2.85 dB at this frequency. For K = 1000, magnitude curve is raised by 60 dB yielding + 57.15 dB at 1.56 rad/s. Thus, the gain

margin is

– 57.15 dB.

Phase margin: Raising the magnitude curve by 54 dB yields 0 dB at 500 rad/s, where the phase curve is -91.03^{o}. Hence, the phase margin is 180^{o}-91.03^{o} = 88.97^{o}.

K = 100:

For K = 1, phase response is 180^{o} at ω = 1.56 rad/s. Magnitude response is -2.85 dB at this frequency. For K = 100, magnitude curve is raised by 40 dB yielding + 37.15 dB at 1.56 rad/s. Thus, the gain margin is

– 37.15 dB.

Phase margin: Raising the magnitude curve by 40 dB yields 0 dB at 99.8 rad/s, where the phase curve is -84.3^{o}. Hence, the phase margin is 180^{o}-84.3^{o} = 95.7^{o}.

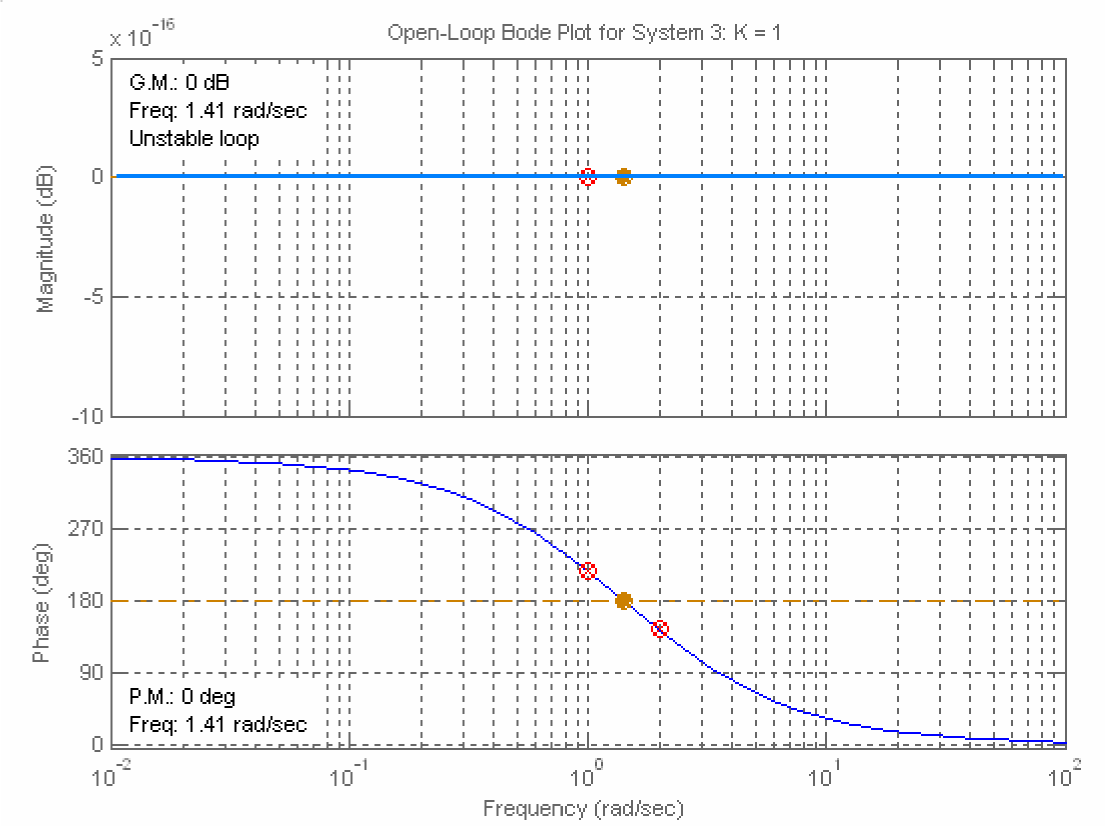
K = 0.1:

For K = 1, phase response is 180^{o} at ω = 1.56 rad/s. Magnitude response is -2.85 dB at this frequency. For K = 0.1, magnitude curve is lowered by 20 dB yielding – 22.85 dB at 1.56 rad/s. Thus, the gain margin is

– 22.85 dB.

Phase margin: Lowering the magnitude curve by 20 dB yields 0 dB at 0.162 rad/s, where the phase curve is -99.8^{o}. Hence, the phase margin is 180^{o}-99.86^{o} = 80.2^{o}.

System 3: Plotting Bode plots for K = 1 yields



K = 1000:

For K = 1, phase response is 180° at ω = 1.41 rad/s. Magnitude response is 0 dB at this frequency.

For K = 1000, magnitude curve is raised by 60 dB yielding 60 dB at 1.41 rad/s. Thus, the gain margin is - 60 dB.

Phase margin: Raising the magnitude curve by 60 dB yields no frequency where the magnitude curve is 0 dB. Hence, the phase margin is infinite.

K = 100:

For K = 1, phase response is 180° at ω = 1.41 rad/s. Magnitude response is 0 dB at this frequency.

For K = 100, magnitude curve is raised by 40 dB yielding 40 dB at 1.41 rad/s. Thus, the gain margin is - 40 dB.

Phase margin: Raising the magnitude curve by 40 dB yields no frequency where the magnitude curve is 0 dB. Hence, the phase margin is infinite.

K = 0.1:

For K = 1, phase response is 180° at ω = 1.41 rad/s. Magnitude response is 0 dB at this frequency.

For K = 0.1, magnitude curve is lowered by 20 dB yielding -20 dB at 1.41 rad/s. Thus, the gain margin is 20 dB.

Phase margin: Lowering the magnitude curve by 20 dB yields no frequency where the magnitude curve is 0 dB. Hence, the phase margin is infinite.

12.

Program:

```
%Enter G(s)************************
numg=1;
deng=poly([0 -3 -12]);
'G(s)'
G=tf(numg,deng)
w=0.01:0.1:100;
%Enter K **************************
K=input('Type gain, K ');
bode(K*G,w)
pause
[M,P]=bode(K*G,w);
%Calculate Gain Margin**************
for i=1:1:length(P);
if P(i)<=-180;
fprintf('\nGain K = %g',K)
fprintf(', Frequency(180 deg) = %g',w(i))
fprintf(', Magnitude = %g',M(i))
fprintf(', Magnitude (dB) = %g',20*log10(M(i)))
fprintf(', Phase = %g',P(i))
Gm=20*log10(1/M(i));
fprintf(', Gain Margin (dB) = %g',Gm)
break
end
end
%Calculate Phase Margin**************
for i=1:1:length(M);
if M(i)<=1;
fprintf('\nGain K = %g',K)
fprintf(', Frequency (0 dB) = %g',w(i))
fprintf(', Magnitude = %g',M(i))
fprintf(', Magnitude (dB) = %g',20*log10(M(i)))
fprintf(', Phase = %g',P(i))
Pm=180+P(i);;
fprintf(', Phase Margin = %g',Pm)
break
end
end

'Alternate program using MATLAB margin function:'

clear
clf
%Bode Plot and Find Points
%Enter G(s)************************
numg=1;
deng=poly([0 -3 -12]);
'G(s)'
G=tf(numg,deng)
w=0.01:0.1:100;
%Enter K **************************
K=input('Type gain, K ');
bode(K*G,w)
[Gm,Pm,Wcp,Wcg]=margin(K*G)
'Gm(dB)'
20*log10(Gm)
```

Computer response:

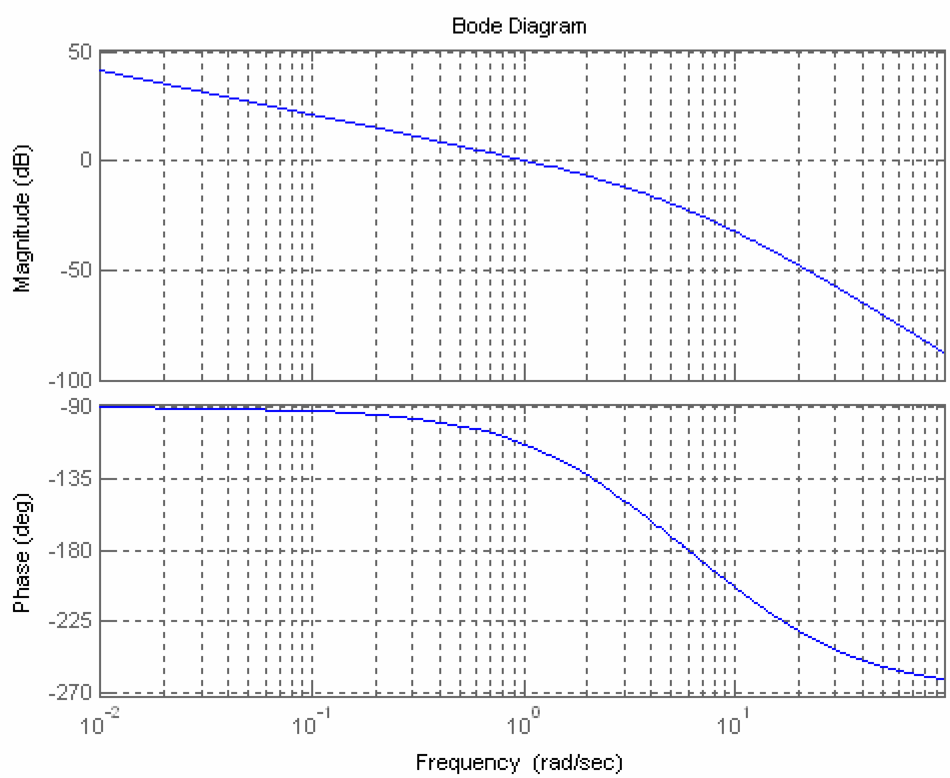
```
ans =

G(s)


Transfer function:
         1
-------------------
s^3 + 15 s^2 + 36 s

Type gain, K 40

Gain K = 40, Frequency(180 deg) = 6.01, Magnitude = 0.0738277, Magnitude
(dB) = -22.6356, Phase = -180.076, Gain Margin (dB) = 22.6356
Gain K = 40, Frequency (0 dB) = 1.11, Magnitude = 0.93481, Magnitude (dB) =
-0.585534, Phase = -115.589, Phase Margin = 64.4107
```




```
Alternate program using MATLAB margin function:


ans =

G(s)


Transfer function:
         1
-------------------
s^3 + 15 s^2 + 36 s

Type gain, K 40

Gm =
```

```
   13.5000

Pm =

   65.8119

Wcp =

     6

Wcg =

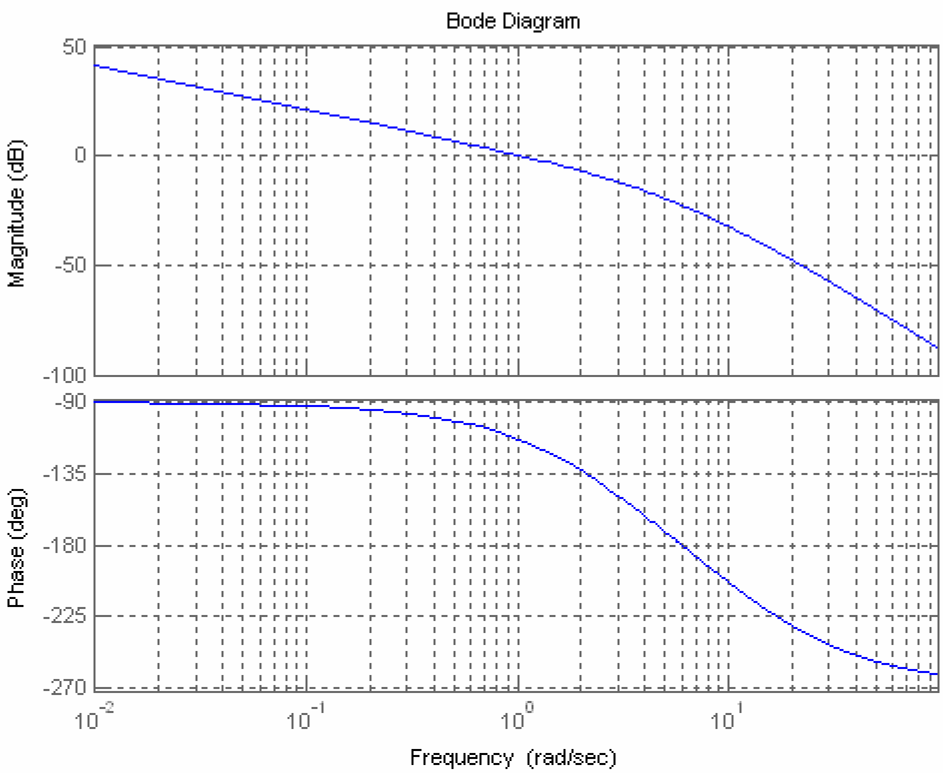
    1.0453

ans =

Gm(dB)

ans =

   22.6067
```



13.

Program:

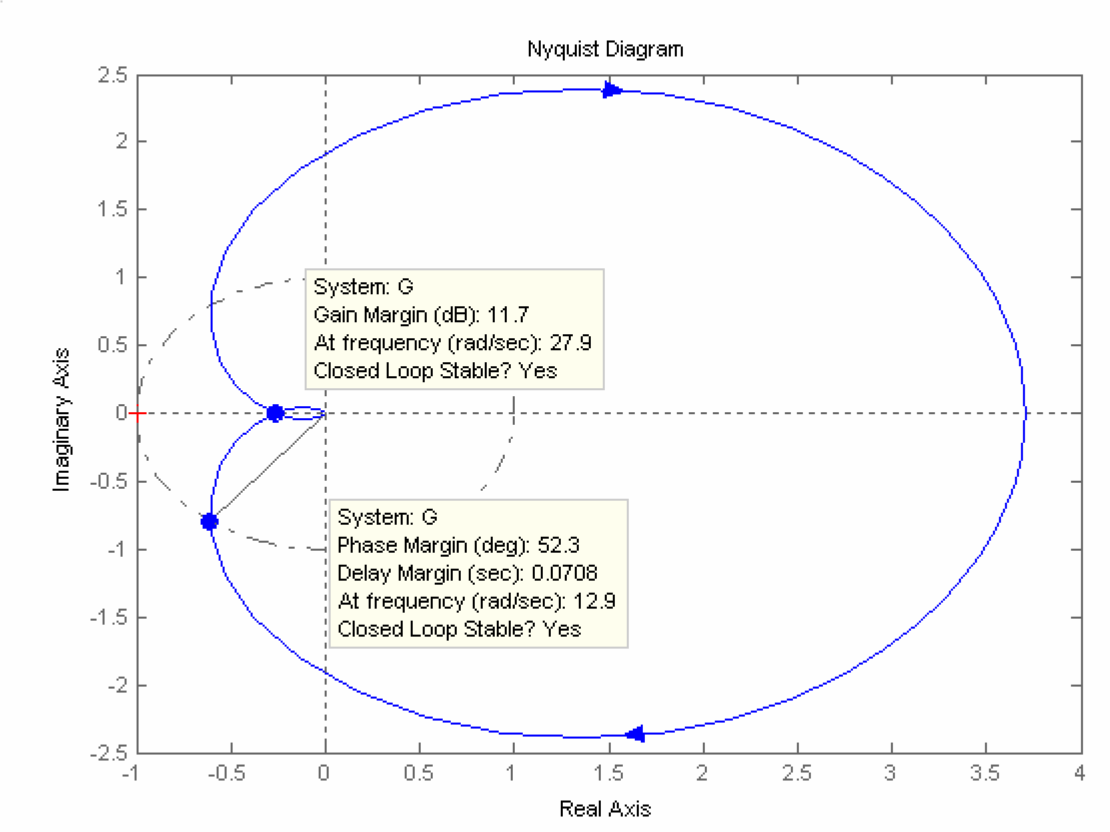
```
numg=10000;
deng=poly([-5 -18 -30]);
G=tf(numg,deng)
Ltiview
```

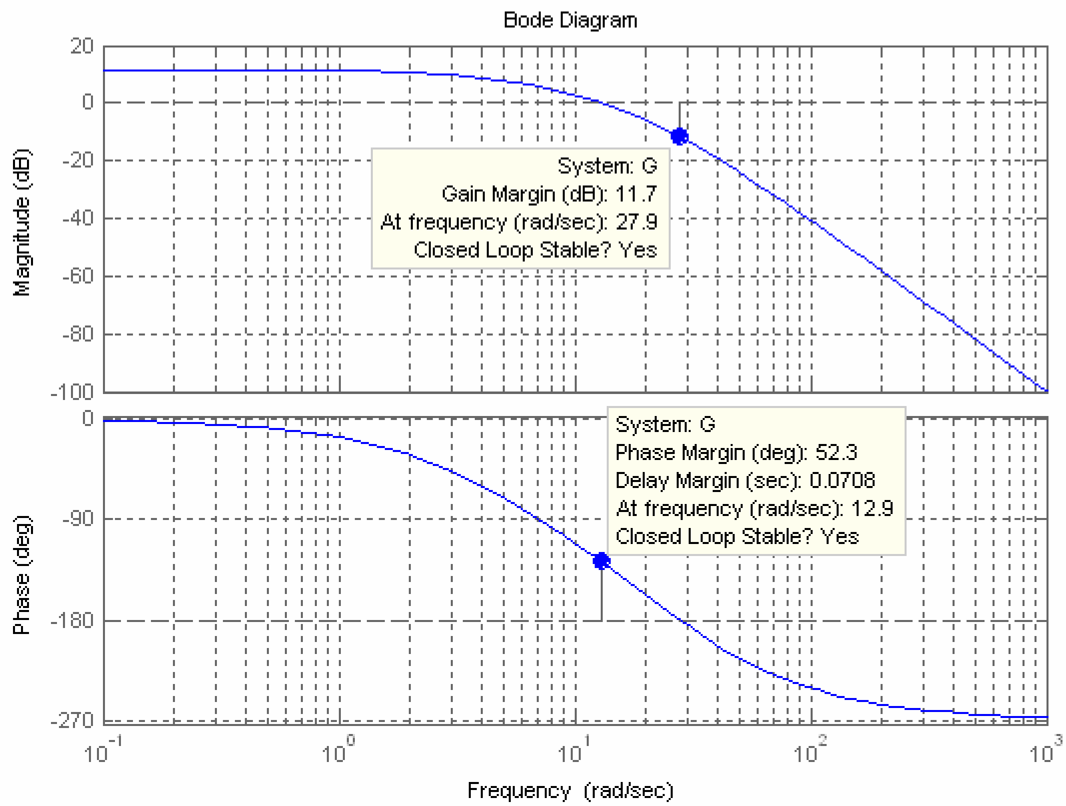
Computer response:

```
ans =

10.4N

Transfer function:
           10000
---------------------------
s^3 + 53 s^2 + 780 s + 2700
```





14.

Squaring Eq. (10.51) and setting it equal to $\left(\frac{1}{\sqrt{2}}\right)^2$ yields

$$\frac{\omega_n{}^4}{(\omega_n{}^2 - \omega^2)^2 + 4\zeta^2\omega_n{}^2\omega^2} = \frac{1}{2}$$

Simplifying,

$$\omega^4 + 2\omega_n{}^2(2\zeta^2 - 1)\omega^2 - \omega_n{}^4 = 0$$

Solving for ω^2 using the quadratic formula and simplifying yields,

$$\omega^2 = \omega_n{}^2\left[- (2\zeta^2 - 1) \pm \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right]$$

Taking the square root and selecting the positive term,

$$\omega = \omega_n\sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

15.

a. Using Eq. (10.55), ω_{BW} = 10.06 rad/s.

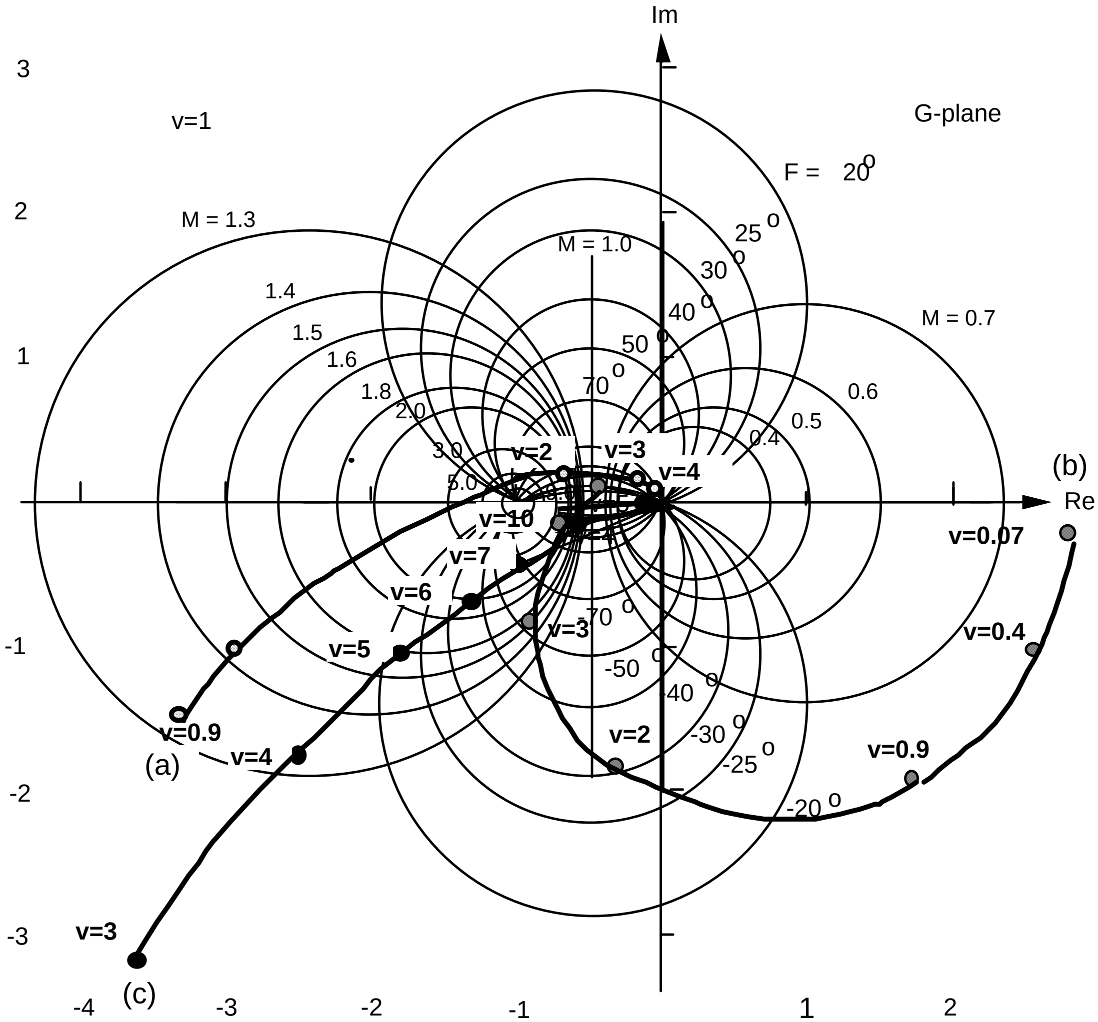
b. Using Eq. (10.56), ω_{BW} = 1.613 rad/s.

c. First find ζ. Since $T_s = \frac{4}{\zeta\omega_n}$ and $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$, $\frac{T_p}{T_s} = \frac{\zeta\pi}{4\sqrt{1-\zeta^2}}$. Solving for ζ with $\frac{T_p}{T_s} =$ 0.5 yields $\zeta = 0.537$. Using either Eq. (10.55) or (10.56) yields $\omega_{BW} = 2.29$ rad/s.

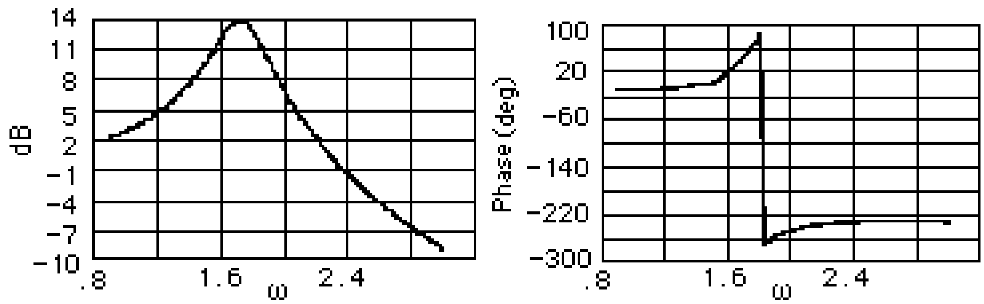
d. Using $\zeta = 0.3$, $\omega_n T_r = 1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1 = 1.3217$. Hence,

$$\omega_n = \frac{1.3217}{T_r} = \frac{1.3217}{4} = 0.3304$$ rad/s. Using Eq. (10.54) yields $\omega_{BW} = 0.4803$ rad/s.

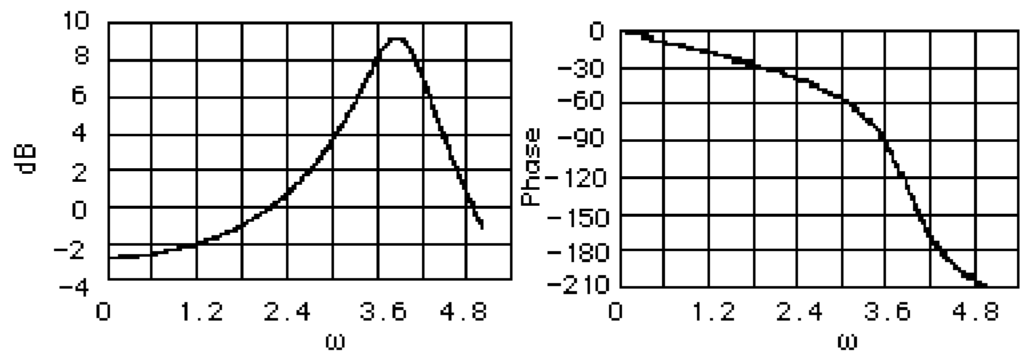
16.



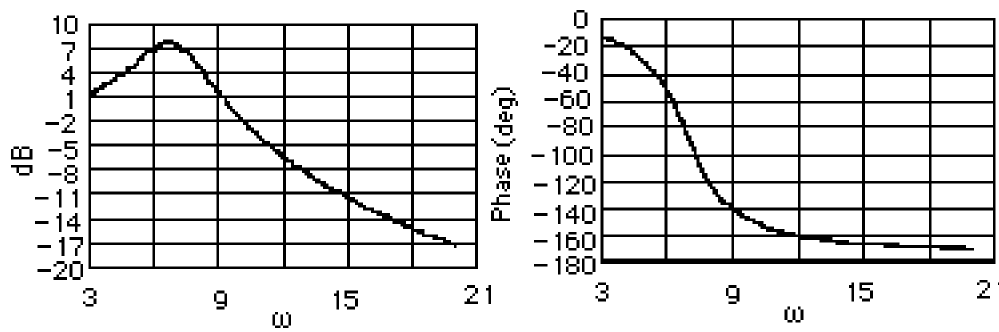
a.


dB
14
11
8
5
2
−1
−4
−7
−10
.8
1.6
2.4
ω
Phase (deg)
100
20
−60
−140
−220
−300
.8
1.6
2.4
ω

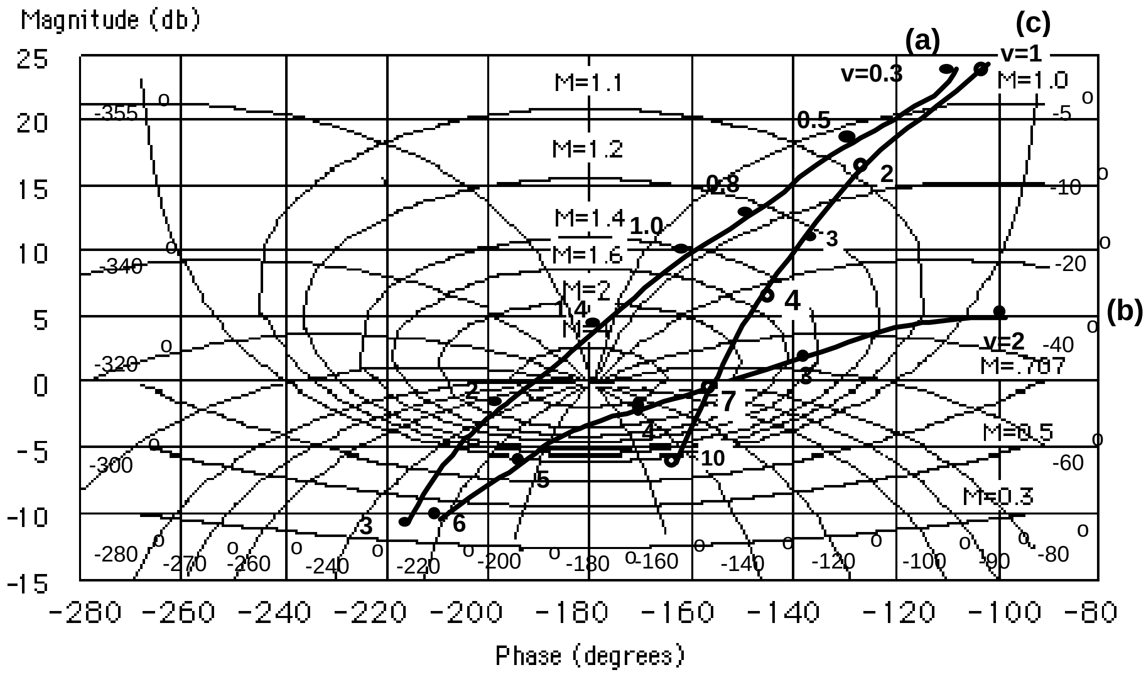
b.


dB
10
8
6
4
2
0
−2
−4
0
1.2
2.4
3.6
4.8
ω
Phase
0
−30
−60
−90
−120
−150
−180
−210
0
1.2
2.4
3.6
4.8
ω

c.


dB
10
7
4
1
−2
−5
−8
−11
−14
−17
−20
3
9
15
21
ω
Phase (deg)
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
3
9
15
21
ω

17.



18.

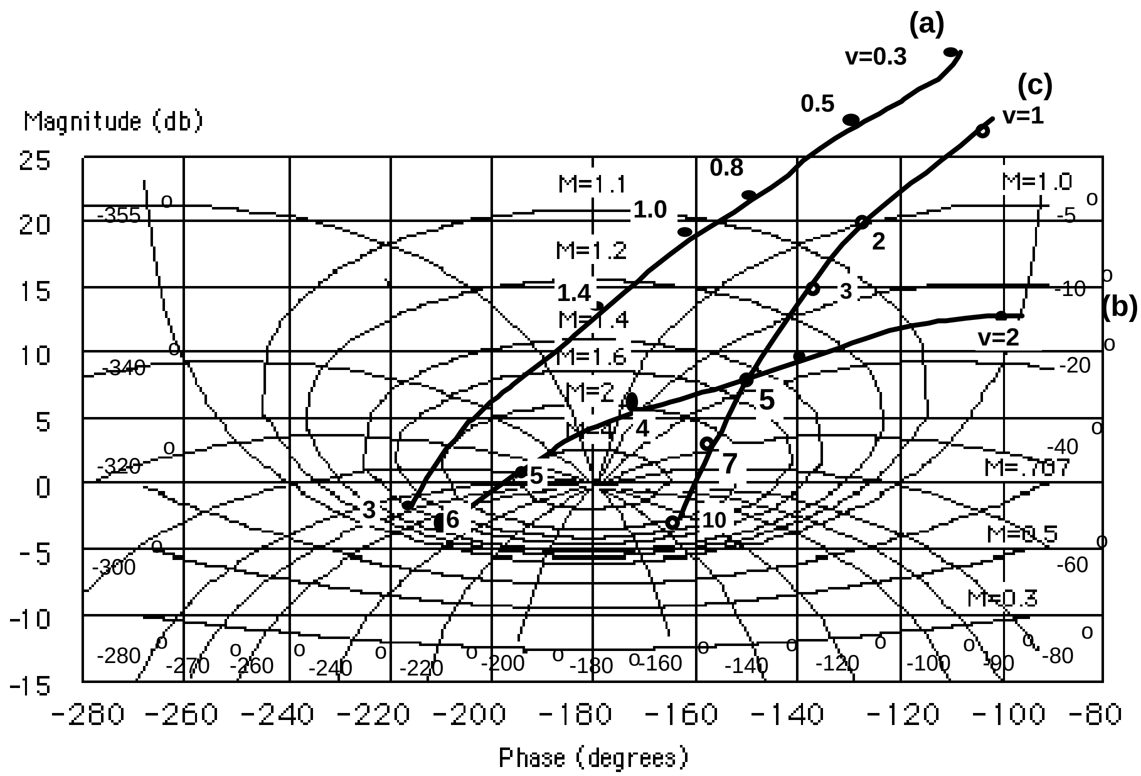
a. The polar plot is approximately tangent to M = 5. Using Figure 10.40, the student would estimate 72% overshoot. However, notice that the polar plot intersects the negative real axis at a magnitude greater than unity. Hence, the system is actually unstable and the estimated percent overshoot is not correct.

b. The polar plot is approximately tangent to M = 3. Using Figure 10.40, we estimate 58% overshoot.

c. The polar plot is approximately tangent to M = 2.5. Using Figure 10.40, we estimate 52% overshoot.

19.

Raise each curve in Problem 17 by (a) 9.54 dB, (b) 7.96 dB, and (c) 3.52 dB, respectively.



Systems (a) and (b) are both unstable since the open-loop magnitude is greater than unity when the open-loop phase is 180°. System (c) is tangent to approximately M = 3. Using Figure 10.40, we estimate 58% overshoot.

20.

Program:

```
%Enter G(s)***********************
numg=[1 5];
deng=[1 4 25 0];
'G(s)'
G=tf(numg,deng)
%Enter K *************************
K=input('Type gain, K ');
'T(s)'
T=feedback(K*G,1)
bode(T)
title('Closed-loop Frequency Response')
[M,P,w]=bode(T);
[Mp i]=max(M);
Mp
MpdB=20*log10(Mp)
wp=w(i)
for i=1:1:length(M);
if M(i)<=0.707;
fprintf('Bandwidth = %g',w(i))
break
end
end
```

Computer response:

```
ans =

G(s)

Transfer function:

      s + 5

------------------

s^3 + 4 s^2 + 25 s


Type gain, K 40

ans =

T(s)

Transfer function:

       40 s + 200

------------------------

s^3 + 4 s^2 + 65 s + 200


Mp =

    6.9745

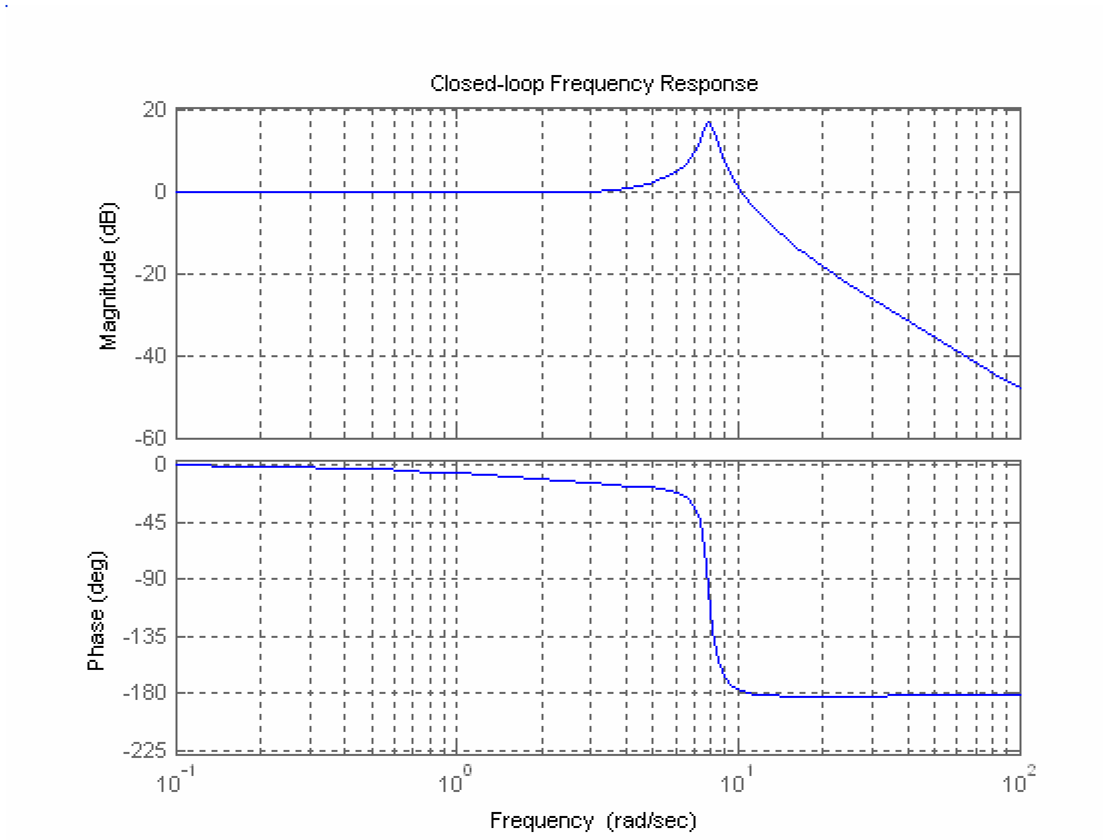
MpdB =

   16.8702

wp =

    7.8822

Bandwidth = 11.4655
```



21.

Program:

```
numg=[7 35];
deng=[1 4 10 0];
G=tf(numg,deng)
bode(G)                               %Make a Bode plot.
title('Open-Loop Frequency Response')
                                      %Add a title to the Bode plot.
[Gm,Pm,Wcp,Wcg]=margin(G);            %Find margins and margin
                                      %frequencies.
'Gain margin(dB); Phase margin(deg.); 0 dB freq. (r/s);'
'180 deg. freq. (r/s)'                %Display label.
margins=[20*log10(Gm),Pm,Wcg,Wcp]     %Display margin data.
Ltiview
```

Computer response:

```
Transfer function:
     7 s + 35
------------------
s^3 + 4 s^2 + 10 s


ans =

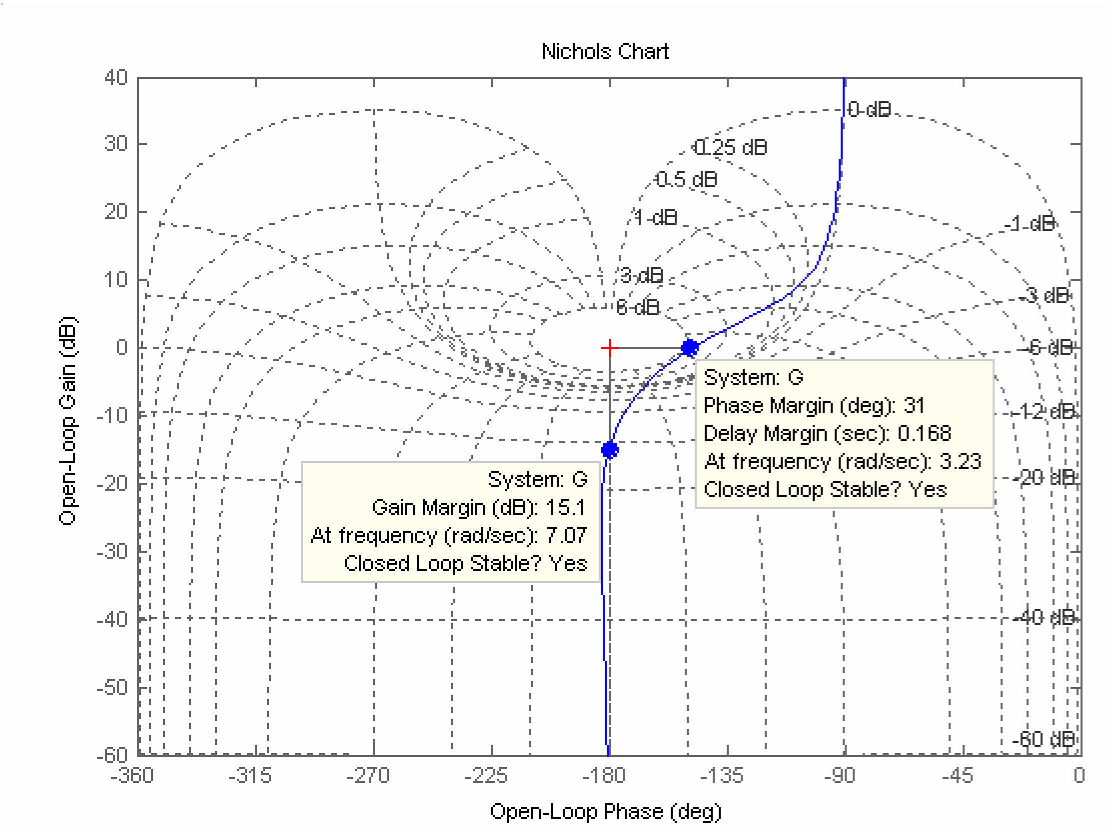
Gain margin(dB); Phase margin(deg.); 0 dB freq. (r/s);


ans =

180 deg. freq. (r/s)
```

```
margins =

   15.1403   31.0397    3.2252    7.0715
```



22.

Program:

```
%Enter G(s)************************************
numg=7*[1 5];
deng=[1 4 10 0];
'Open-Loop System'
'G(s)'
G=tf(numg,deng)
clf
w=.10:1:10;
nichols(G,w)
ngrid
title('Nichols Plot')
[M,P]=nichols(G,w);
for i=1:1:length(M);
if M(i)<=0.45;
BW=w(i);
break
end
end
pause
MpdB=input('Enter Mp in dB from Nichols Plot  ');
Mp=10^(MpdB/20);
z2=roots([4,-4,(1/Mp^2)]);%Since Mp=1/sqrt(4z^2(1-z^2))
z1=sqrt(z2);
z=min(z1);
Pos=exp(-z*pi/(sqrt(1-z^2)));
```

```
Ts=(4/(BW*z))*sqrt((1-z^2)+sqrt(4*z^4-4*z^2+2));
Tp=(pi/(BW*sqrt(1-z^2)))*sqrt((1-z^2)+sqrt(4*z^4-4*z^2+2));
'Closed-Loop System'
'T(s)'
T=feedback(G,1)
bode(T)
title('Closed-Loop Frequency Resposne Plots')
fprintf('\nDamping Ratio = %g',z)
fprintf(', Percent Overshoot = %g',Pos*100)
fprintf(', Bandwidth = %g',BW)
fprintf(', Mp (dB) = %g',MpdB)
fprintf(', Mp = %g',Mp)
fprintf(', Settling Time = %g',Ts)
fprintf(', Peak Time = %g',Tp)
pause
step(T)
title('Closed-Loop Step Response')
```

Computer response:

```
ans =

Open-Loop System

ans =

G(s)

Transfer function:
     7 s + 35
------------------
s^3 + 4 s^2 + 10 s

Enter Mp in dB from Nichols Plot  6

ans =

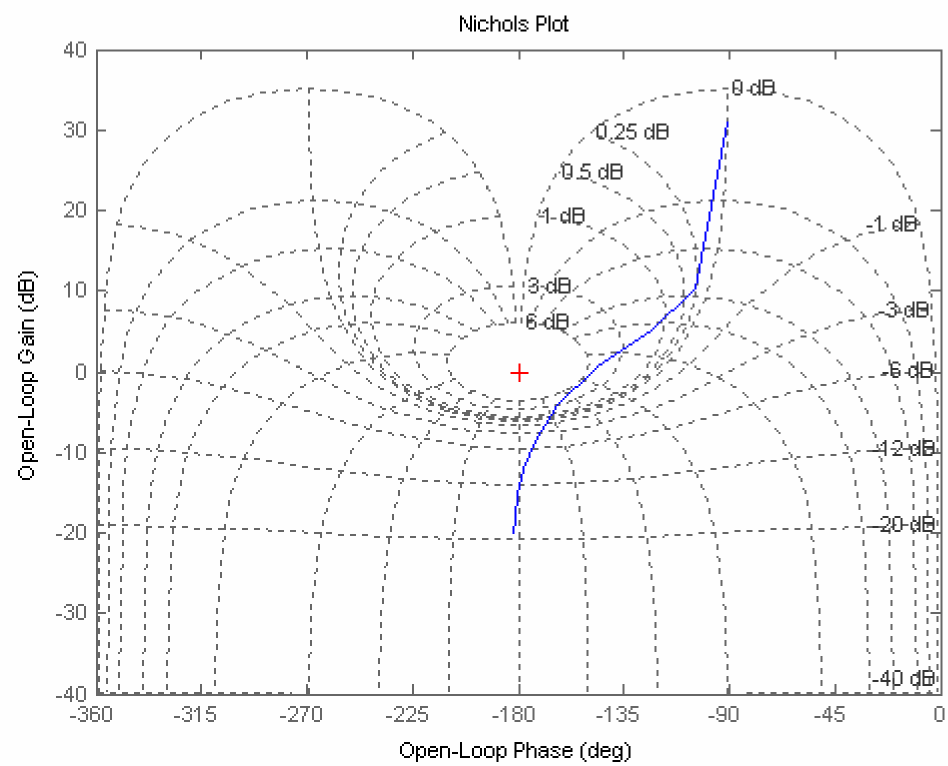
Closed-Loop System

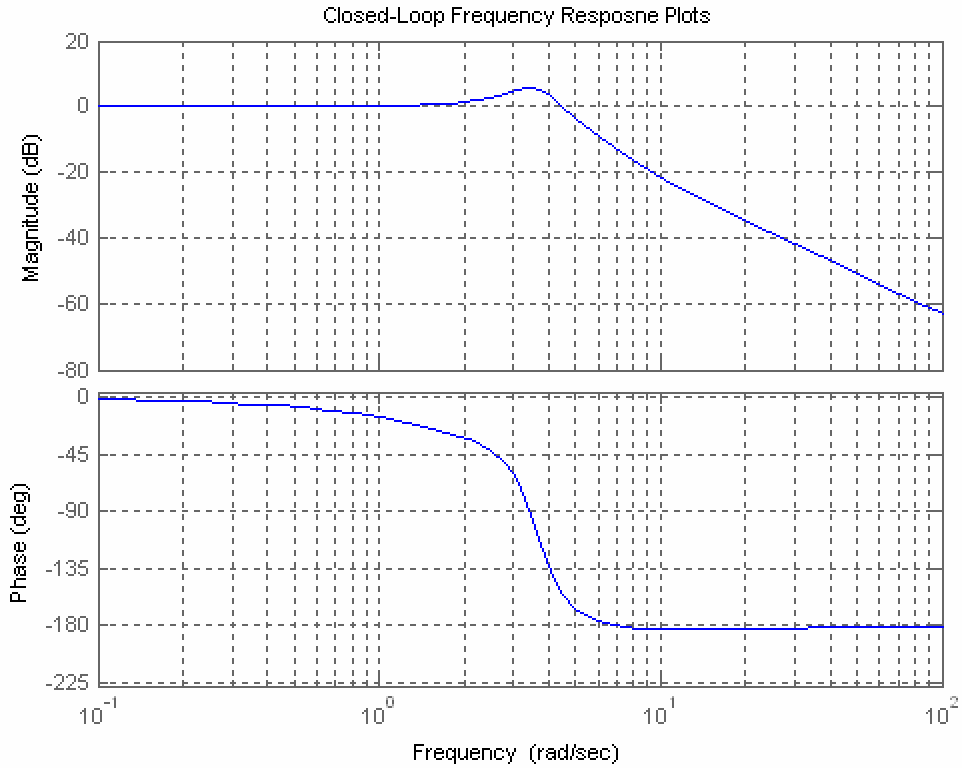
ans =

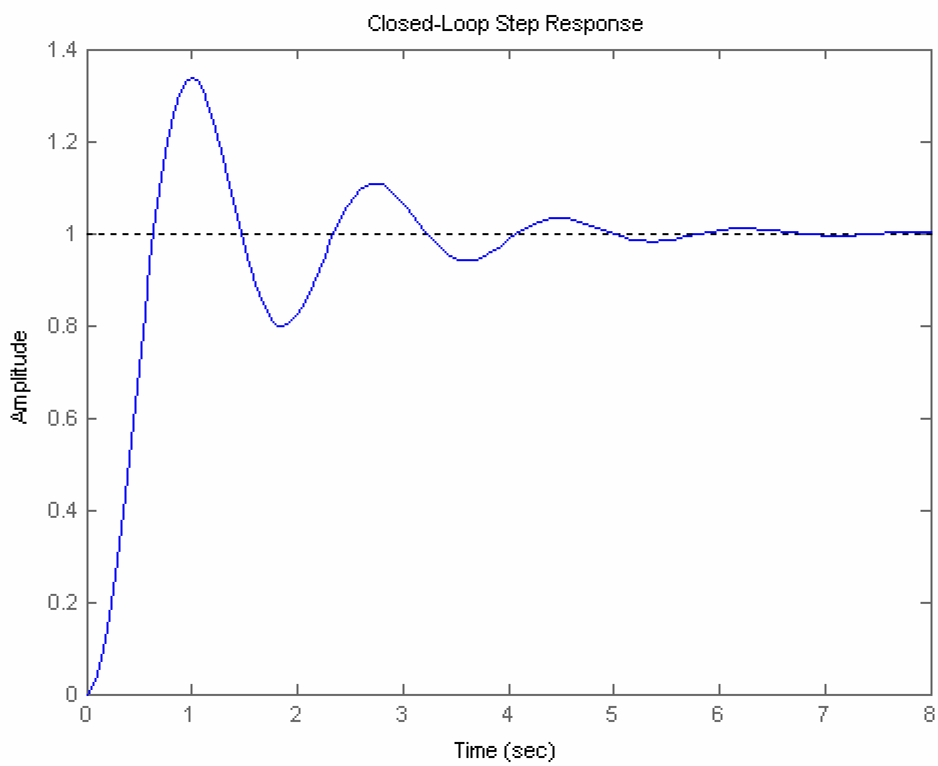
T(s)


Transfer function:
       7 s + 35
-----------------------
s^3 + 4 s^2 + 17 s + 35
```

```
Damping Ratio = 0.259481, Percent Overshoot = 42.9946, Bandwidth = 5.1, Mp
0.957852
```




Closed-Loop Frequency Resposne Plots
Magnitude (dB)
20
0
-20
-40
-60
-80
Phase (deg)
0
-45
-90
-135
-180
-225
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)


Closed-Loop Step Response
Amplitude
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
6
7
8
Time (sec)

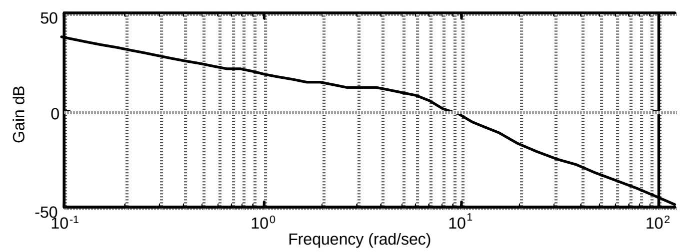
23.

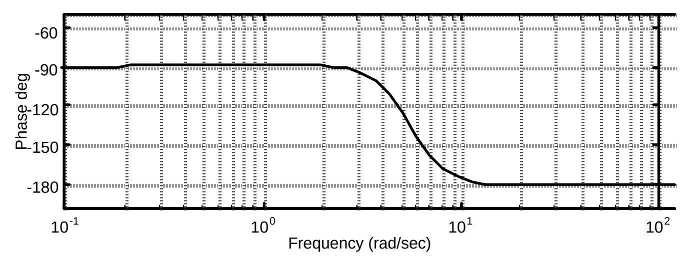
System 1: Using non-asymptotic frequency response plots, the zero dB crossing is at 9.7 rad/s at a phase of –163.2º. Therefore the phase margin is 180º – 163.2º = 16.8º. |G(jω)| is down 7 dB at 14.75 rad/s. Therefore the bandwidth is 14.75 rad/s. Using Eq. (10.73), ζ = 0.15. Using Eq. (4.38), %OS = 62.09%. Eq. (10.55) yields T_s = 2.76 s, and Eq. (10.56) yields T_p = 0.329 s.

System 2: Using non-asymptotic frequency response plots, the zero dB crossing is at 6.44 rad/s at a phase of -150.73º. Therefore the phase margin is 180º - 150.73º = 29.27º. |G(jω)| is down 7 dB at 10.1 rad/s. Therefore the bandwidth is 10.1 rad/s. Using Eq. (10.73), ζ = 0.262. Using Eq. (4.38), %OS = 42.62%. Eq. (10.55) yields T_s = 2.23 s, and Eq. (10.56) yields T_p = 0.476 s.

24.

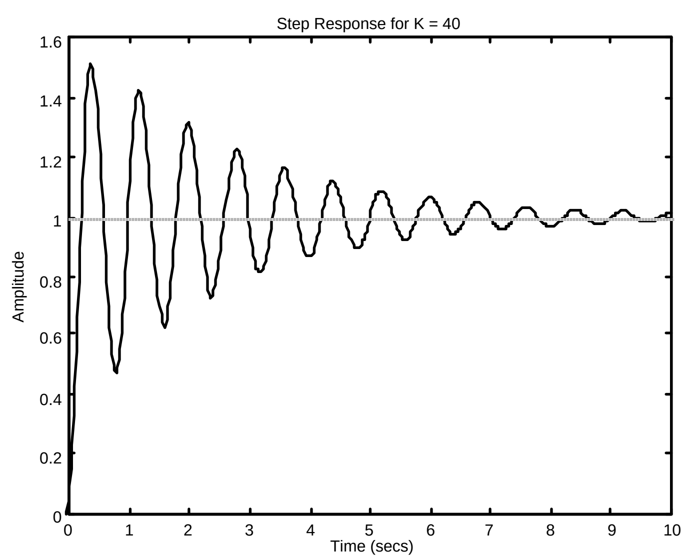
a.





b. Zero dB frequency = 7.8023; Looking at the phase diagram at this frequency, the phase margin is 8.777 degrees. Using Eq. (10.73) or Figure 10.48, ζ = 0.08. Thus, %OS = 77.7.

c.



25.

From the Bode plots: Gain margin = 14.96 dB; phase margin = 49.57^0; 0 dB frequency = 2.152 rad/s; 180^0 frequency = 6.325 rad/s; bandwidth(@-7 dB point) = 3.8 rad/s. From Eq. (10.73) $\zeta = 0.48$; from Eq. (4.38) %OS = 17.93; from Eq. (10.55) $T_s = 2.84$ s; from Eq. (10.56) $T_p = 1.22$ s.

26.

Program:

```
G=zpk([-2],[0 -1 -4],100)
%G=zpk([-3 -5],[0 -2 -4 -6],50)
G=tf(G)
bode(G)
title('System 1')
%title('System 2')
pause
%Find Phase Margin
[Gm,Pm,Wcg,Wcp]=margin(G);
w=1:.01:20;
[M,P,w]=bode(G,w);
%Find Bandwidth
for k=1:1:length(M);
   if 20*log10(M(k))+7<=0;
   'Mag'
   20*log10(M(k))
   'BW'
   wBW=w(k)
   break
end
end
%Find Damping Ratio,Percent Overshoot, Settling Time, and Peak Time
for z= 0:.01:10
Pt=atan(2*z/(sqrt(-2*z^2+sqrt(1+4*z^4))))*(180/pi);
if (Pm-Pt)<=0
   z;
   Po=exp(-z*pi/sqrt(1-z^2));
   Ts=(4/(wBW*z))*sqrt((1-2*z^2)+sqrt(4*z^4-4*z^2+2));
   Tp=(pi/(wBW*sqrt(1-z^2)))*sqrt((1-2*z^2)+sqrt(4*z^4-4*z^2+2));
fprintf('Bandwidth = %g ',wBW)
fprintf('Phase Margin = %g',Pm)
fprintf(', Damping Ratio = %g',z)
fprintf(', Percent Overshoot = %g',Po*100)
fprintf(',Settling Time = %g',Ts)
fprintf(', Peak Time = %g',Tp)
   break
end
end
T=feedback(G,1);
step(T)
title('Step Response System 1')
%title('Step Response System 2')
```

Computer response:

```
Zero/pole/gain:
  100 (s+2)
-------------
s (s+1) (s+4)


Transfer function:
   100 s + 200
-----------------
s^3 + 5 s^2 + 4 s


ans =
```

```
Mag

ans =

   -7.0007

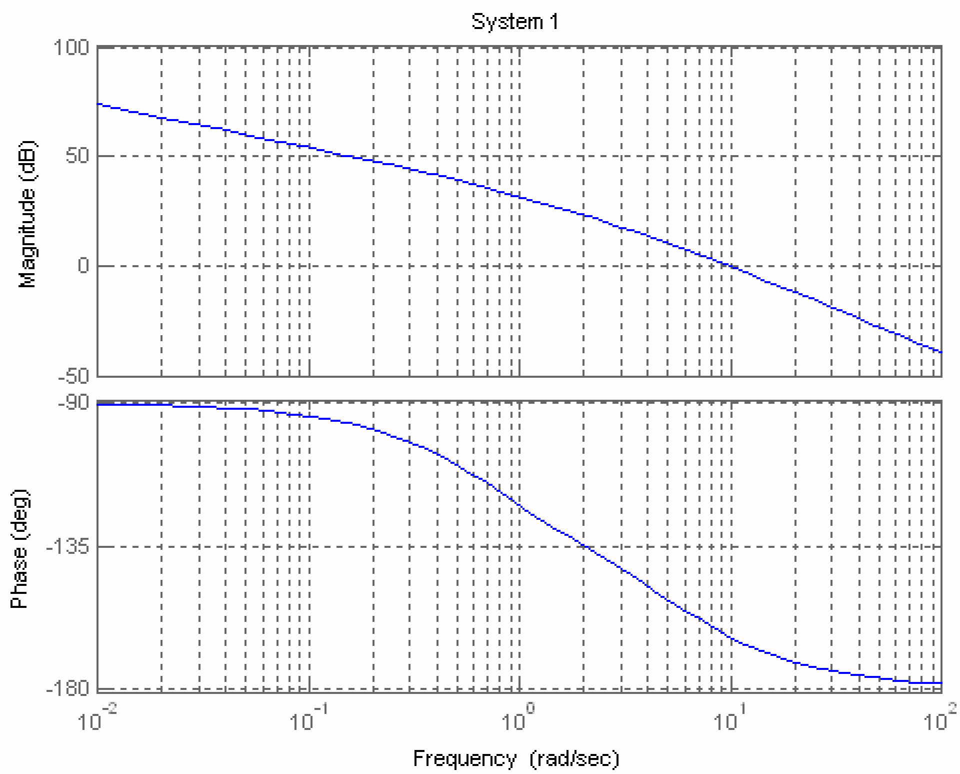

ans =

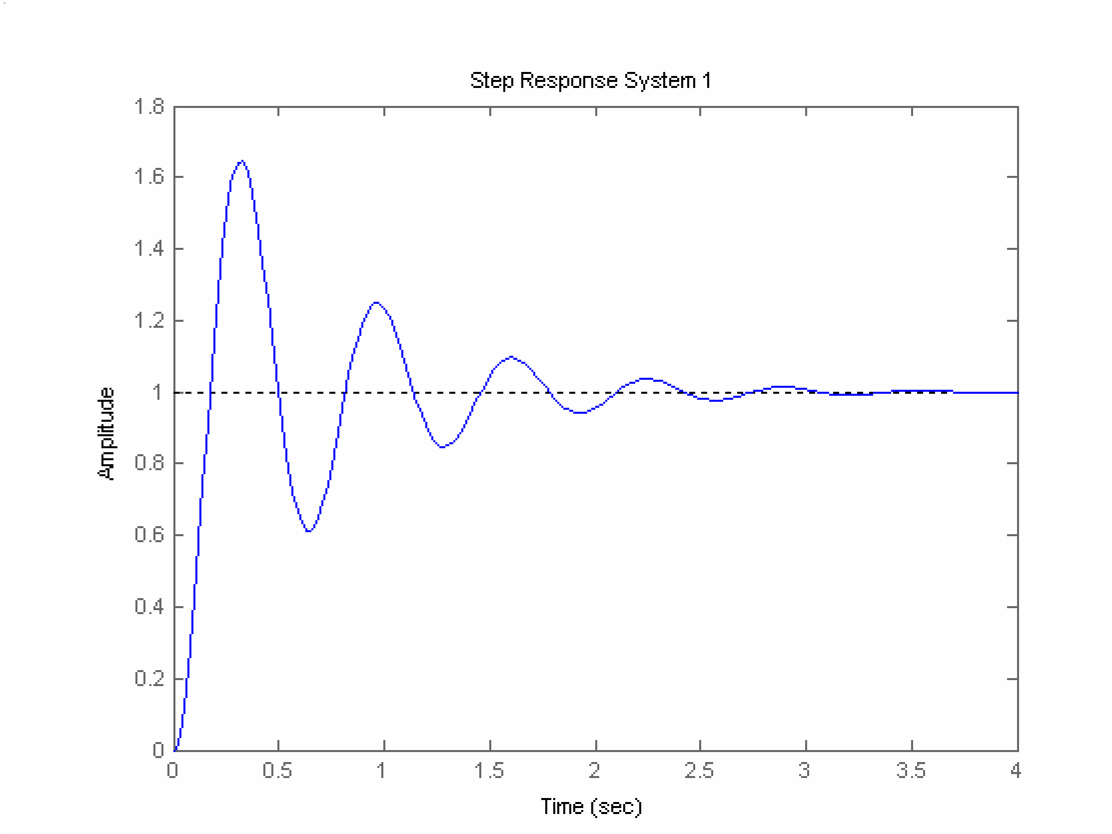
BW


wBW =

   14.7500

Bandwidth = 14.75 Phase Margin = 16.6617, Damping Ratio = 0.15, Percent
Overshoot = 62.0871,Settling Time = 2.76425, Peak Time = 0.329382
```






```
Zero/pole/gain:

  50 (s+3) (s+5)

-------------------

s (s+2) (s+4) (s+6)

Transfer function:

    50 s^2 + 400 s + 750

----------------------------

s^4 + 12 s^3 + 44 s^2 + 48 s

ans =

Mag

ans =

   -7.0026

ans =
```

```
BW

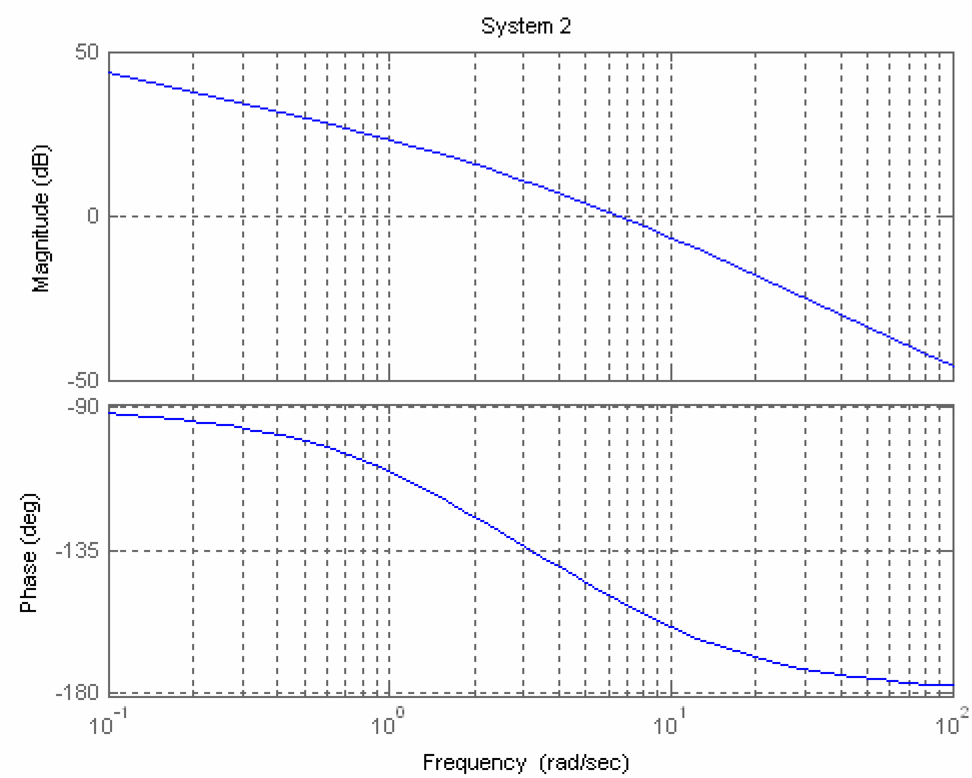
wBW =

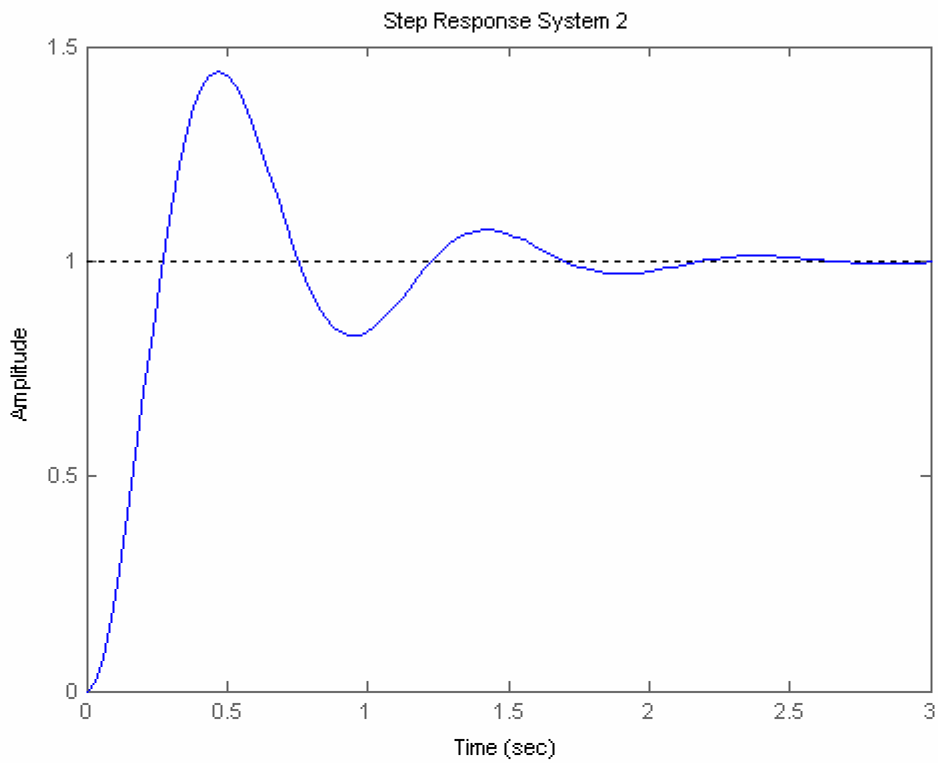
   10.1100

Bandwidth = 10.11 Phase Margin = 29.2756, Damping Ratio = 0.27, Percent Overshoot

= 41.439,Settling Time = 2.1583, Peak Time =

0.475337
```





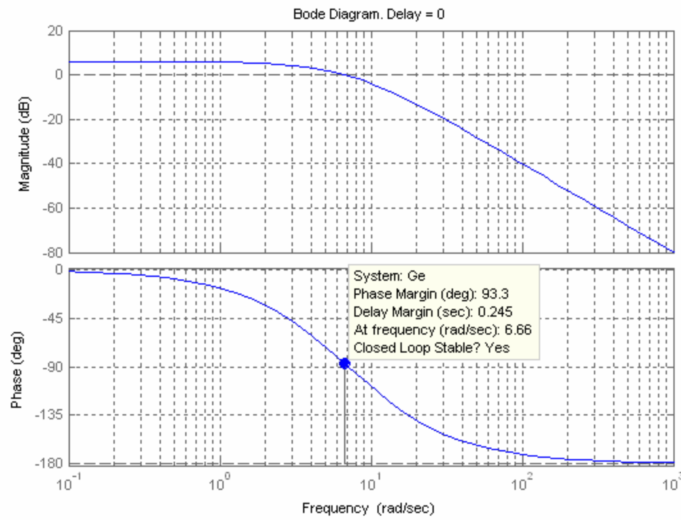
27.

The phase margin of the given system is 20°. Using Eq. (10.73), $\zeta = 0.176$. Eq. (4.38) yields 57% overshoot. The system is Type 1 since the initial slope is - 20 dB/dec. Continuing the initial slope down to the 0 dB line yields $K_v = 4$. Thus, steady-state error for a unit step input is zero; steady state error for a unit ramp input is $\frac{1}{K_v} = 0.25$; steady-state error for a parabolic input is infinite.

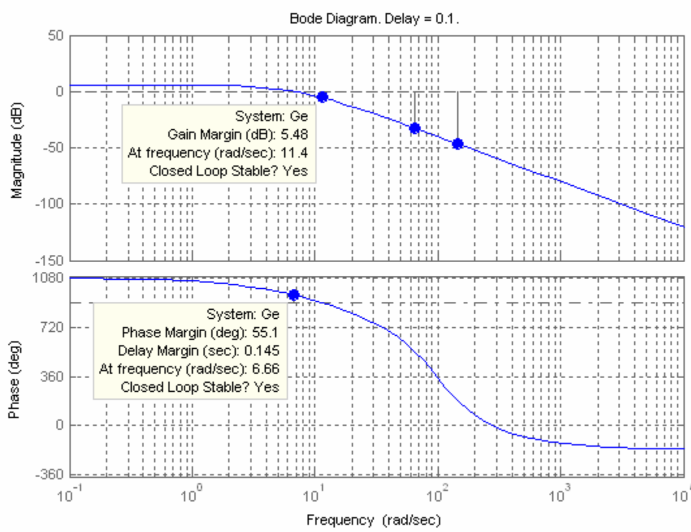
28.

The magnitude response is the same for all time delays and crosses zero dB at 0.5 rad/s. The following is a plot of the magnitude and phase responses for the given time delays:

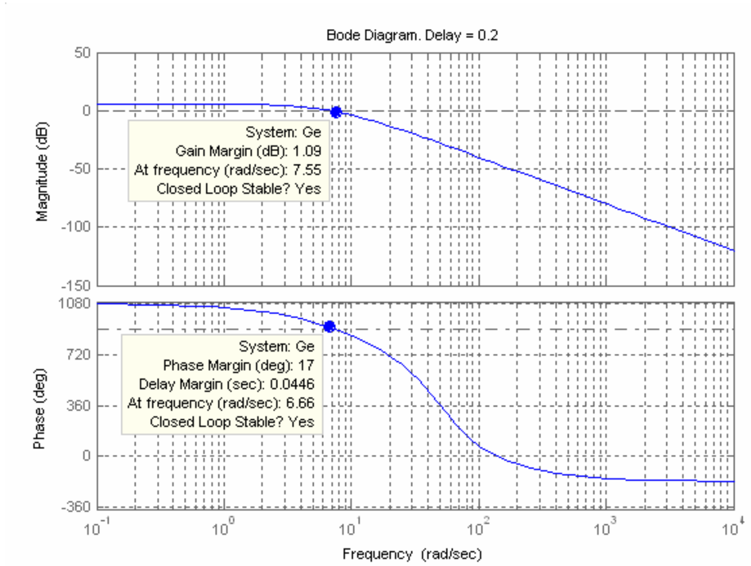
a.



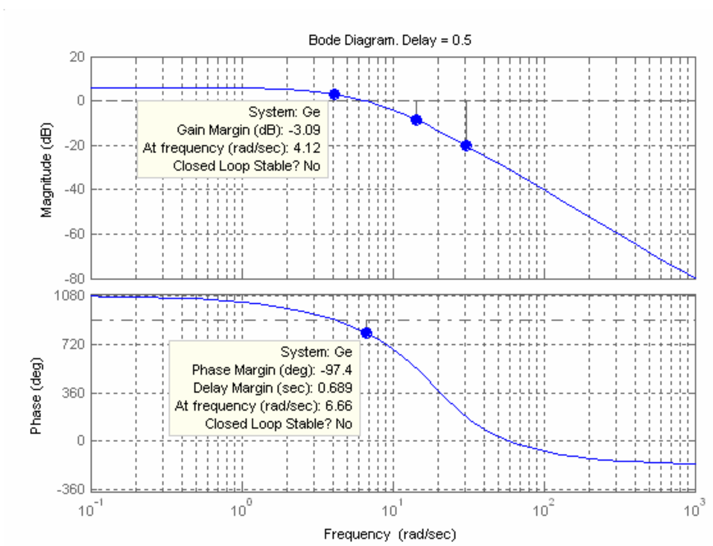
For T = 0, $\Phi_M = 93.3^o$; System is stable.



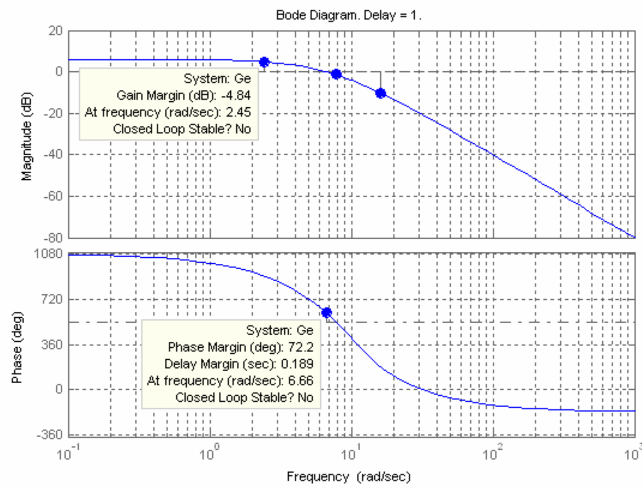
For T = 0.1, $\Phi_M = 55.1^o$; System is stable.



For T = 0.2, $\Phi_M = 17^o$; System is stable.



For T = 0.5, $\Phi_M = -97^o$; System is unstable.



For T = 1, $\Phi_M = 72.2^o$; System is unstable because the gain margin is -4.84 dB.

b.

For T = 0, the phase response reaches 180^o at infinite frequency. Therefore the gain margin is infinite. The system is stable.

For T = 0.1, the phase response is -180^o at 11.4 rad/s. The magnitude response is -5.48 dB at 11.4 rad/s. Therefore, the gain margin is 5.48 dB. The system is stable.

For T = 0.2, the phase response is -180^o at 7.55 rad/s. The magnitude response is -1.09 dB at 7.55 rad/s. Therefore, the gain margin is 1.09 dB and the system is stable.

For T = .5, the phase response is -180^o at 4.12 rad/s. The magnitude response is +3.09 dB at 4.12 rad/s. Therefore, the gain margin is – 3.09 dB and the system is unstable.

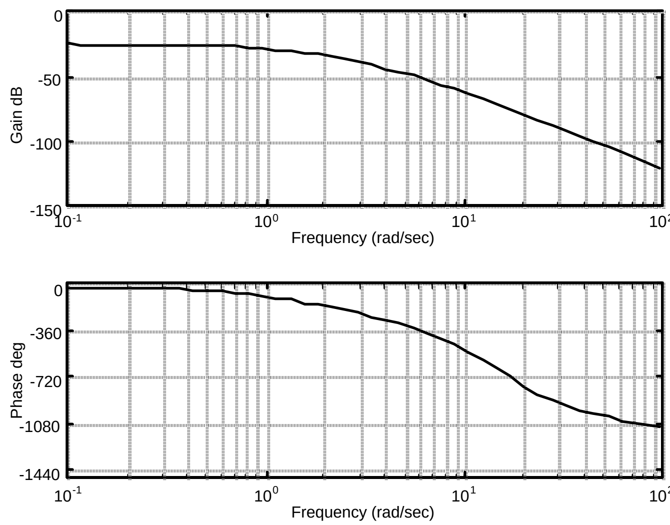
For T = 1, the phase response is -180^o at 2.45 rad/s. The magnitude response is +4.84 dB at 2.45 rad/s. Therefore, the gain margin is -4.84 dB and the system is unstable.

c.
T = 0; T = 0.1; T = 0.2

d.
T = 0.5, -3.09 dB; T = 1, - 4.84 dB;

29.

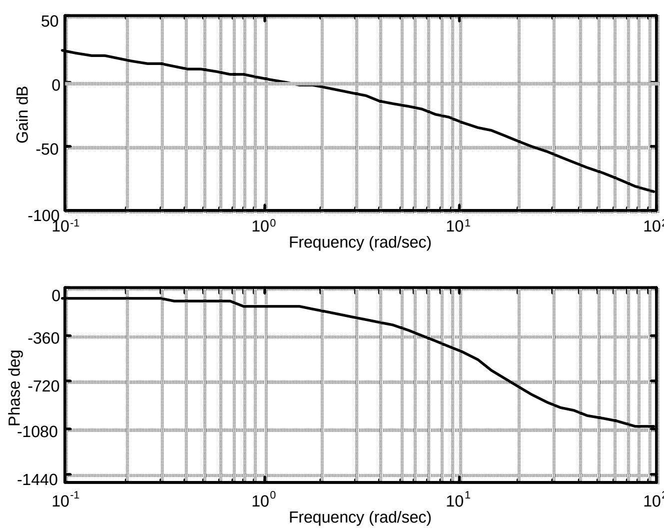
The Bode plots for K = 1 and 0.5 second delay is:



The phase is -180° at 2.12 rad/s. At this frequency, the gain is -34.76 dB. Thus the gain can be raised by 34.76 dB = 54.71. Hence for stability, $0<K<54.71$.

30.

The Bode plots for K = 40 and a delay of 0.5 second is shown below.



The magnitude curve crosses zero dB at a frequency of 1.0447 rad/s. At this frequency, the phase plot shows a phase margin of 35.74 degrees. Using Eq. (10.73) or Figure 10.48, $\zeta = 0.33$. Thus, %OS = 33.3.

31.

Program:

```
%Enter G(s)************************
numg1=1;
deng1=poly([0 -3 -12]);
'G1(s)'
G1=tf(numg1,deng1)
[numg2,deng2]=pade(0.5,5);
'G2(s) (delay)'
```

```
G2=tf(numg2,deng2)
'G(s)=G1(s)G2(s)'
G=G1*G2
%Enter K ***************************
K=input('Type gain, K ');
T=feedback(K*G,1);
step(T)
title(['Step Response for K = ',num2str(K)])
```

Computer response:

```
ans =

G1(s)

Transfer function:

         1

-------------------

s^3 + 15 s^2 + 36 s

ans =

G2(s) (delay)

Transfer function:

-s^5 + 60 s^4 - 1680 s^3 + 2.688e004 s^2 - 2.419e005 s + 9.677e005

------------------------------------------------------------------

s^5 + 60 s^4 + 1680 s^3 + 2.688e004 s^2 + 2.419e005 s + 9.677e005

ans =

G(s)=G1(s)G2(s)

Transfer function:

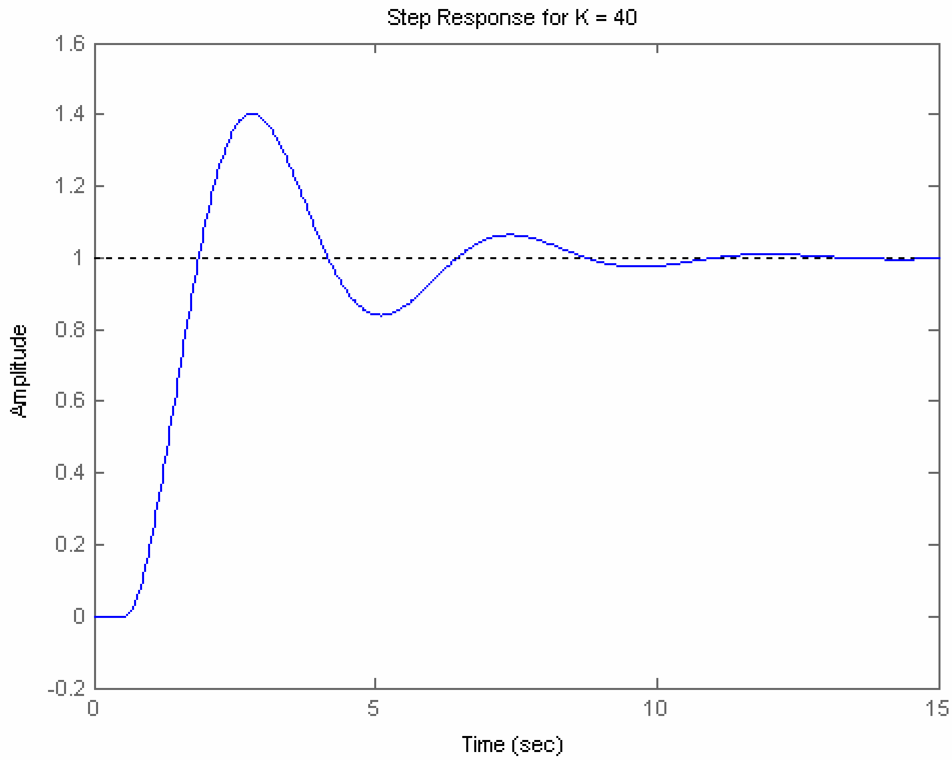
    -s^5 + 60 s^4 - 1680 s^3 + 2.688e004 s^2 - 2.419e005 s + 9.677e005

--------------------------------------------------------------------------

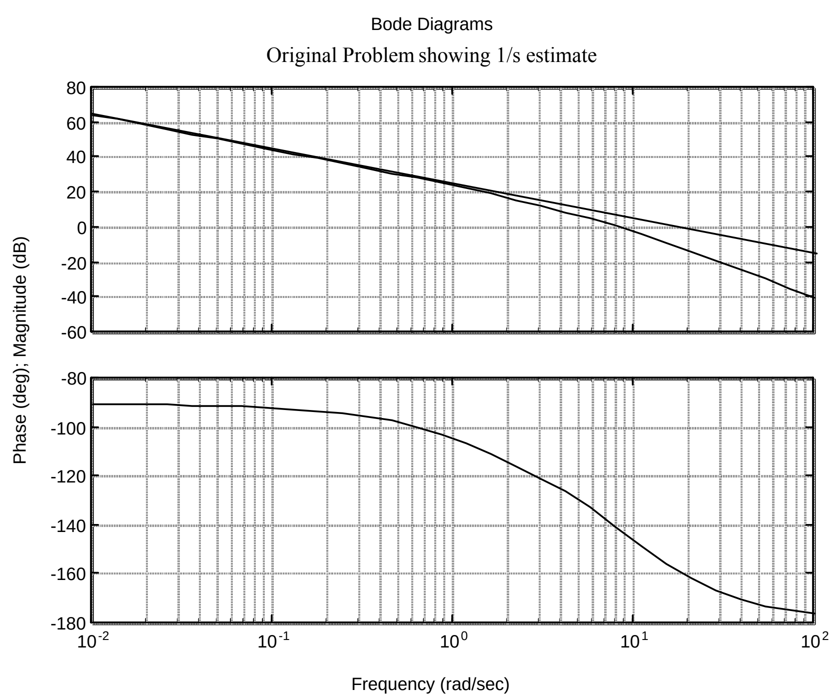
s^8 + 75 s^7 + 2616 s^6 + 5.424e004 s^5 + 7.056e005 s^4 + 5.564e006 s^3

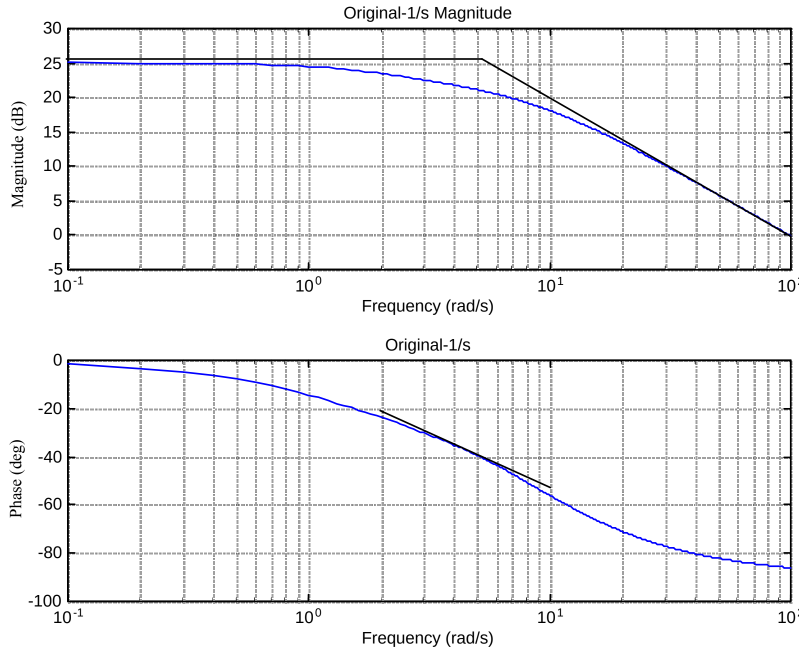
                                             + 2.322e007 s^2 + 3.484e007 s
```

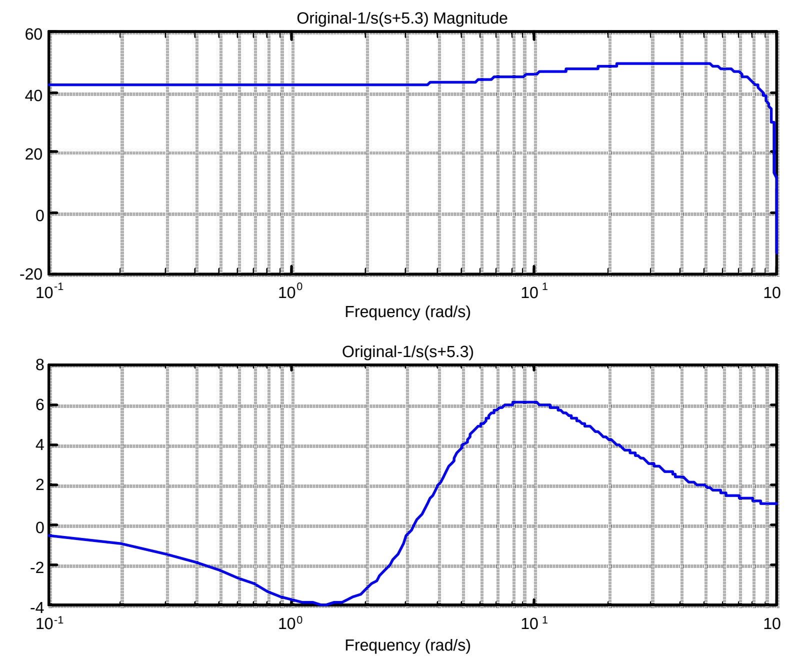
```
Type gain, K 40
```



32.







Estimated $K = 41$ dB $= 112$. Therefore, final estimate is $G(s) = \dfrac{112}{s(s+5.3)}$.

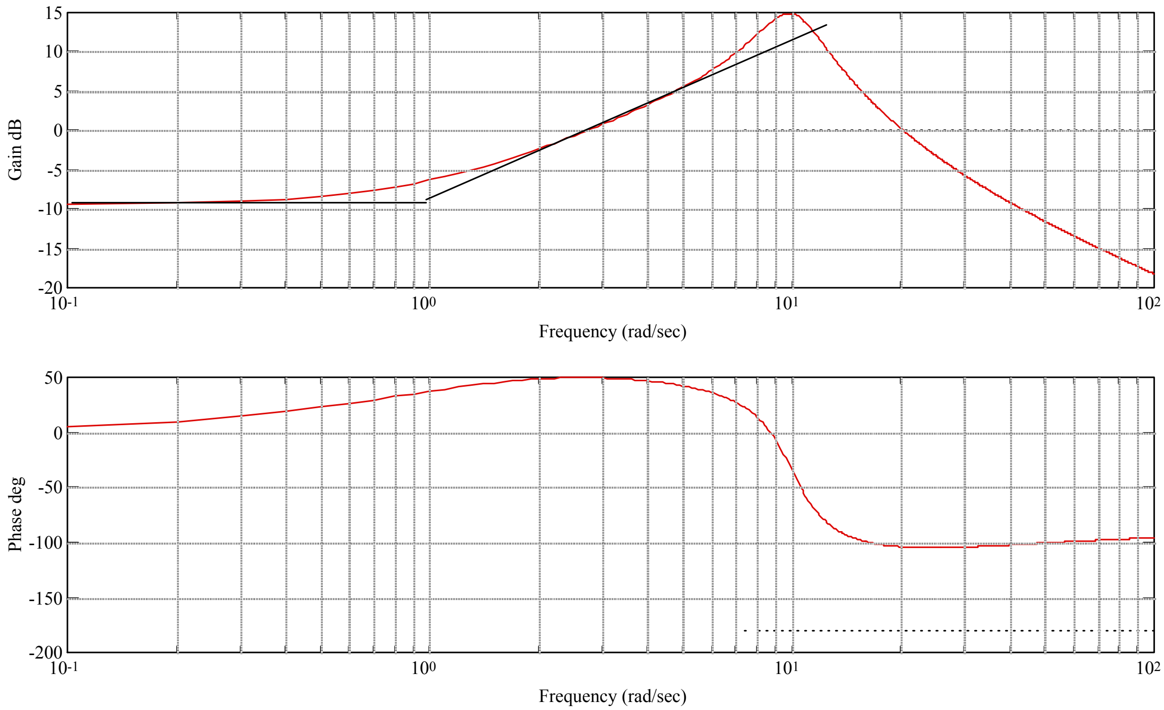
33.

Program:

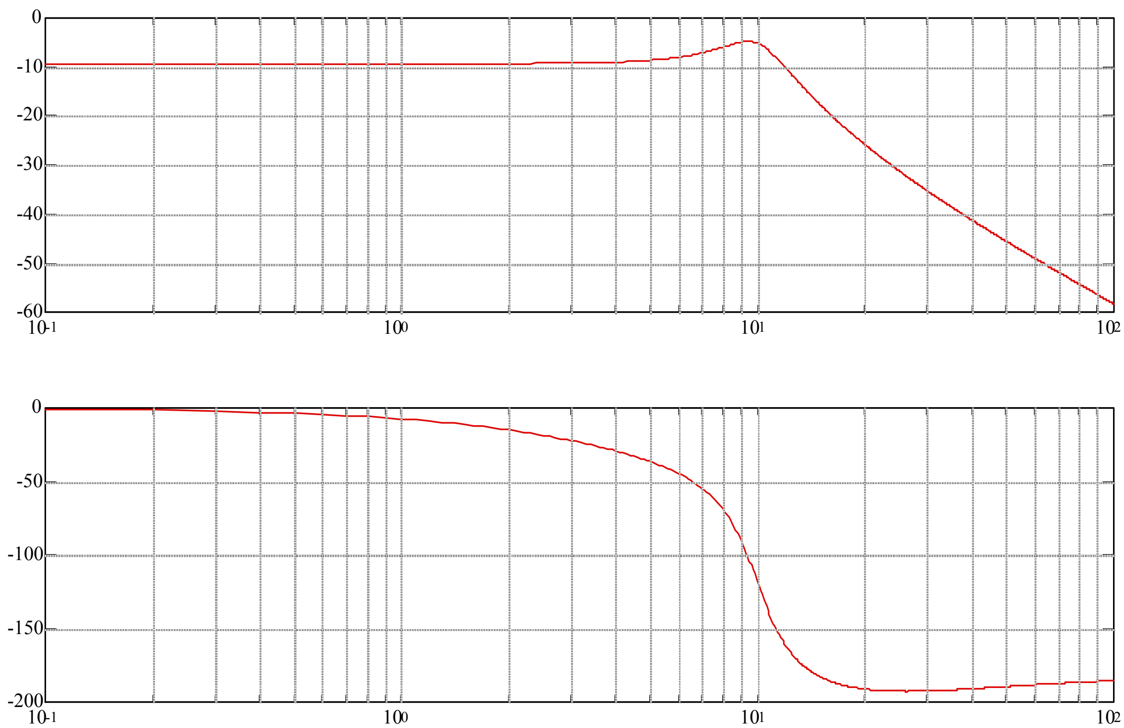
```
%Generate total system Bode plots - numg0,deng0 - M0,P0
clf
numg0=12*poly([-1 -20]);
deng0=conv([1 7],[1 4 100]);
G0=tf(numg0,deng0);
w=0.1:0.1:100;
```

```
[M0,P0]=bode(G0,w);
M0=M0(:,:);
P0=P0(:,:);
[20*log10(M0),P0,w];
bode(G0,w)
pause
%Subtract (s+1) [numg1,deng1] and generate Bode plot-M2,P2
numg1=[1 1];
deng1=1;
G1=tf(numg1,deng1);
[M1,P1]=bode(G1,w);
M1=M1(:,:);
P1=P1(:,:);
M2=20*log10(M0)-20*log10(M1);
P2=P0-P1;
clf
subplot(2,1,1)
semilogx(w,M2)
grid
subplot(2,1,2)
semilogx(w,P2)
grid
pause
%Subtract10^2/(s^2+2*0.3*10s+10^2) [numg2,deng2] and generate Bode plot-
M4,P4
numg2=100;
deng2=[1 2*0.3*10 10^2];
G2=tf(numg2,deng2);
[M3,P3]=bode(G2,w);
M3=M3(:,:);
P3=P3(:,:);
M4=M2-20*log10(M3);
P4=P2-P3;
clf
subplot(2,1,1)
semilogx(w,M4)
grid
subplot(2,1,2)
semilogx(w,P4)
grid
pause
%Subtract(8.5/23)(s+23)/(s+8.5) [numg3,deng3] and generate Bode plot-M6,P6
numg3=(8.5/23)*[1 23];
deng3=[1 8.5];
G3=tf(numg3,deng3);
[M5,P5]=bode(G3,w);
M5=M5(:,:);
P5=P5(:,:);
M6=M4-20*log10(M5);
P6=P4-P5;
clf
subplot(211)
semilogx(w,M6)
grid
subplot(212)
semilogx(w,P6)
grid
```

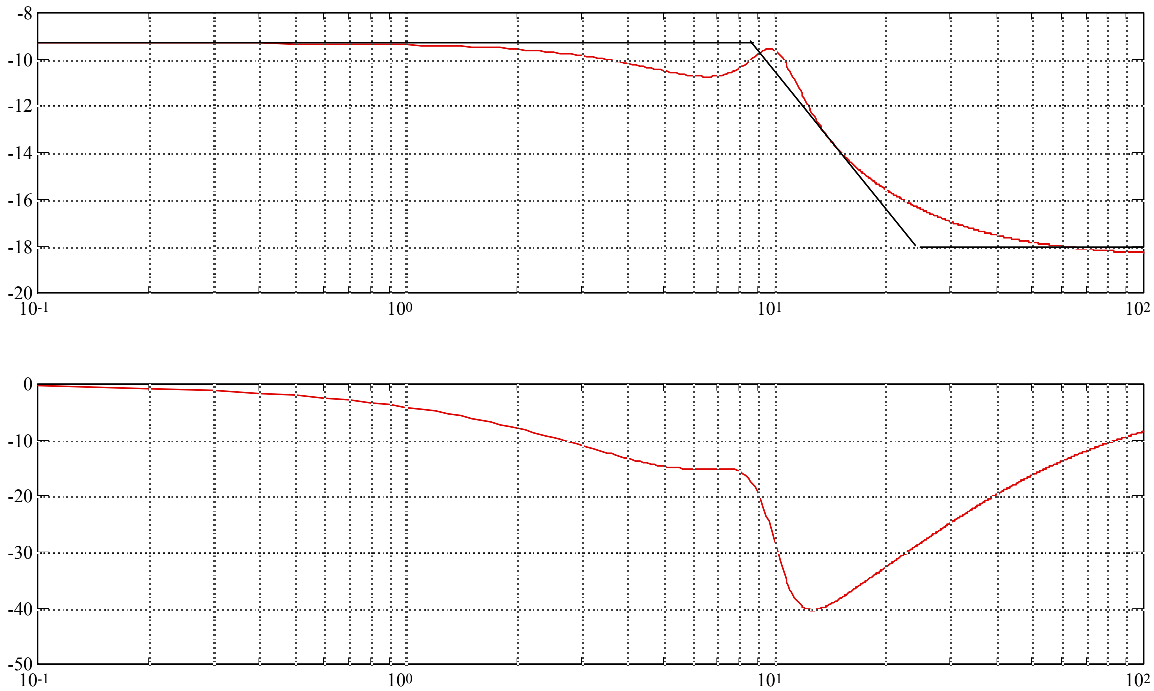
Computer responses and analysis:



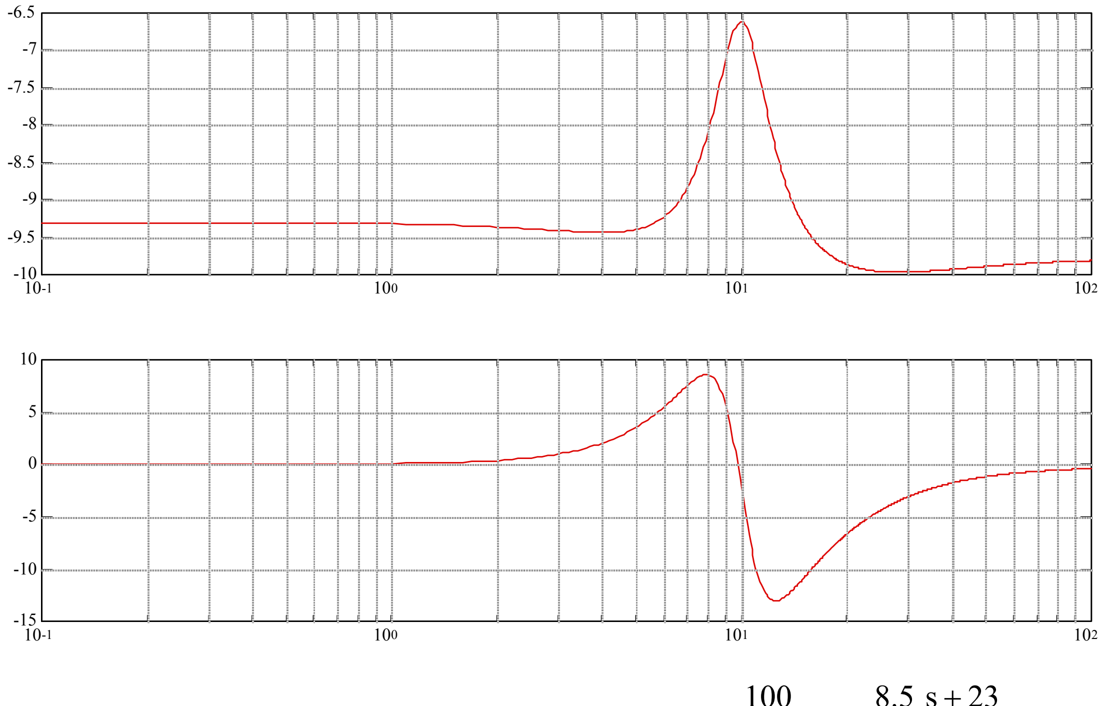
Original data showing estimate of a component, (s+1)



Original data minus (s+1) showing estimate of $(10^2/(s^2+2*0.3*10s+10^2))$



Original data minus $(s+1)(10^2/(s^2+2*0.3*10s+10^2))$ showing estimate of $(8.5/23)(s+23)/(s+8.5)$

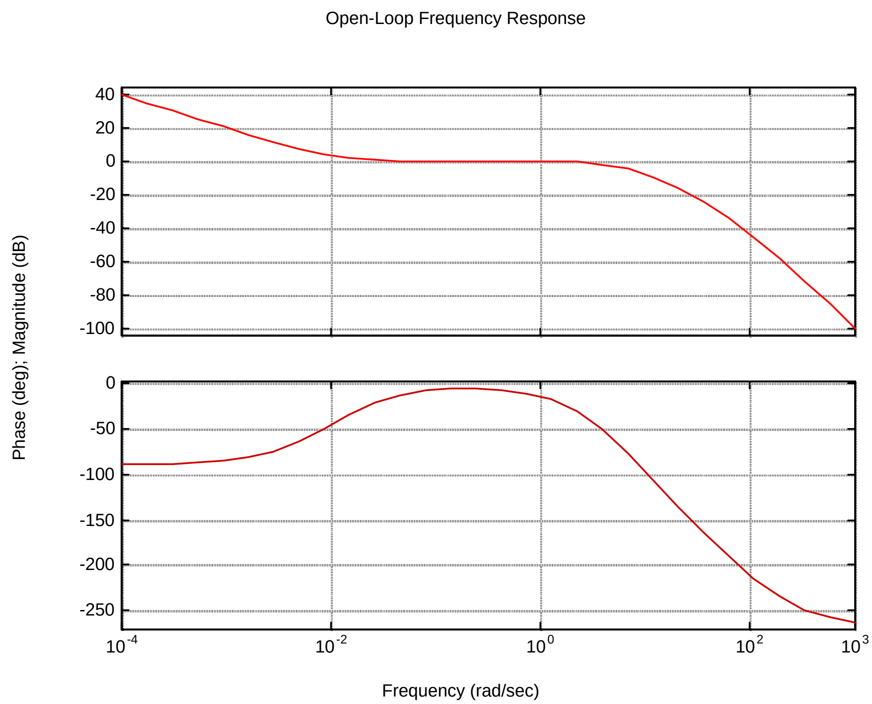


Original data minus final estimate of $G(s) = (s+1) * \frac{100}{s^2 + 6s + 100} * \frac{8.5}{23} \frac{s+23}{s+8.5}$

Thus the final estimate is $G(s) = (s+1) * \frac{100}{s^2 + 6s + 100} * \frac{8.5}{23} \frac{s+23}{s+8.5}$ * K. Since the original plot starts from -10 dB, 20 log K = -10, or K = 0.32.

34.

a.

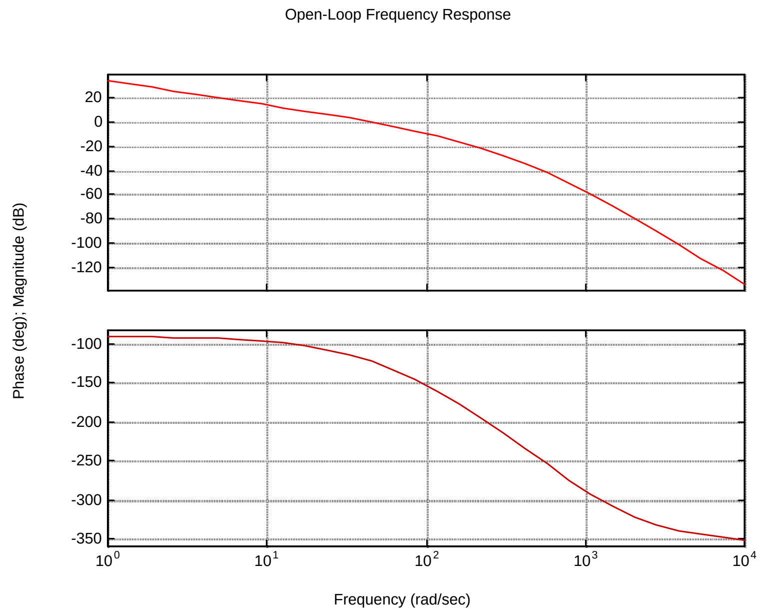


From the Bode plot: Gain margin = 29 52 dB; phase margin = 157.5^0; 0 dB frequency = 1.63 rad/s; 180^0 frequency = 49.8 rad/s.

b. System is stable since it has 180^0 of phase with a magnitude less than 0 dB.

35.

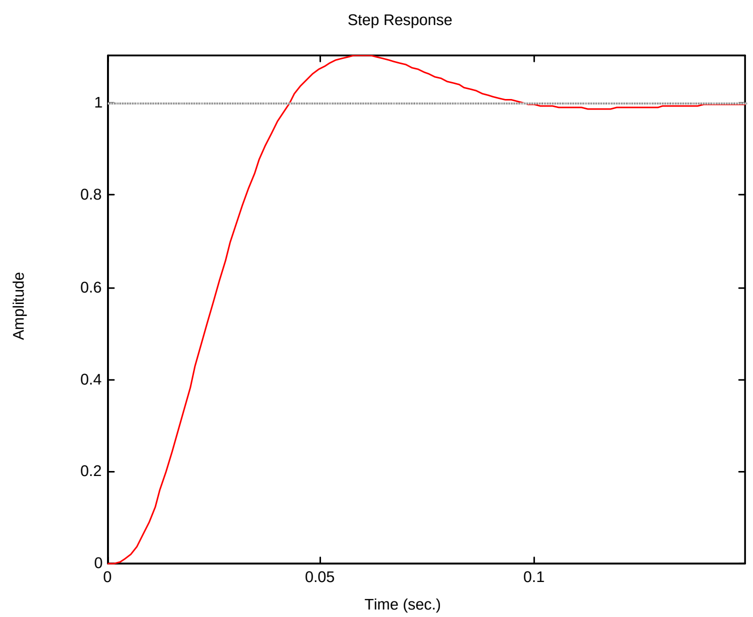
a.



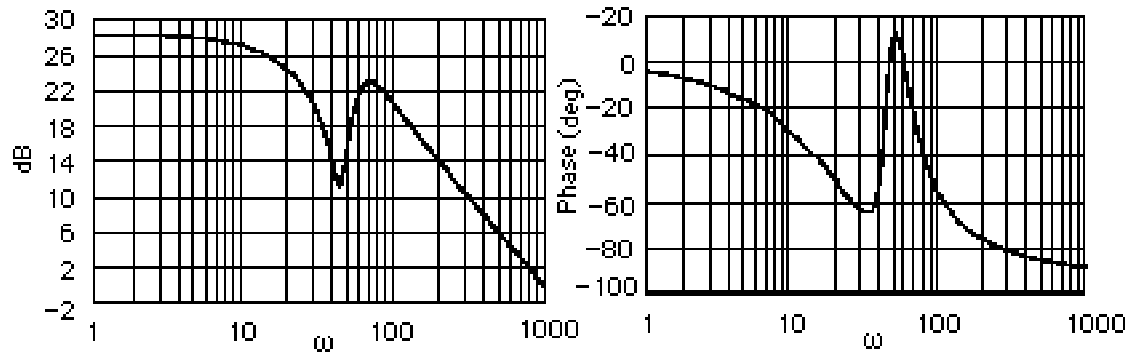
From the Bode plot: Gain margin = 17.1 dB; phase margin = 57.22^0; 0 dB frequency = 45.29 rad/s; 180^0 frequency = 169.03 rad/s; bandwidth(@-7 dB open-loop) = 85.32 rad/s.

b. From Eq. (10.73) $\zeta = 0.58$; from Eq. (4.38) %OS = 10.68; from Eq. (10.55) $T_s = 0.0949$ s; from Eq. (10.56) $T_p = 0.0531$ s.

c.



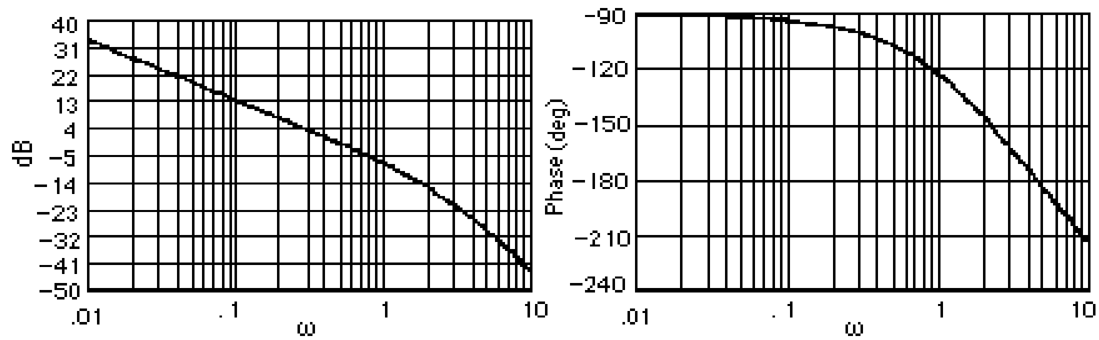
36.



Resonance at 70 rad/s.

37.

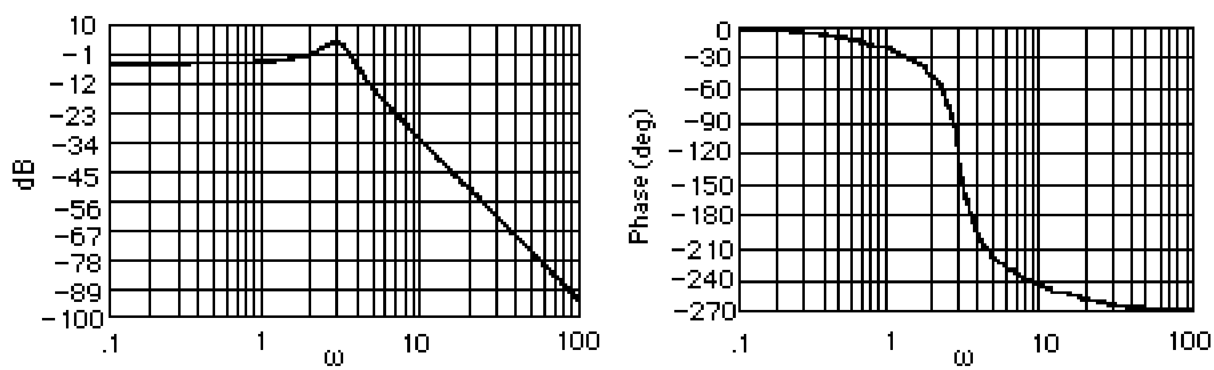
$G(s) = \frac{10}{s(s+2)(s+10)}$. Plotting the Bode plots,



The gain is zero dB at 0.486 rad/s and the phase angle is -106.44. Thus, the phase margin is 180^o - 106.44^o = 73.56^o . Using Eq. (10.73), $\zeta = 0.9$. Using Eq. (4.38), %OS = 0.15%.

38.

$G(s) = \frac{22.5}{(s+4)(s^2+0.9s+9)}$. Plotting the Bode plots,

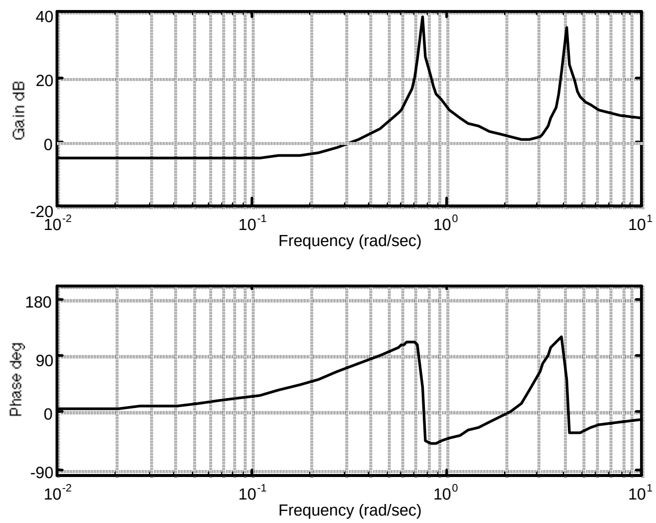


The phase response is 180^o at $\omega = 3.55$ rad/s, where the gain is -1.17 dB. Thus, the gain margin is 1.17 dB. Unity gain is at $\omega = 2.094$ rad/s, where the phase is - 49.85^o and at $\omega = 3.452$ rad/s, where

the phase is -173.99°. Hence the phase margin is measured at ω = 3.452 rad/s and is 180°- 173.99° = 6.01°. Using Eq. (10.73), ζ = 0.0525. Eq. (4.38) yields %OS = 84.78%.

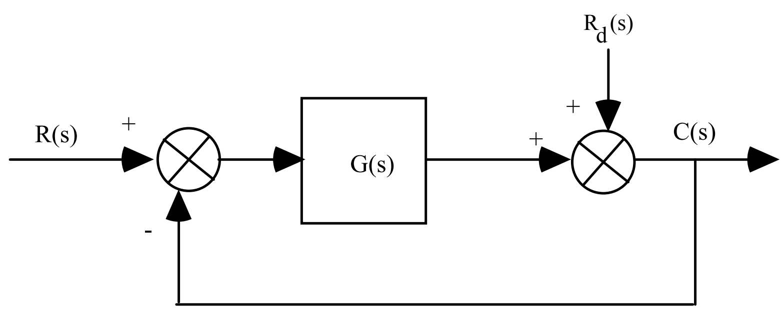
39.

a.



The frequencies that will be reduced occur at the peaks of the magnitude plot. The frequencies at the peaks are 4.14 rad/s and 0.754 rad/s.

b. Consider a system with a disturbance, R_d at the output of a system:



The transfer function relating C(s) to R_d(s) is $\frac{C(s)}{R_d(s}=\frac{1}{1+G(s)}$. Therefore,

$$C(s)=\frac{1}{1+\frac{N_G}{D_G}}*\frac{N_{Rd}}{D_{Rd}}=\frac{D_G}{D_G+N_G}*\frac{N_{Rd}}{D_{Rd}}$$

Thus, if the poles of G(s) match the poles of R_d ($D_G = D_{Rd}$) there will be cancellation and the dynamics of the disturbance will be reduced. Thus, if the dynamics of R_d is oscillation, add poles in cascade with G(s) that have the same dynamics. Since the poles yield large gain at these bending frequencies a zero is placed near the poles so that the filter will have minimal effect on the transient

response (similar to placing a zero near a pole for a lag compensator). This arrangement of poles and zeros is called a dipole. Also note that a high gain at the bending frequency yields negative feedback for the output to subtract from R_d. Care should be exercised through analysis and simulation to be sure that the system's response to an input, other than the disturbance, is not adversely affected by the additional poles.

40.

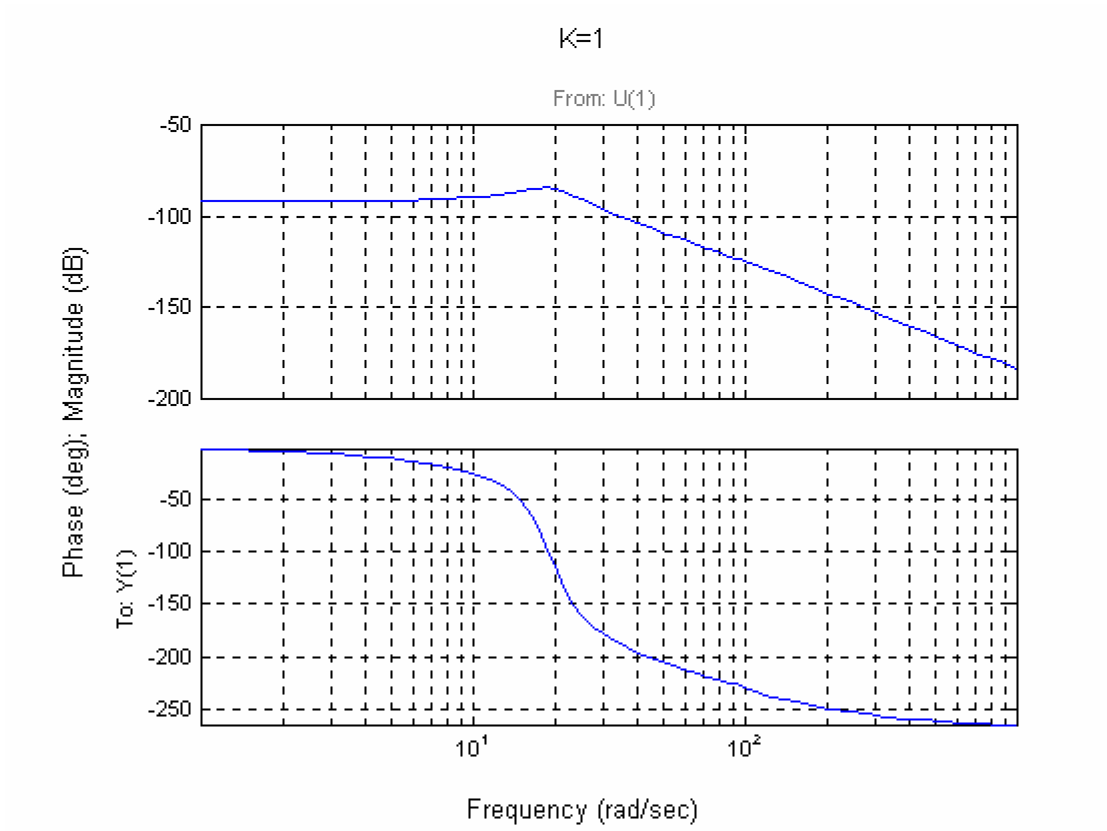
a. From Chapter 8,

$$G_e(s) = \frac{0.6488K\,(s+53.85)}{(s^2 + 8.119s + 376.3)\,(s^2 + 15.47s + 9283)}$$

Cascading the notch filter,

$$G_{et}(s) = \frac{0.6488K\,(s+53.85)(s^2 + 16s + 9200)}{(s^2 + 8.119s + 376.3)\,(s^2 + 15.47s + 9283)(s+60)^2}$$

Plotting the Bode plot,



From the Bode plot: Gain margin = 96.74 dB; phase margin = ∞; 0 dB frequency = N/A; 180^0 frequency = 30.44 rad/s.

b. K = 96.74 dB = 68732

c. In Chapter 6 K = 188444. The difference is due to the notch filter.

E L E V E N

Design via Frequency Response

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Gain Design

a. The required phase margin for 25% overshoot (ζ = 0.404), found from Eq. (10.73), is 43.49°.

From the solution to the Case Study Challenge problem of Chapter 10, $G(s) = \frac{50.88K}{s(s+1.32)(s+100)}$.

Using the Bode plots for K = 1 from the solution to the Case Study Challenge problem of Chapter 10, we find the required phase margin at ω = 1.35 rad/s, where the magnitude response is -14 dB. Hence, K = 5.01 (14 dB).

b.

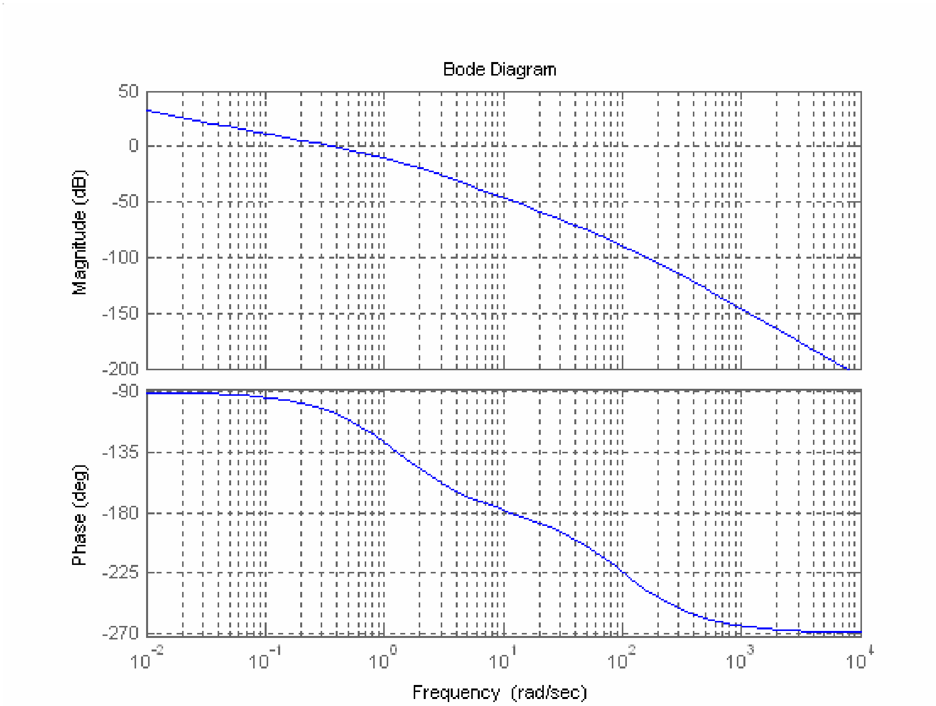
Program:

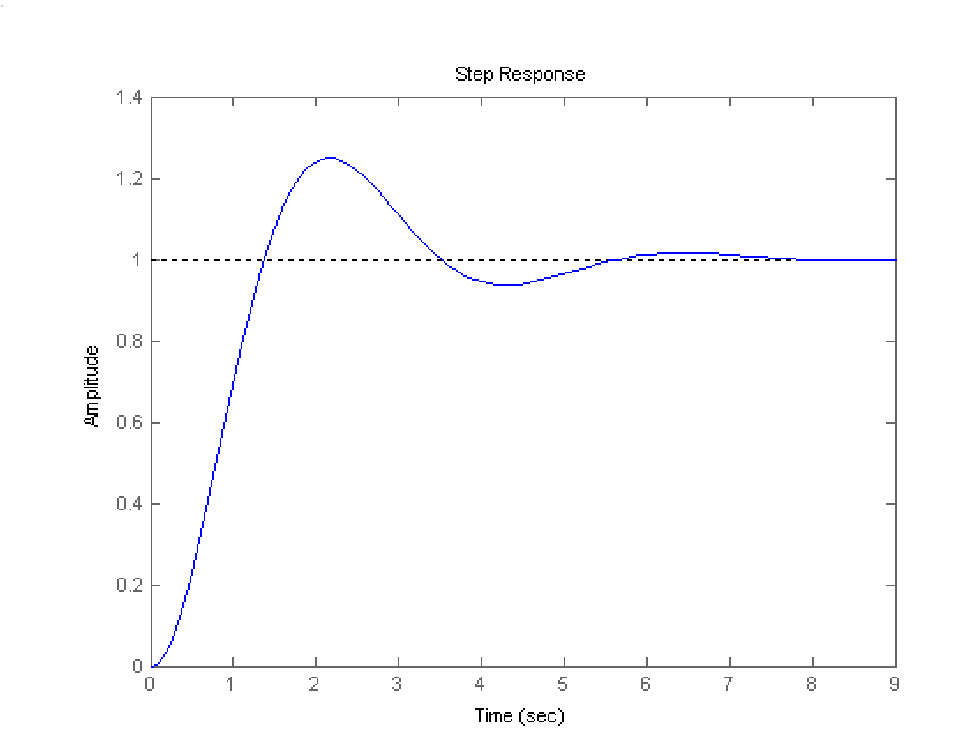
```
%Input system
numg=50.88;
deng=poly([0 -1.32 -100]);
G=tf(numg,deng);
%Percent Overshoot to Damping Ratio to Phase Margin
Po=input('Type %OS  ');
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
Pm=atan(2*z/(sqrt(-2*z^2+sqrt(1+4*z^4))))*(180/pi);
fprintf('\nPercent Overshoot = %g',Po)
fprintf(', Damping Ratio = %g',z)
fprintf(', Phase Margin = %g',Pm)
%Get Bode data
bode(G)
pause
w=0.01:0.05:1000;%Step size can be increased if memory low.
[M,P]=bode(G,w);
M=M(:,:);
P=P(:,:);
Ph=-180+Pm;
for i=1:1:length(P);
if P(i)-Ph<=0;
M=M(i);
K=1/M;
fprintf(', Frequency  = %g',w(i))
fprintf(', Phase  = %g',P(i))
fprintf(', Magnitude = %g',M)
fprintf(', Magnitude (dB) = %g',20*log10(M))
fprintf(', K = %g',K)
break
end
end
T=feedback(K*G,1);
step(T)
```

Computer response:

```
Type %OS  25

Percent Overshoot = 25, Damping Ratio = 0.403713, Phase Margin = 43.463,
Frequency  = 1.36, Phase  = -136.634, Magnitude = 0.197379, Magnitude (dB)
= -14.094, K = 5.06641
```





Antenna Control: Cascade Compensation Design

a. From the solution to the previous Case Study Challenge in this chapter, $G(s) = \frac{50.88K}{s(s+1.32)(s+100)}$.

For $K_v = 20$, K = 51.89. Hence, the gain compensated system is

$$G(s) = \frac{2640.16}{s(s+1.32)(s+100)}$$

Using Eq. (10.73), 15% overshoot (i.e. $\zeta = 0.517$) requires a phase margin of 53.18°. Using the Bode plots for K = 1 from the solution to the Case Study Challenge problem of Chapter 10, we find the required phase margin at $\omega = 0.97$ rad/s where the phase is -126.82°.

To speed up the system, we choose the compensated phase margin frequency to be 4.6 * 0.97 = 4.46 rad/s. Choose the lag compensator break a decade below this frequency, or $\omega = 0.446$ rad/s.

At the phase margin frequency, the phase angle is -166.067°, or a phase margin of 13.93°. Using 5° leeway, we need to add 53.18° - 13.93° + 5° = 44.25°. From Figure 11.8, $\beta = 0.15$, or $\gamma = \frac{1}{\beta}$ = 6.667. Using Eq. (11.15), the lag portion of the compensator is

$$G_{Lag}(s) = \frac{(s+0.446)}{(s+\frac{0.446}{6.667})} = \frac{s+0.446}{s+0.0669} .$$

Using Eqs. (11.9) and (11.15), $T_2 = \frac{1}{\omega_{max}\sqrt{\beta}}$ = 0.579. From Eq. (11.15), the lead portion of the compensator is

$$G_{Lead}(s) = \frac{s+1.727}{s+11.51}$$

The final forward path transfer function is

$$G(s)G_{Lag}(s)G_{Lead}(s) = \frac{2640.16(s+0.446)(s+1.727)}{s(s+1.32)(s+100)(s+0.0669)(s+11.51)}$$

b.

Program:

```
%Input system ****************************
K=51.89;
numg=50.88*K;
deng=poly([0 -1.32 -100]);
G=tf(numg,deng);
Po=15;
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
%Determine required phase margin**************
Pmreq=atan(2*z/(sqrt(-2*z^2+sqrt(1+4*z^4))))*(180/pi)
phreq=Pmreq-(180)%required phase
w=0.1:0.01:10;
[M,P]=bode(G,w);
for i=1:1:length(P);%search for phase angle
if P(i)-phreq<=0;
ph=P(i)
w(i)
break
end
end
wpm=4.6*w(i)
```

```
[M,P]=bode(G,wpm);%Find phase at wpm
Pmreqc=Pmreq-(180+P)+5%Find contribution required from compensator+5
beta=(1-sin(Pmreqc*pi/180))/(1+sin(Pmreqc*pi/180))
%Design lag compensator***************
zclag=wpm/10;
pclag=zclag*beta;
Kclag=beta;
%Design lead compensator**********
zclead=wpm*sqrt(beta);
pclead=zclead/beta;
Kclead=1/beta;
%Create compensated forward path*********
numgclag=Kclag*[1 zclag];
dengclag=[1 pclag];
'Gclag(s)'
Gclag=tf(numgclag,dengclag);
Gclagzpk=zpk(Gclag)
numgclead=Kclead*[1 zclead];
dengclead=[1 pclead];
'Gclead(s)'
Gclead=tf(numgclead,dengclead);
Gcleadzpk=zpk(Gclead)
Gc=Gclag*Gclead;
'Ge(s)=G(s)*Gclag(s)*Gclead(s)'
Ge=Gc*G;
Gezpk=zpk(Ge)
%Test lag-lead compensator****************
T=feedback(Ge,1);
bode(Ge)
title('Lag-lead Compensated')
[Gm,Pm,wcp,wcg]=margin(Ge);
'Compensated System Results'
fprintf('\nResulting Phase Margin = %g',Pm)
fprintf(', Resulting Phase Margin Frequency = %g',wcg)
pause
step(T)
title('Lag-lead Compensated')
```

Computer response:

```
Pmreq =

   53.1718

phreq =

 -126.8282

ph =

 -126.8660

ans =

    0.9700

wpm =
    4.4620
Pmreqc =

   44.2468

beta =

    0.1780

ans =
```

```
Gclag(s)

Zero/pole/gain:
0.17803 (s+0.4462)
------------------
   (s+0.07944)

ans =

Gclead(s)

Zero/pole/gain:
5.617 (s+1.883)
---------------
   (s+10.58)

ans =

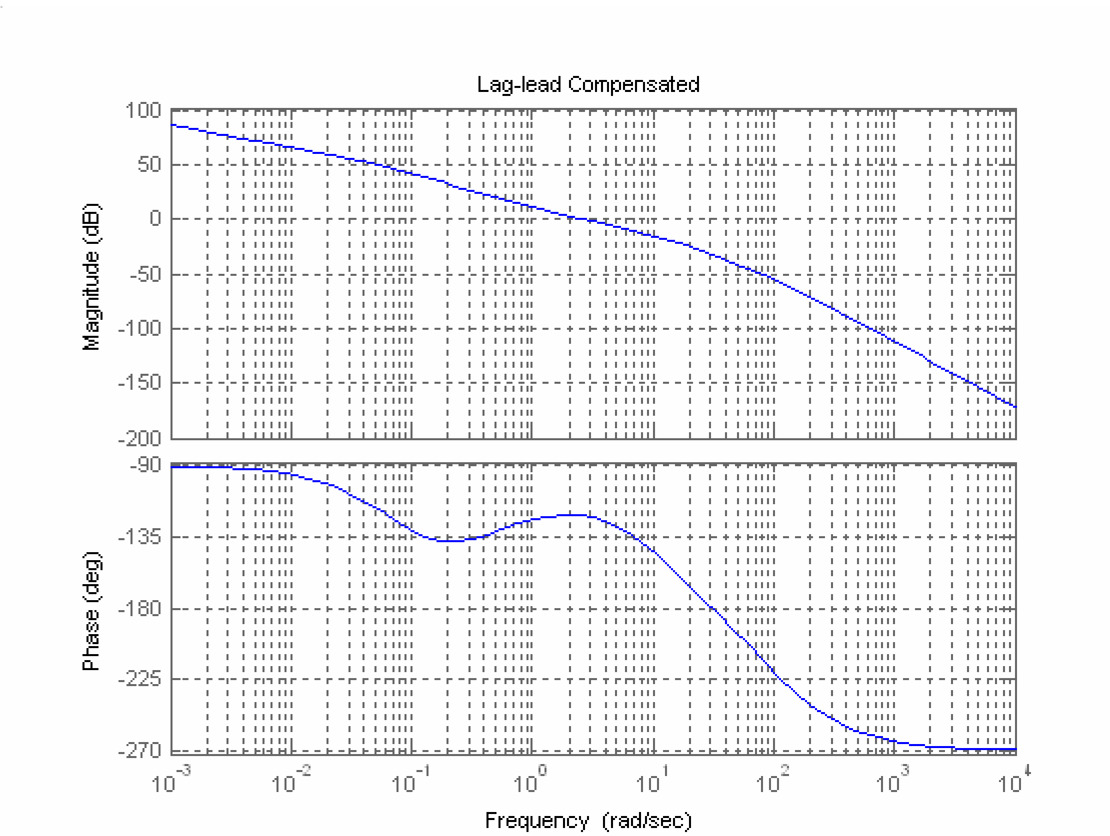
Ge(s)=G(s)*Gclag(s)*Gclead(s)

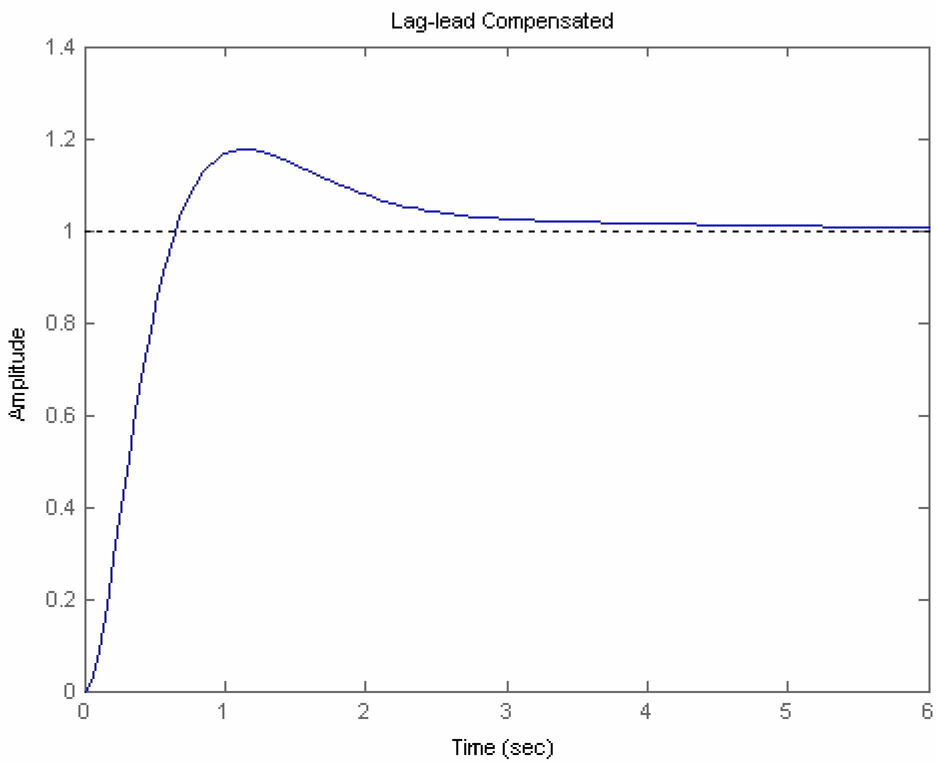

Zero/pole/gain:
     2640.1632 (s+1.883) (s+0.4462)
----------------------------------------
s (s+100) (s+10.58) (s+1.32) (s+0.07944)

ans =

Compensated System Results

Resulting Phase Margin = 57.6157, Resulting Phase Margin Frequency =
2.68618»
```





Answers to Review Questions

1. Steady-state error requirements can be designed simultaneously with transient response requirements.

2. Via the phase margin

3. The lag compensator is a low pass filter. Thus, while the low frequency gain is increased, the high-frequency gain at 180 ^{o} is decreased to make the system stable.

4. The lag network affects the phase angle at low frequencies, but not at high frequencies. For the compensated system, the phase plot is about the same as that of the uncompensated system around and above the phase margin frequency yielding the same transient response.

5. To compensate for the slight negative angle that the lag compensator has near the phase margin frequency

6. Compensated system has higher low-frequency gain than the uncompensated system designed to yield the same transient response; compensated and uncompensated system have the same phase margin frequency; the compensated system has lower gain around the phase margin frequency; the compensated and uncompensated system's have approximately the same phase values around the phase margin frequency.

7. The lead network is a high pass filter. It raises the gain at high frequencies. The phase margin frequency is increased.

8. Not only is the magnitude curve increased at higher frequencies, but so is the phase curve. Thus the 180^{o} point moves up in frequency with the increase in gain.

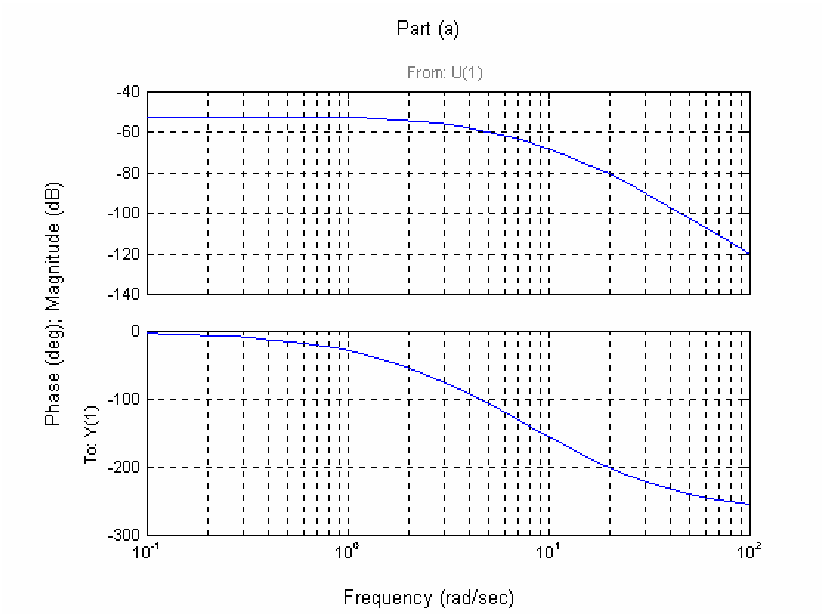
9. To correct for the negative phase angle of the uncompensated system

10. When designing the lag portion of a lag-lead compensator, we do not worry about the transient design. The transient response will be considered when designing the lead portion of a lag-lead compensator.

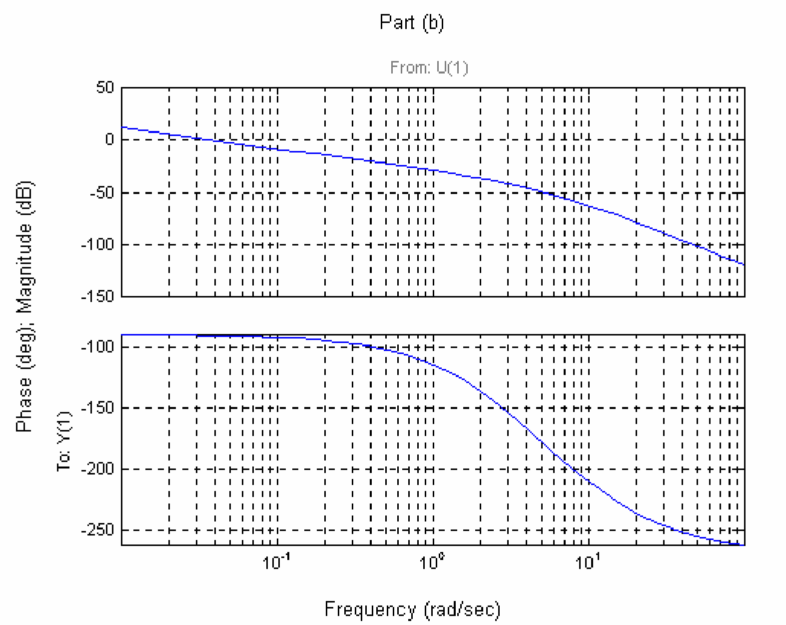
SOLUTIONS TO PROBLEMS

1.

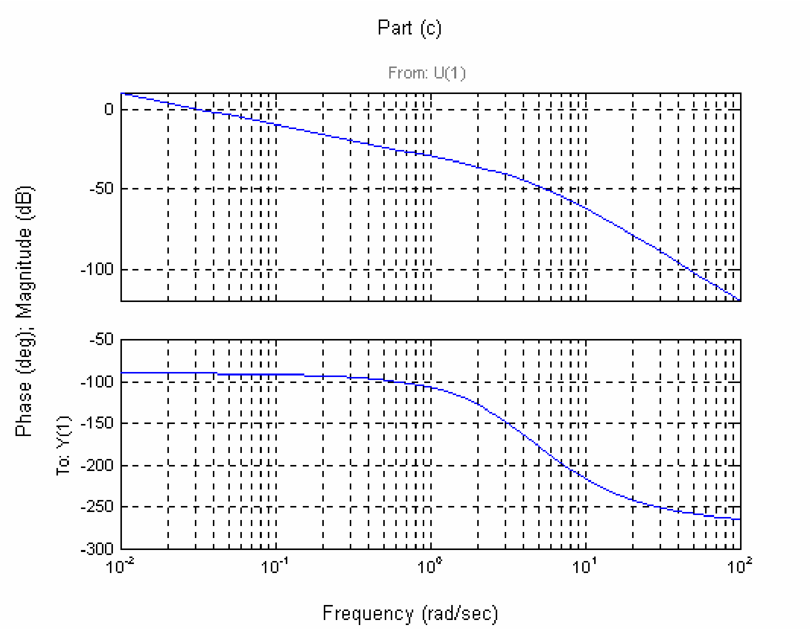
a. Plot Bode plots for K = 1; angle is 180^{o} at ω = 14.39 rad/s where the magnitude is –74.29 dB. Therefore a 64.29 dB (or K = 1639) increase will yield a 10 dB gain margin.



b. Plot Bode plots for K = 1; angle is 180^{o} at ω = 5.196 rad/s where the magnitude is –50.21 dB. Therefore a 40.21 dB (or K = 102.4) increase will yield a 10 dB gain margin.



c. Plot Bode plots for K = 1; angle is 180° at ω = 5.233 rad/s where the magnitude is –48.58 dB. Therefore a 38.58 dB (or K = 84.92) increase will yield a 10 dB gain margin.



2.

a. For a 40° phase margin, the phase must be -140° when the magnitude plot is zero dB. The phase is -140° at ω = 8.097 rad/s. At this frequency, the magnitude curve is –65.02 dB. Thus a 65.02 dB increase (K = 1782) will yield a 40° phase margin.

b. For a 40° phase margin, the phase must be -140° when the magnitude plot is zero dB. The phase is -140° at ω = 2.201 rad/s. At this frequency, the magnitude curve is – 37.60 dB. Thus a 37.60 dB increase (K = 75.86) will yield a 40° phase margin.

c. For a 40^o phase margin, the phase must be -140^o when the magnitude plot is zero dB. The phase is -140^o at ω = 2.653 rad/s. At this frequency, the magnitude curve is – 38.78 dB. Thus a 38.78 dB increase (K = 86.9) will yield a 40^o phase margin.

3.

20% overshoot => ζ = 0.456 => ϕ_M = 48.15^o.

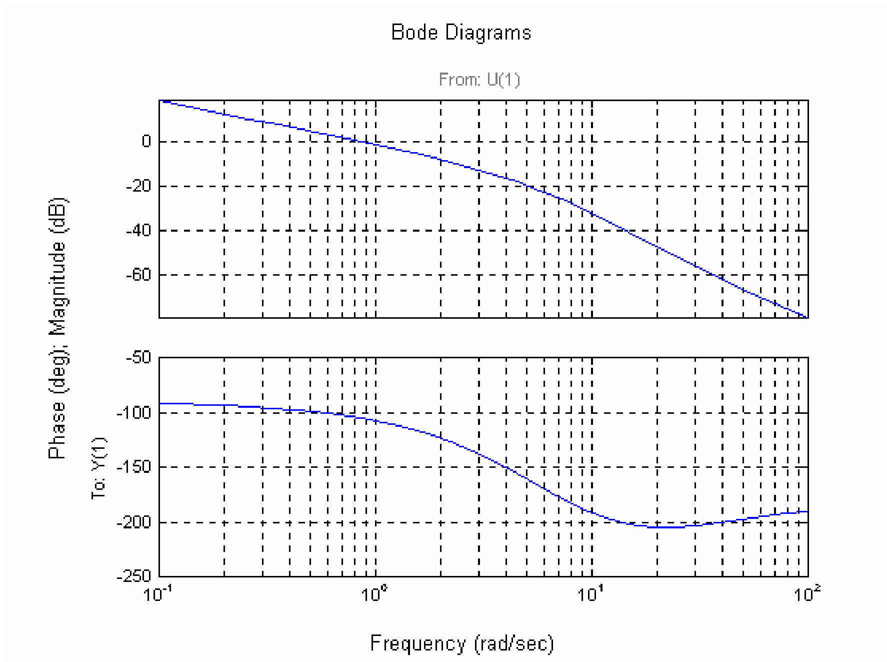
a. Looking at the phase diagram, where ϕ_M = 48.15^o (i.e. ϕ = -131.85^o), the phase margin frequency = 3.105 rad/s. At this frequency, the magnitude curve is -48.3 dB. Thus the magnitude curve has to be raised by 48.3 dB (K = 260).

b. Looking at the phase diagram, where ϕ_M = 48.15^o (i.e. ϕ = -131.85^o), the phase margin frequency = 6.462 rad/s. At this frequency, the magnitude curve is – 63.04 dB. Thus the magnitude curve has to be raised by 63.04 dB (K = 1419).

c. Looking at the phase diagram, where ϕ_M = 48.15^o (i.e. ϕ = -131.85^o), the phase margin frequency = 6.939 rad/s. At this frequency, the magnitude curve is – 64.42 dB. Thus the magnitude curve has to be raised by 64.42 dB (K = 1663).

4.

a. Bode plots for K = 1:



Using Eqs. (4.39) and (10.73) a percent overshoot = 15 is equivalent to a ζ = 0.517 and ϕ_M = 53.17^0. The phase-margin frequency = 2.2 rad/s where the phase is $53.17^0 - 180^0 = -126.83^0$. The magnitude = -8.966 dB, or 0.0.3562. Hence K = 1/ 0.3562 = 2.807.

b.

Program:

```
G=zpk([-20 -25],[0 -5 -8 -14],1)
K=2.807
T=feedback(K*G,1);
```

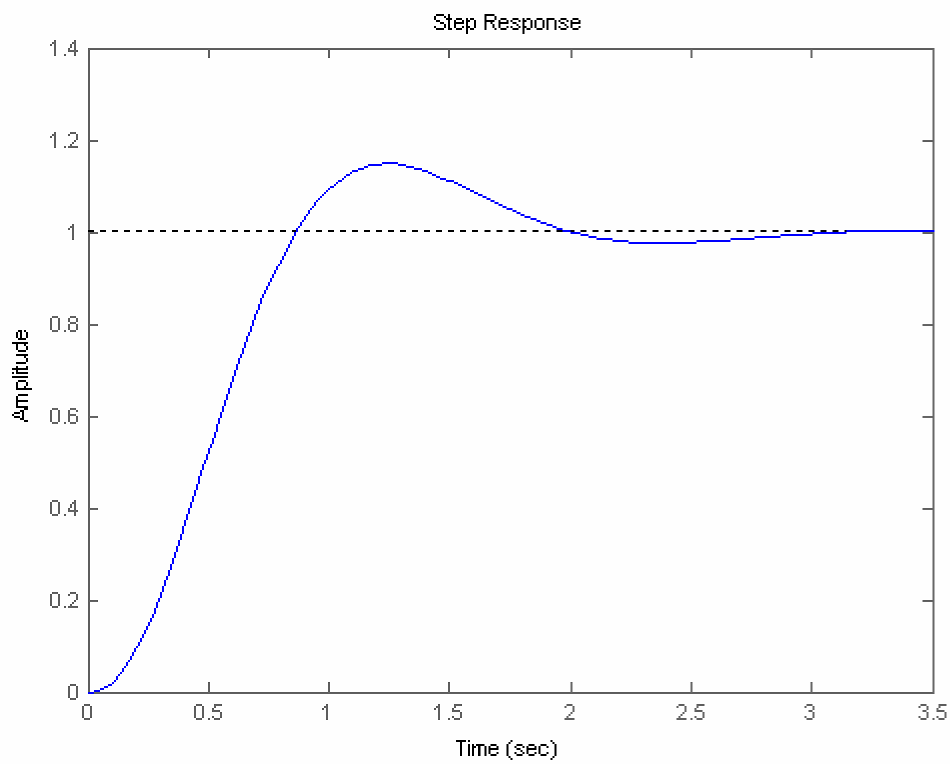
step(T)

Computer response:

```
Zero/pole/gain:
   (s+20) (s+25)
--------------------
s (s+5) (s+8) (s+14)

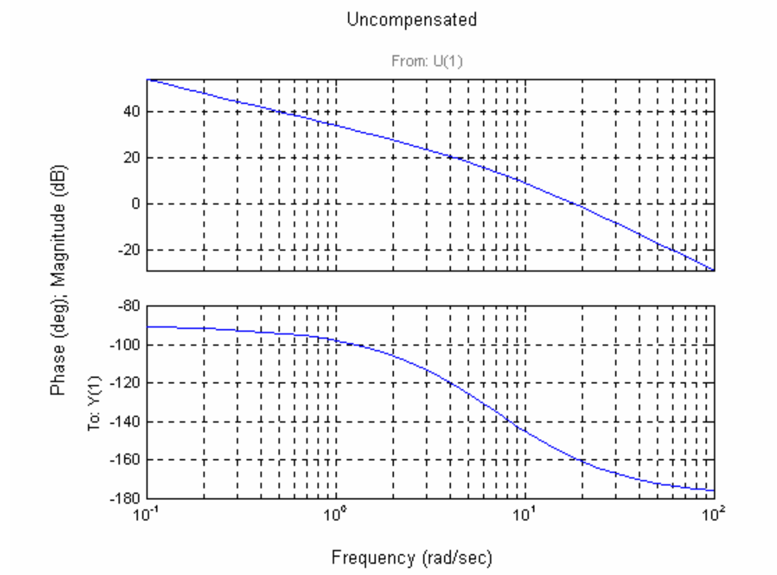
K =

    2.8070
```

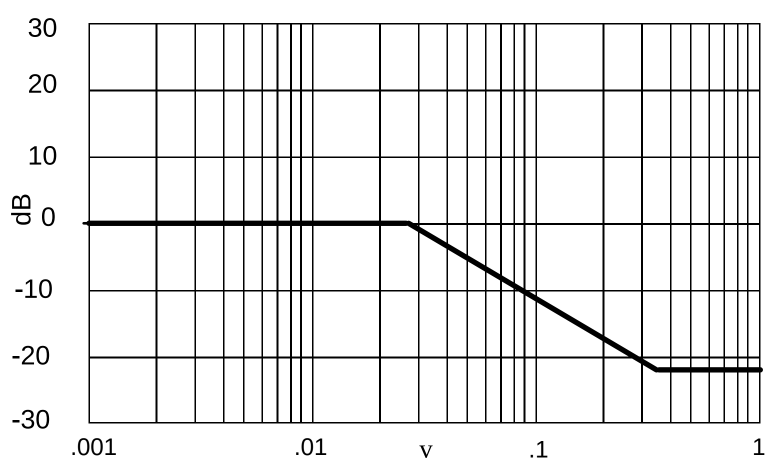


5.

For $K_V = 50$, $K = 350$. Plot the Bode plots for this gain.



Also, since %OS = 15%, $\zeta = 0.517$. Using Eq. (10.73), $\phi_M = 53.17^o$. Increasing ϕ_M by 10^o we will design for a phase margin of 63.17^o. The phase margin frequency is where the phase angle is $63.17 - 180^o = -116.83^o$, or $\omega_{\phi M} = 3.54$ rad/s. At this frequency, the magnitude is 22 dB. Start the magnitude of the compensator at - 22 dB and draw it to 1 decade below $\omega_{\phi M}$.



Then begin +20 dB/dec until zero dB is reached. Read the break frequencies as 0.028 rad/s and 0.354 rad/s from the Bode plot and form a lag transfer function that has unity dc gain:

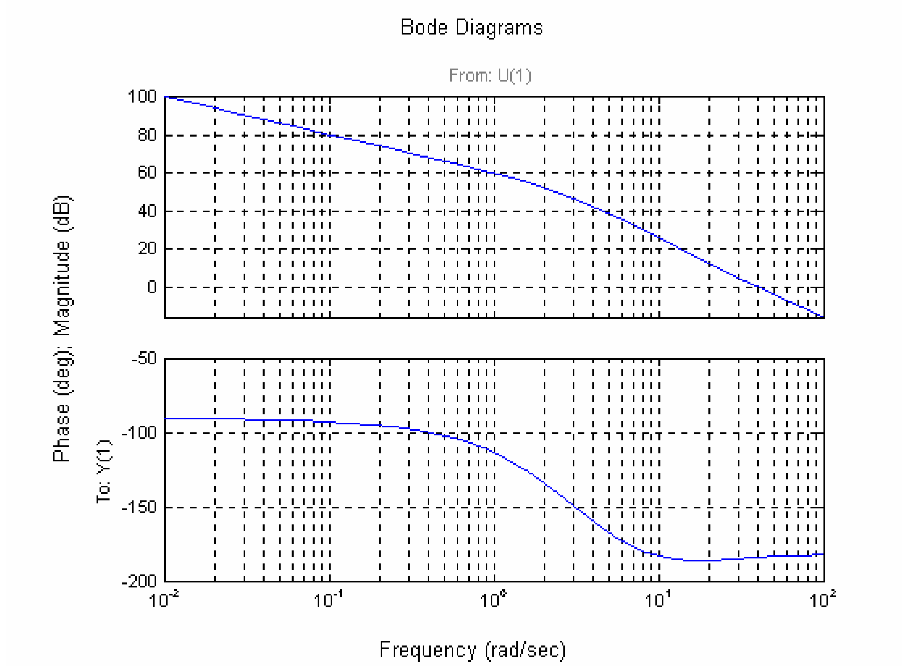
$$G_c(s) = 0.0791 \frac{s + 0.354}{s + 0.028}$$

The compensated forward path is

$$G(s) = \frac{350 * 0.0791(s + 0.354)}{s(s + 7)(s + 0.028)} = \frac{27.69(s + 0.354)}{s(s + 7)(s + 0.028)}$$

6.

a. For K_V = 1000, K = 1473. Plotting the Bode for this value of K:



Using Eqs. (4.39) and (10.73) a percent overshoot = 15 is equivalent to a ζ = 0.517 and ϕ_M = 53.17.

Using an extra 10^o, the phase margin is 63.17^o. The phase-margin frequency = 1.21 rad/s. At this frequency, the magnitude = 57.55 dB = 754.2. Hence the lag compensator K = 1/754.2 = 0.001326. Following Steps 3 and 4 of the lag compensator design procedure in Section 11.3,

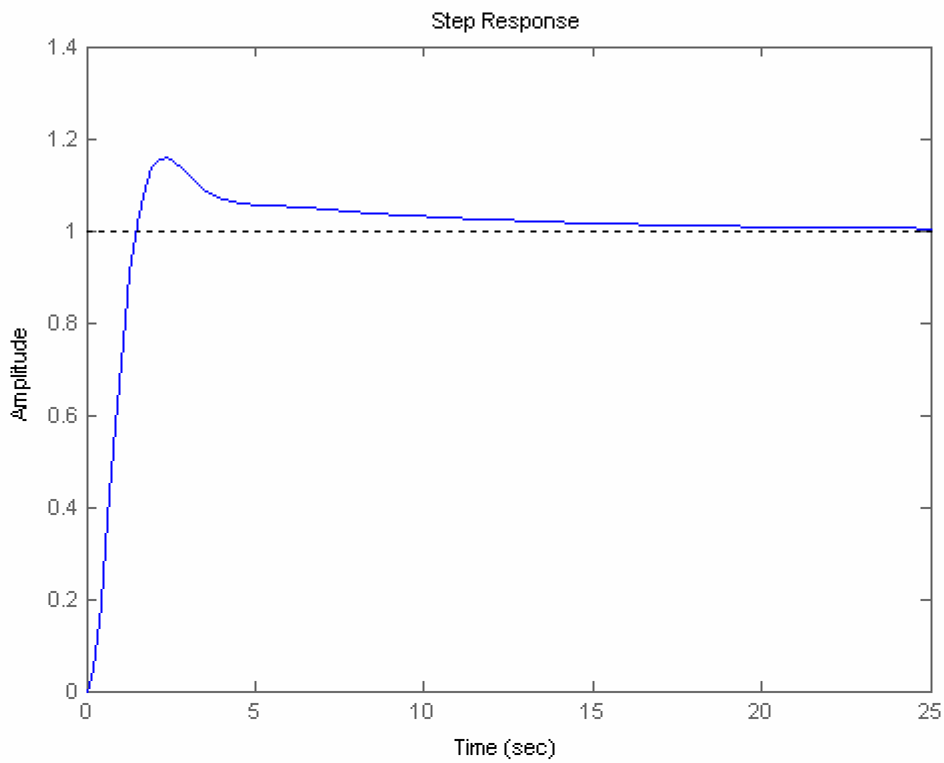
$$G_{lag}(s) = 0.001326 \frac{s + 0.121}{s + 0.0001604}$$

b.

Program:

```
%Input system
numg=1473*poly([-10 -11]);
deng=poly([0 -3 -6 -9]);
G=tf(numg,deng);
numc=0.001326*[1 0.121];
denc=[1 0.0001604];
Gc=tf(numc,denc);
Ge=G*Gc;
T=feedback(Ge,1);
step(T)
```

Computer response:



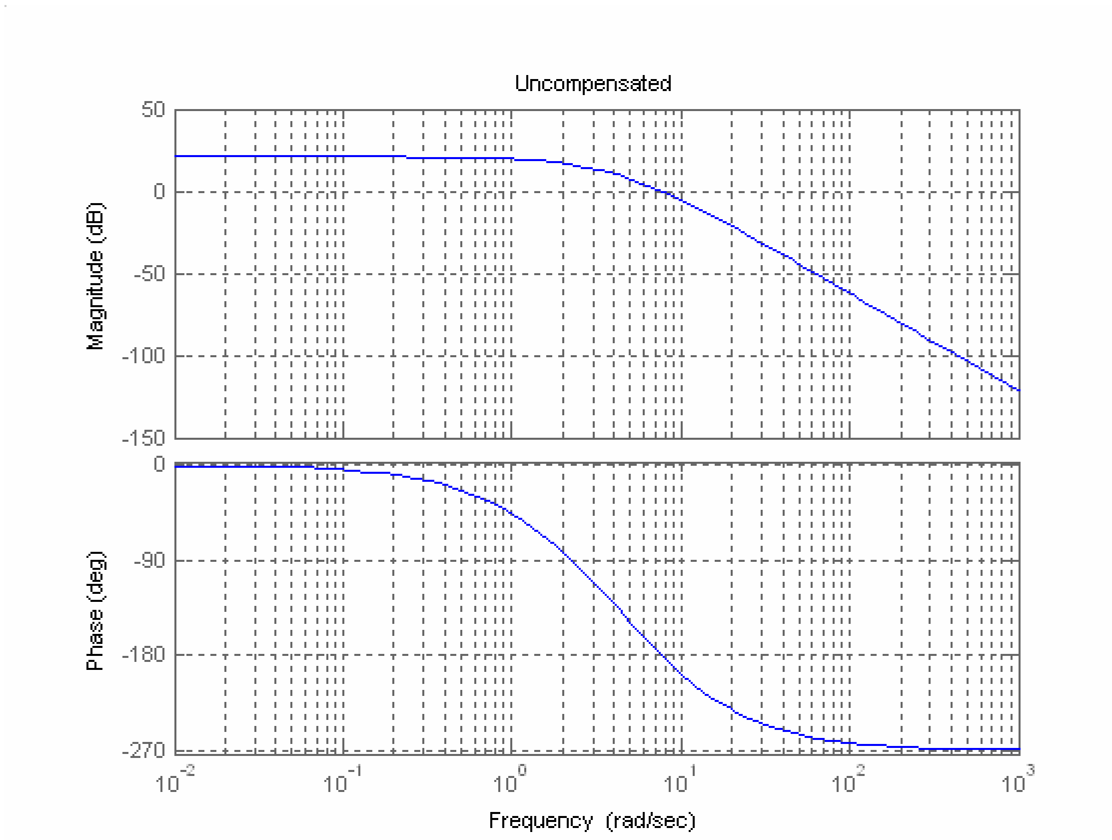
7.

Uncompensated system:

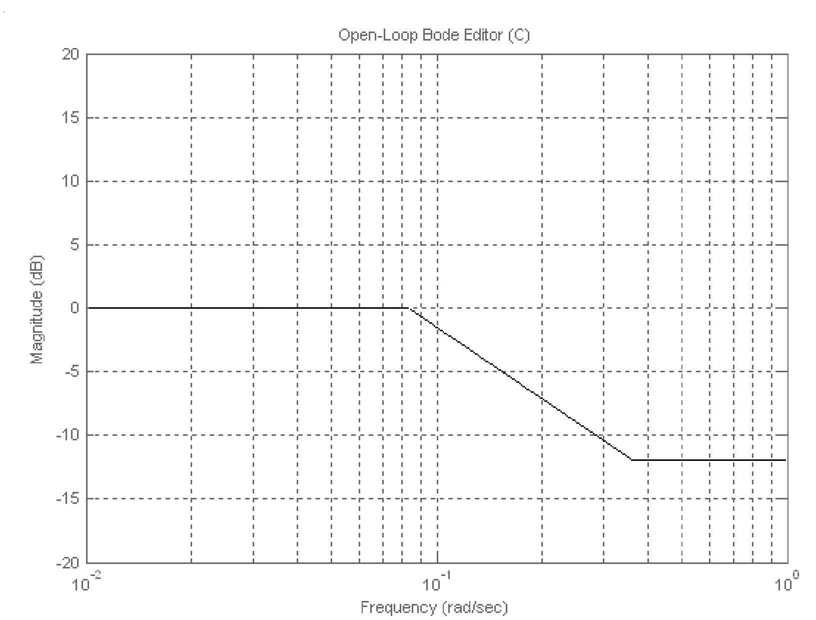
Searching along the 121.1° line (15% overshoot), find the dominant pole at -2.15 ± j3.56 with K = 97.7. Therefore, the uncompensated static error constant is $K_{vo} = \frac{97.7}{70} = 1.396$. On the frequency response curves, plotted for K = 97.7, unity gain occurs at ω = 1.64 rad/s with a phase angle of -71°. Therefore the uncompensated phase margin is 180° - 71° = 109°.

Compensated system:

The old steady-state error, $e_{step}(\infty) = \frac{1}{1+K_{po}} = \frac{1}{1+\frac{97.7}{70}} = 0.4174$. For a 5 times improvement in steady-state error, $e_{step}(\infty) = \frac{1}{1+K_{pn}} = 0.0835$, yielding , $K_{pn} = 10.98 = \frac{K}{70}$. Thus $K = 768.6$. Plotting the Bode plots at this gain,



Adding 5°, the desired phase margin for 15% overshoot is 58.17°, or a phase angle of -121.83°. This phase angle occurs at ω = 3.505 rad/s. At this frequency the magnitude plot is +12 dB. Start the magnitude of the compensator at - 12 dB and draw it to 1 decade below $\omega_{\Phi M}$.



Then, begin +20 dB/dec until zero dB is reached. Read the break frequencies as 0.08797 rad/s and 0.3505 rad/s from the Bode plot and form a lag transfer function that has unity dc gain,

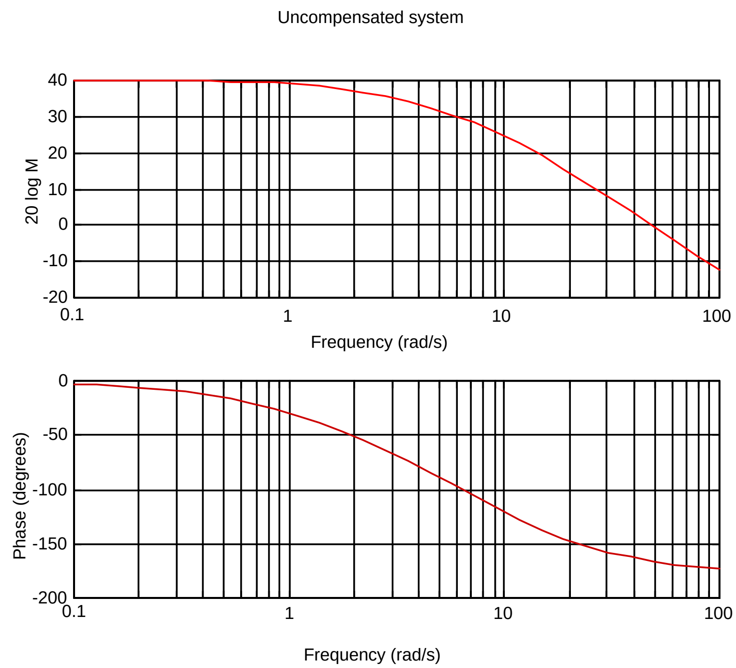
$$G_c(s) = 0.251\frac{s+0.3505}{s+0.08797}$$

The compensated forward path is

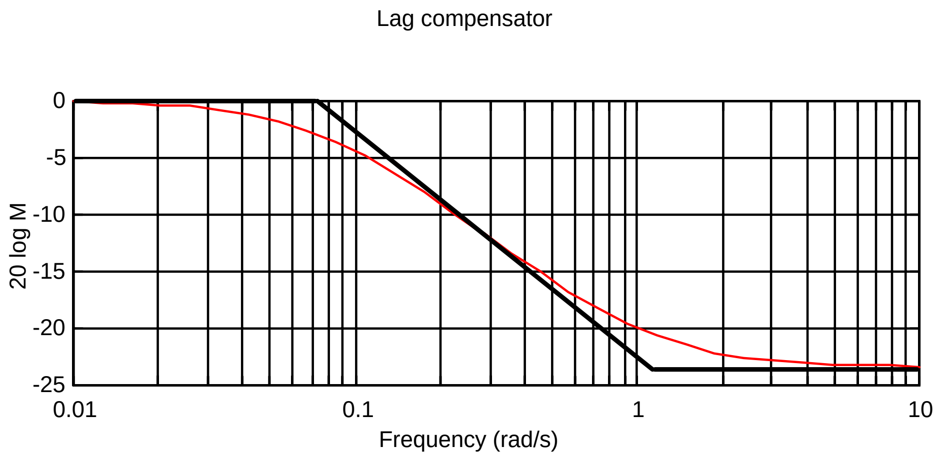
$$G(s) = 0.251*768.6\frac{(s+0.3505)}{(s+2)(s+5)(s+7)(s+0.08797)} = \frac{192.91(s+0.3505)}{(s+2)(s+5)(s+7)(s+0.08797)}.$$

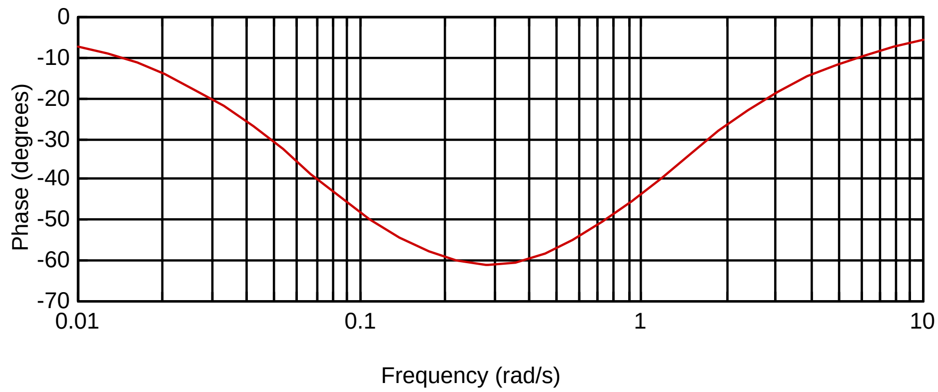
8.

For $K_p = 100 = \frac{K(4)}{(2)(6)(8)}$, K = 2400. Plotting the Bode plot for this gain,



We will design the system for a phase margin 10^0 larger than the specification. Thus $\phi_m = 55^0$. The phase margin frequency is where the phase angle is $-180^0 + 55^0 = -125^0$. From the Bode plot this frequency is $\omega_{\phi_m} = 11$ rad/s. At this frequency the magnitude is 23.37 dB. Start the magnitude of the lag compensator at –23.37 dB and draw it to 1 decade below $\omega_{\phi_m} = 11$, or 1.1 rad/s. Then begin a +20 dB/dec climb until 0 dB is reached. Read the break frequencies as 0.0746 rad/s and 1.1 rad/s from the Bode plot as shown below.





Ensuring unity dc gain, the transfer function of the lag is $G_{lag}(s) = 0.06782\dfrac{(s+1.1)}{(s+0.0746)}$. The compensated forward-path transfer function is thus the product of the plant and the compensator, or

$$G_e(s) = \frac{162.8(s+4)(s+1.1)}{(s+2)(s+6)(s+8)(s+0.0746)}$$

9.

From Example 11.1, K = 58251 yields 9.48% overshoot or a phase margin of 59.19°. Also,

$$G(s) = \frac{58251}{s(s+36)(s+100)}$$

Allowing for a 10° contribution from the PI controller, we want a phase margin of 69.19°, or a phase angle of -180° + 69.19° = -110.81°. This phase angle occurs at ω = 9.8 rad/s where the magnitude is 4 dB. Thus, the PI controller should contribute - 4 dB at ω = 9.8 rad/s. Selecting a break frequency a decade below the phase margin frequency,

$$G_c(s) = \frac{s+0.98}{s}$$

This function has a high-frequency gain of zero dB. Since we want a high-frequency gain of -4 dB (a gain of 0.631),

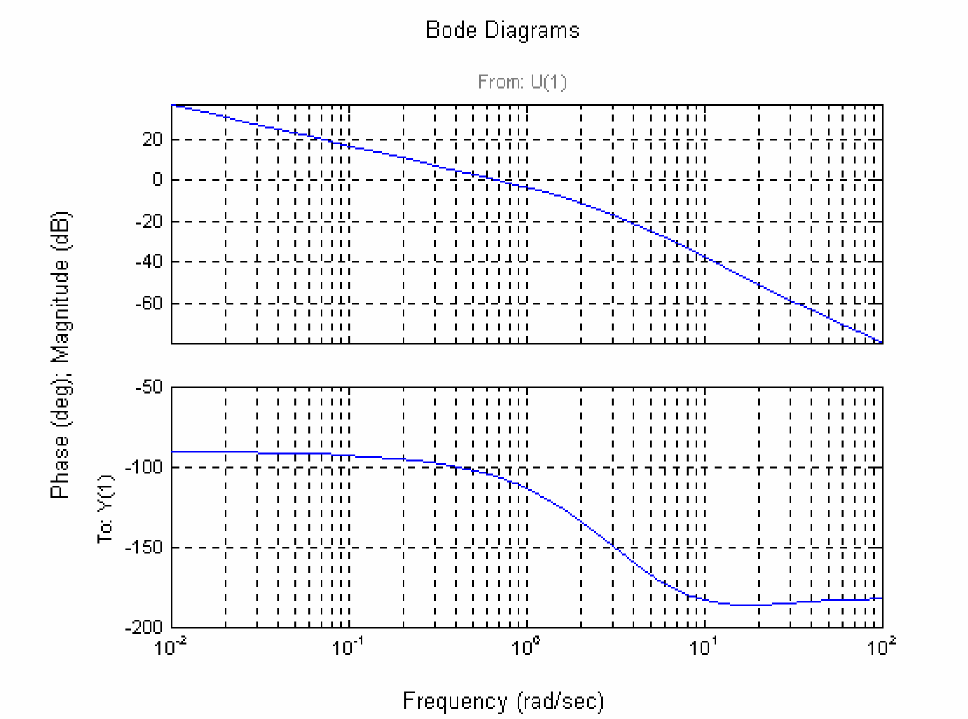
$G_c(s) = 0.631 \frac{s+0.98}{s}$

The compensated forward path is

$G(s) = \frac{58251*0.631(s+0.98)}{s(s+36)(s+100)} = \frac{36756.38(s+0.98)}{s(s+36)(s+100)}$

10.

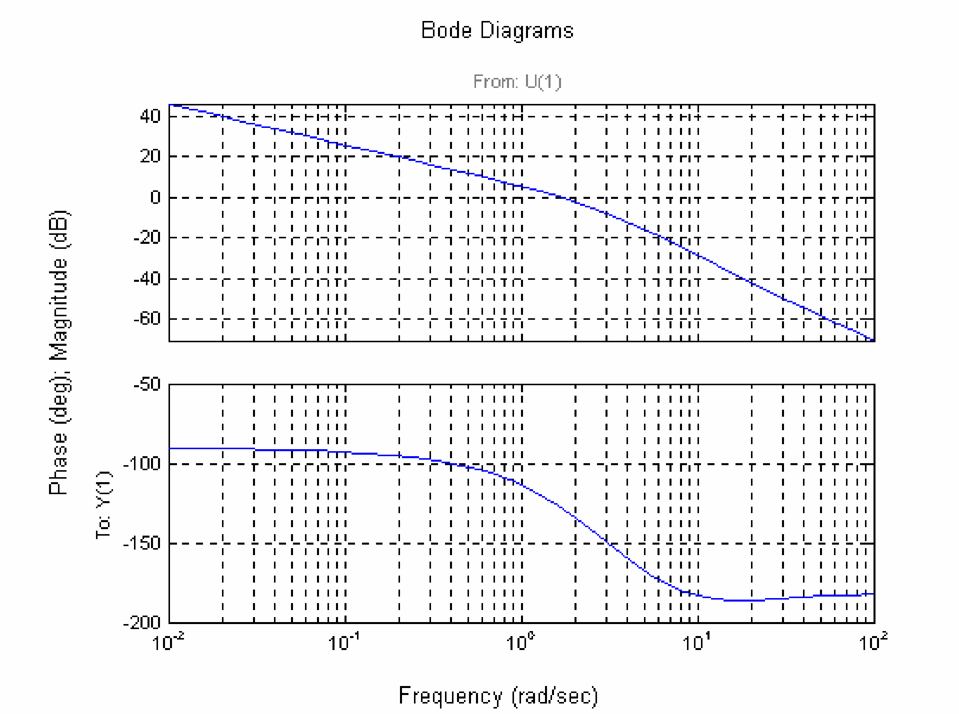
Bode plots for K = 1:



Using Eqs. (4.39) and (10.73) a percent overshoot = 15 is equivalent to a $\zeta = 0.517$ and $\phi_M = 53.17^o$. The phase-margin frequency = 1.66 rad/s. The magnitude = -9.174 dB = 0.3478. Hence K = 1/ 0.3478 = 2.876.

b.

Bode plots for K = 2.876.



Adding 10^{o} to the phase margin yields 63.17. Thus, the required phase is $-180^0 + 63.17^0 = -116.83^0$, which occurs at a frequency of 1.21 rad/s. The magnitude = 3.366 dB = 1.473. Hence, the lag compensator K = 1/ 1.473 = =0.6787. Selecting the break a decade below the phase-margin frequency,

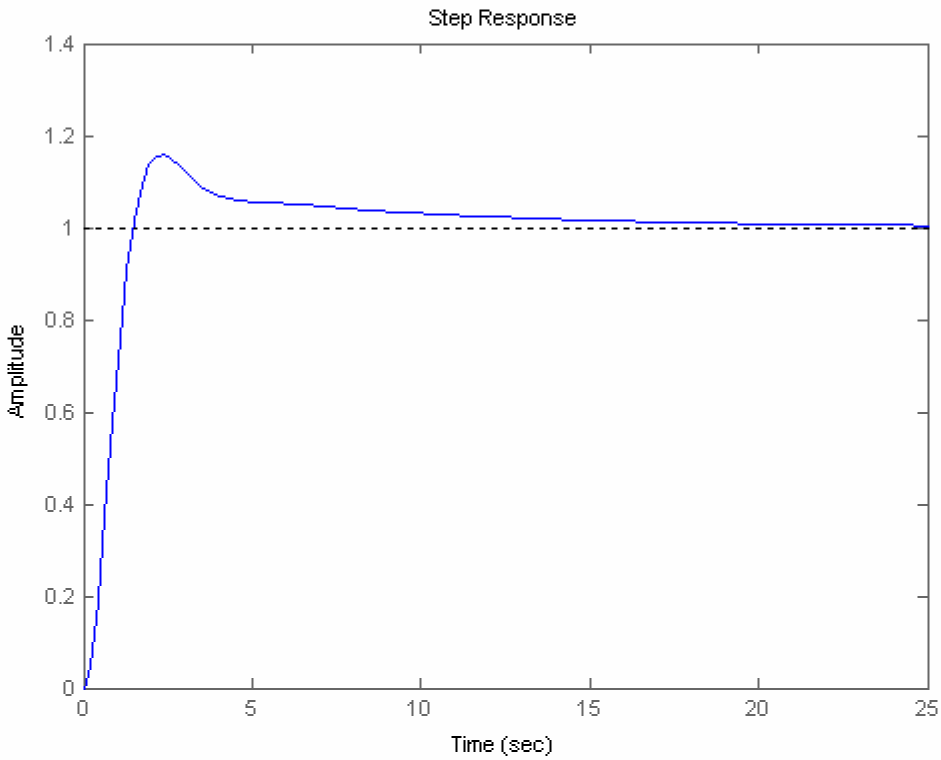
$$G_c(s) = 0.6787\,\frac{s+0.121}{s}$$

c.

Program:

```
%Input system
numg=2.876*poly([-10 -11]);
deng=poly([0 -3 -6 -9]);
G=tf(numg,deng);
numc=0.6787*[1 0.121];
denc=[1 0];
Gc=tf(numc,denc);
Ge=G*Gc;
T=feedback(Ge,1);
step(T)
```

Computer response:



11.

Program:

```
%PI Compensator Design via Frequency Response
%Input system
G=zpk([],[-5 -10],1);
G=tf(G);
%Percent Overshoot to Damping Ratio to Phase Margin
Po=input('Type %OS  ');
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
Pm=atan(2*z/(sqrt(-2*z^2+sqrt(1+4*z^4))))*(180/pi)+10;
fprintf('\nPercent Overshoot = %g',Po)
fprintf(', Damping Ratio = %g',z)
fprintf(', Phase Margin = %g',Pm)
%Get Bode data
bode(G)
title('Uncompensated')
pause
%Find frequency at desired phase margin and the gain at this frequency
w=logspace(-1,2,10000);
%w=.1:0.1:100;
[M,P,w]=bode(G,w);
Ph=-180+Pm
for i=1:1:length(P);
if P(i)-Ph<=0
Mag=M(i)
wf=w(i);
fprintf(', Frequency  = %g',wf)
fprintf(', Phase  = %g',P(i))
fprintf(', Magnitude = %g',Mag)
fprintf(', Magnitude (dB) = %g',20*log10(Mag))
break
end
end
```

```
%Design PI compensator
%Break frequency is a decade below phase margin frequency
wh=wf/10;
%Magnitude is reciprocal of magnitude of G at the phase margin frequency
%so net magnitude is 0 dB at the phase margin frequency
Kc=1/Mag
'PI Compensator'
Gpi=tf(Kc*[1 wh],[1 0])
bode(Gpi)
title(['PI compensator'])
pause
'G(s)Gpi(s)'
Ge=series(G,Gpi);
Ge=zpk(Ge)
bode(Ge)
title('PI Compensated')
[Gm,Pm,Wcp,Wcg]=margin(Ge);
'Gain margin(dB); Phase margin(deg.); 0 dB freq. (r/s);'
'180 deg. freq. (r/s)'
margins=[20*log10(Gm),Pm,Wcg,Wcp]
pause
T=feedback(Ge,1);
step(T)
title('PI Compensated')
```

Computer response:

```
Type %OS  25

Percent Overshoot = 25, Damping Ratio = 0.403713, Phase Margin = 53.463
Ph =

 -126.5370


Mag =

    0.0037

, Frequency  = 14.5518, Phase  = -126.54, Magnitude = 0.00368082, Magnitude
(dB) = -48.6811
Kc =

  271.6786


ans =

PI Compensator


Transfer function:
271.7 s + 395.3
---------------
       s


ans =

G(s)Gpi(s)


Zero/pole/gain:
271.6786 (s+1.455)
------------------
  s (s+10) (s+5)
```

```
ans =

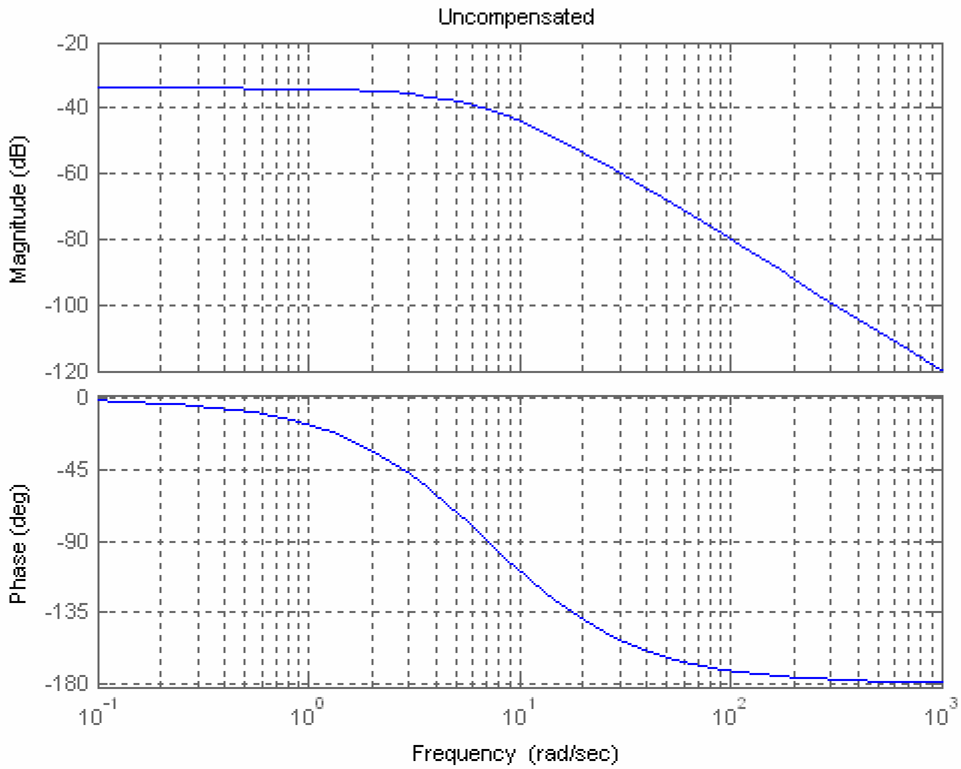
Gain margin(dB); Phase margin(deg.); 0 dB freq. (r/s);

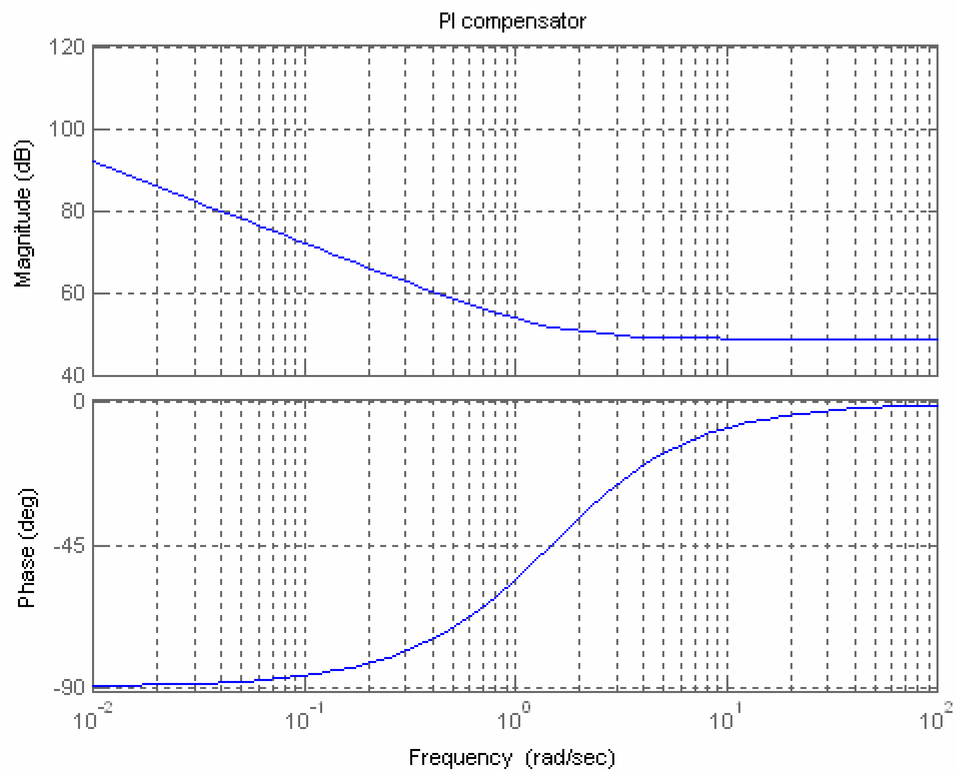
ans =

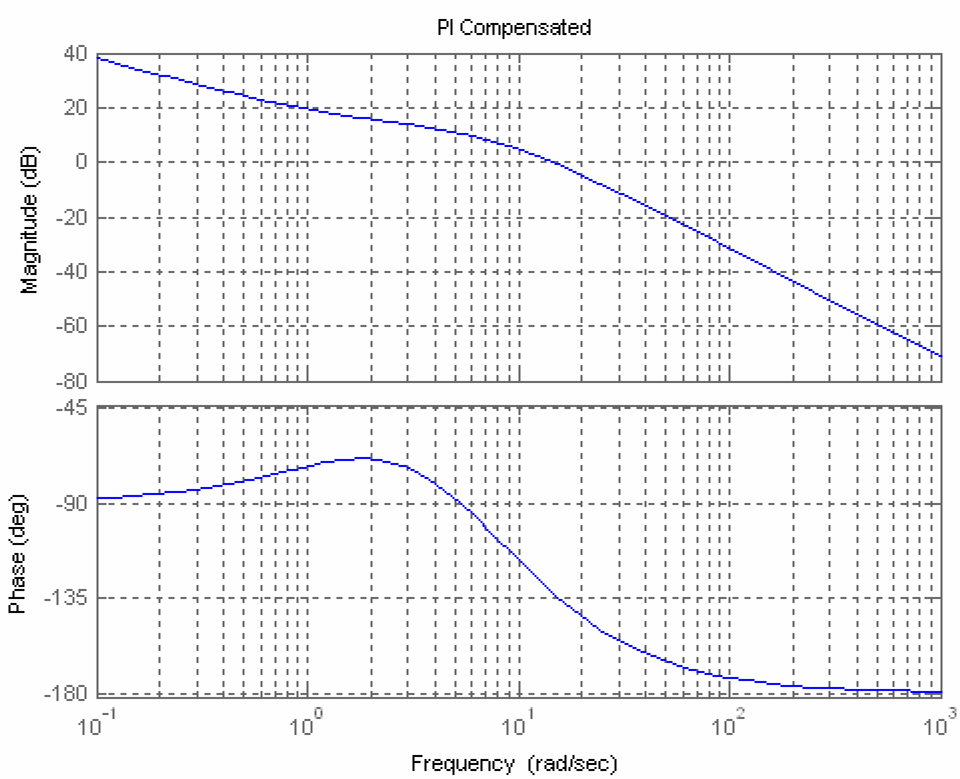
180 deg. freq. (r/s)

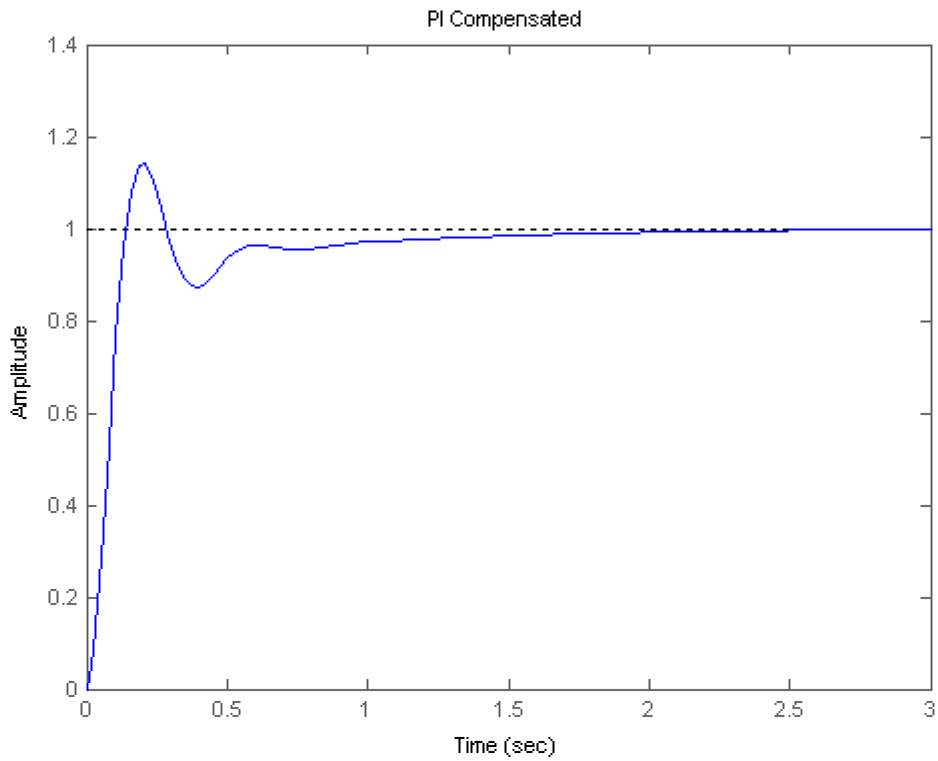
margins =

       Inf   47.6277   14.5975       Inf
```



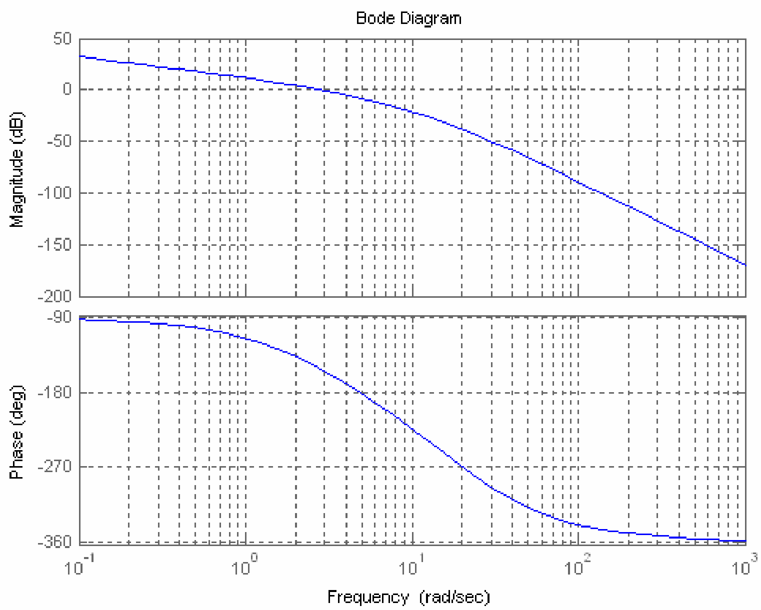

PI compensator
Magnitude (dB)
120
100
80
60
40
Phase (deg)
0
-45
-90
Frequency (rad/sec)


PI Compensated
Magnitude (dB)
40
20
0
-20
-40
-60
-80
Phase (deg)
-45
-90
-135
-180
Frequency (rad/sec)



12.

For $K_v = 4$, $\frac{K}{900} = 4$, or $K = 3600$. Plot the Bode diagrams.



The magnitude curve crosses zero dB at ω = 2.83 rad/s. with a phase angle of 152.1°, which yields an

uncompensated phase margin of 27.9°. Thus, we need an additional 12.1° plus an additional amount to compensate for the fact that the phase margin frequency will increase. Assume a lead network with a phase contribution of 22.1°. Using Eqs. (11.11), and (11.12),

The value of beta is:	0.453
The |G(jwmax)| for the compensator is:	1.485
or in db:	3.44

The magnitude curve has a gain of -3.44 dB at ω = 3.625 rad/s. Therefore, choose this frequency as the new phase margin frequency. Using Eqs. (11.9) and (11.6), the compensator transfer function has the following specifications:

T	0.41
zero	-2.44
pole	-5.38
gain	2.21

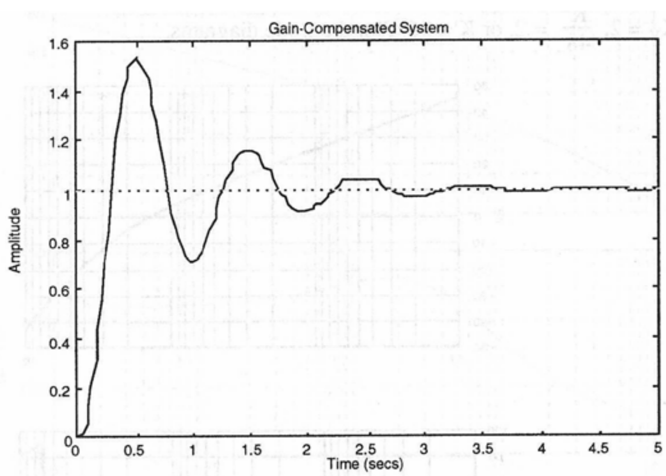
The compensated forward path is

$$G(s) = \frac{3600*2.21(s+2.44)}{s(s+3)(s+15)(s+20)(s+5.38)} = \frac{7956(s+2.44)}{s(s+3)(s+15)(s+20)(s+5.38)}$$

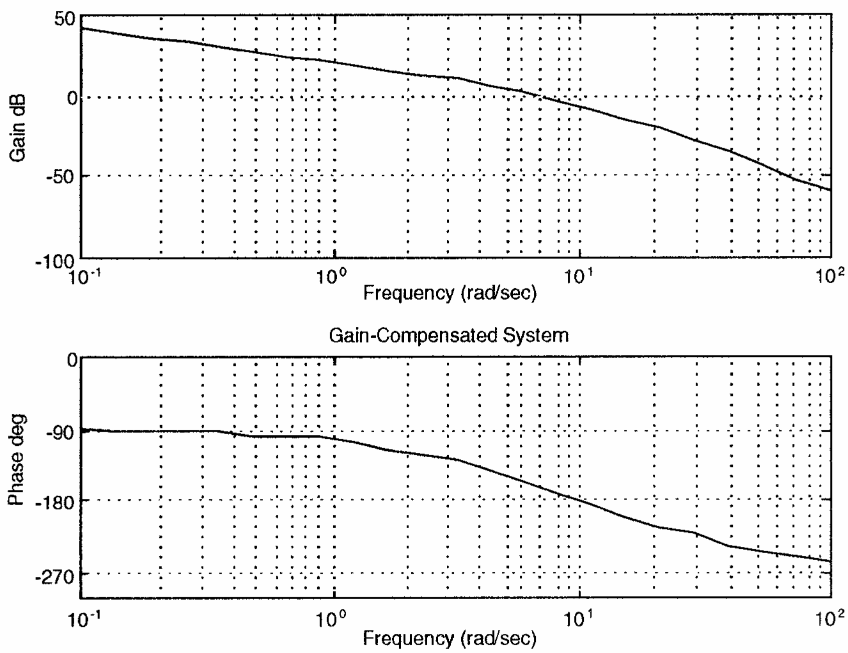
A Bode plot of G(s) shows a phase margin of 37.8°. Thus, a redesign is necessary to meet the exact requirement. This redesign can be done by adding a larger correction factor to the phase required from the lead compensator, See Control Solutions for the redesign.

13.

a. Gain-compensated time response:



Bode plots for K = 1000 (Kv = 10):

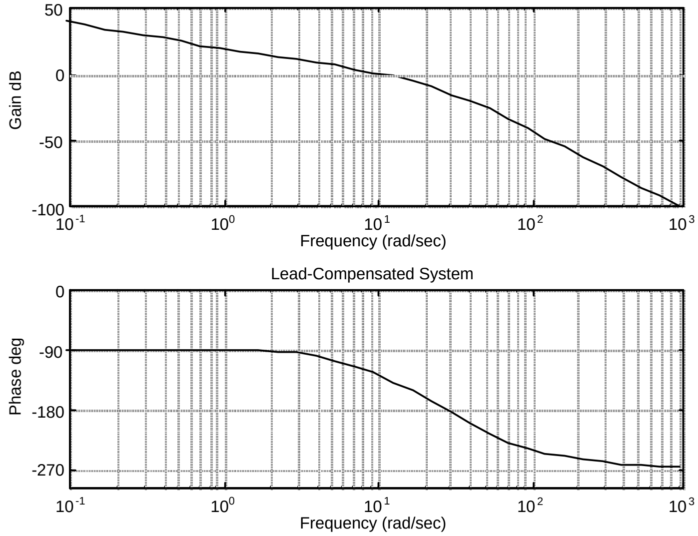


The specifications for the gain compensated system are: K = 1000, percent overshoot = 10, ζ = 0.591155, peak time = 0.5 s, current phase margin = 22.5362°.

To meet the requirements: required phase margin (Eq. 10.73) = 58.5931°, required phase margin with correction factor of 20° = 78.5931, required bandwidth (Eq. 10.56) = 9.03591, required phase contribution from compensator = 78.5931° - 22.5362° = 56.0569°, compensator beta (Eq. 11.11) = 0.0931398, new phase margin frequency (Eq. 11.12) = 11.51.

Now design the compensator: Compensator gain Kc = 1/β = 10.7366, compensator zero (Eq. 11.12) = -3.51272, compensator pole = z_c/β = -37.7144.

Lead-compensated Bode plots:



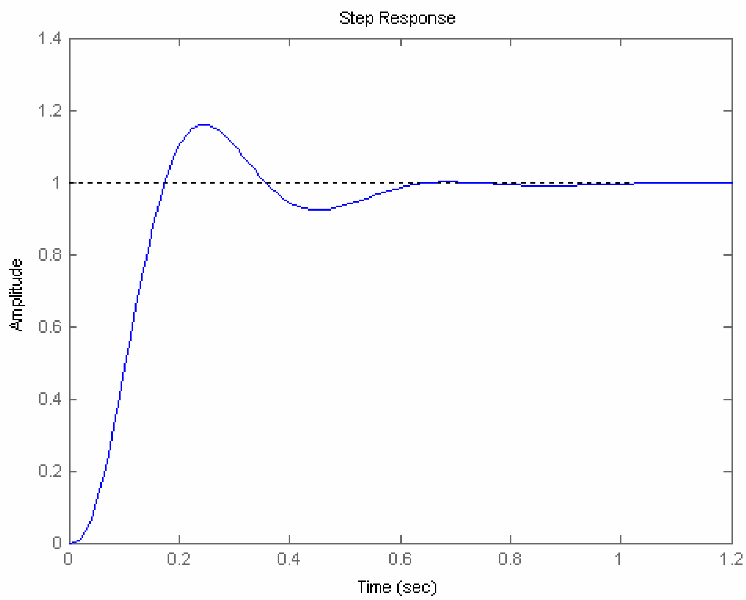
Lead-compensated phase margin = 50.2352.

b.

Program:

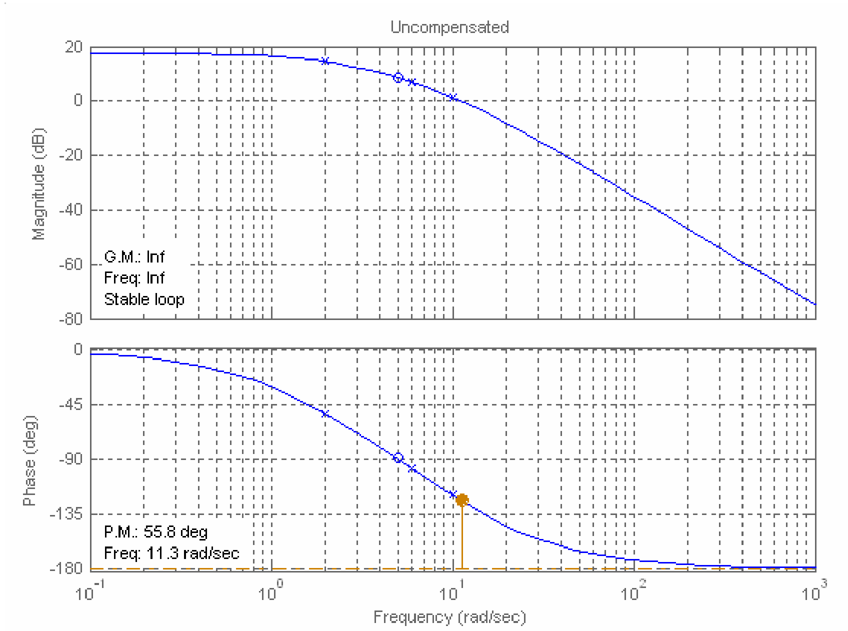
```
numg=1000;
deng=poly([0 -5 -20]);
G=tf(numg,deng);
numc=[1 3.51272];
denc=[1 37.7144];
Gc=tf(numc,denc);
Ge=G*10.7366*Gc;
T=feedback(Ge,1);
step(T)
```

Computer response:



14.

Uncompensated system: Searching the $\zeta = 0.456$ line (20% overshoot), find the dominant poles Q = -6.544 ± j12.771 with a gain of 178.21. Hence, $T_s = \frac{4}{\zeta\omega_n}$ = 0.611 second, $K_p = \frac{178.21*5}{2*6*10} = 7.425$. The Bode plot for the uncompensated system is:

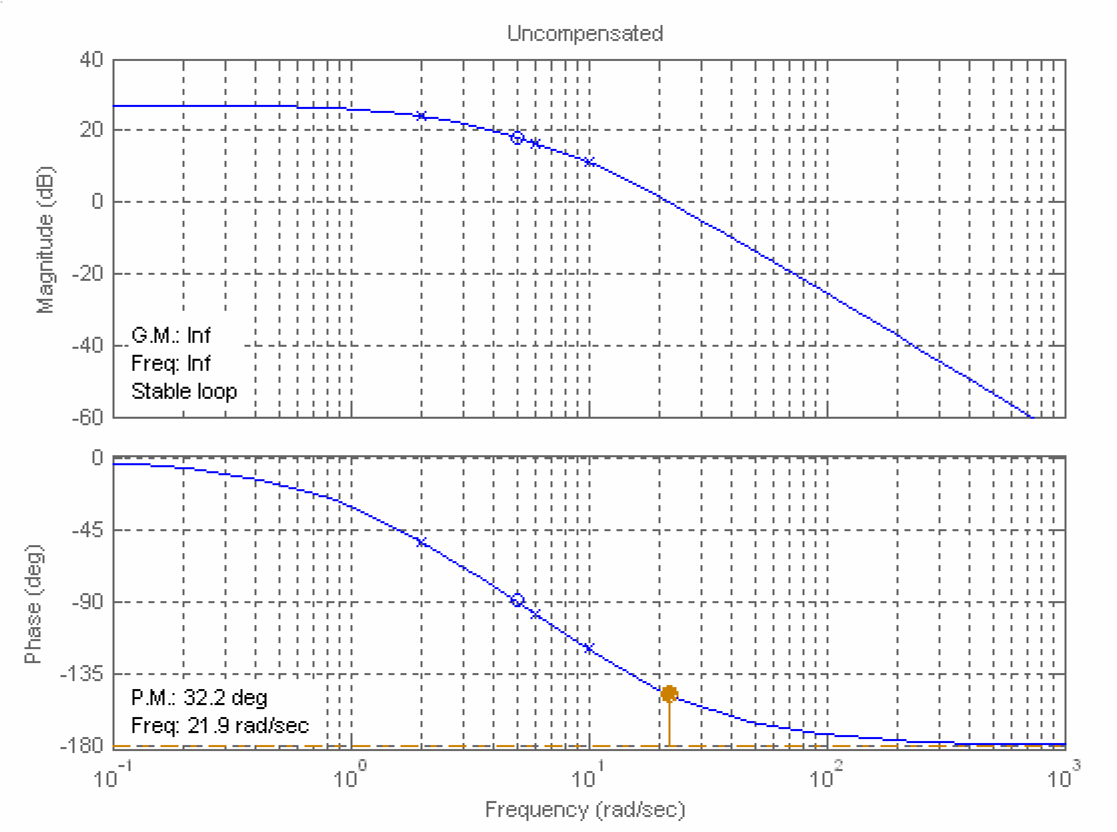


The uncompensated system has a phase margin of 55.8° and a phase margin frequency of 11.3 rad/s.

Compensated system: For a threefold improvement in K_p, K_{pn} = 22.28. Therefore, K = 3*178.21 = 534.63. For a twofold reduction in settling time, the new dominant poles are Q_n = 2Q = - 13.09 ± j25.54. The gain adjusted system is

$$G_c(s) = \frac{534.63(s+5)}{(s+2)(s+6)(s+10)}$$

Plotting the Bode diagrams for the gain compensated system,



At unity gain the phase is -147.8° at ω = 21.9 rad/s. Thus, the gain compensated phase margin is 180° – 147.8° = 32.2°. Using Eq. (10.73) with ζ = 0.456 (i.e. 20% overshoot), the required Φ_M = 48.15°. We add 15.95° plus a correction factor of 5° to the phase margin of the gain compensated system for a total additional phase of 20.95°. Using Eqs. (11.11), and (11.12),

The value of beta is:	0.473
The |G(jwmax)| for the compensator is:	1.45
or in db:	3.25

The magnitude curve has a gain of -3.25 dB at ω = 26.9 rad/s. Therefore, choose this frequency as the new phase margin frequency. Using Eqs. (11.9) and (11.6), the compensator transfer function has the following specifications:

T	0.054
zero	-18.51
pole	-39.1
gain	2.11

The compensated forward path is

$$G(s)=\frac{534.63*2.11(s+5)(s+18.51)}{(s+2)(s+6)(s+10)(s+39.1)}=\frac{1128.1(s+5)(s+18.51)}{(s+2)(s+6)(s+10)(s+39.1)}$$

A simulation of the system shows percent overshoot = 23.2%, settling time = 0.263, phase margin = 48.4^o, phase margin frequency = 26.7 r/s.

15.

If $G(s) = \dfrac{144000}{s(s+36)(s+100)}$, $K_V = 40$. Also, for a 0.1 second peak time, and $\zeta = 0.456$ (20% overshoot), Eq. (10.56) yields a required bandwidth of 46.59 rad/s. Using Eq. (10.73), the required phase margin is 48.15^o. Let us assume that we raise the phase margin frequency to 39 rad/s. The phase angle of the uncompensated system at this frequency is -158.6^o. To obtain the required phase margin, the compensator must contribute 26.75^o more at 39 rad/s. Assume the following form for the compensator: $G_C(s) = K'K_D(s+\frac{1}{K_D})$. The angle contributed by the compensator is

$\phi_C = \tan^{-1}\dfrac{\omega}{1/K_D} = 26.75^o$. Letting $\omega = 39$ rad/s, $K_D = 0.0129$. Hence, the compensator is

$G_C(s) = 0.0129\,(s+77.37)$. The compensated forward path is

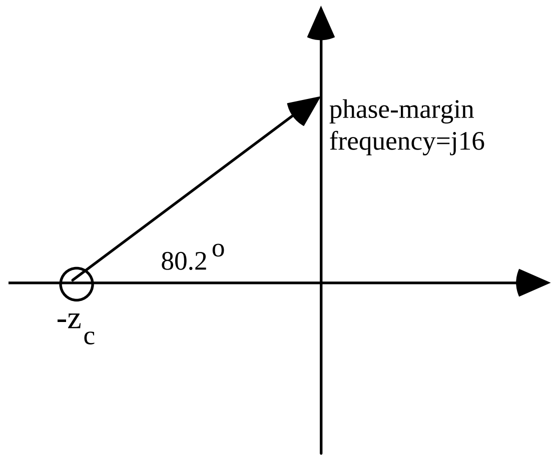
$$G(s)=\frac{144000*0.0129(s+77.37)}{s(s+36)(s+100)}=\frac{1857.6(s+77.37)}{s(s+36)(s+100)}$$

The closed-loop bandwidth is approximately 50 rad/s, which meets the requirements.
The lag compensated forward path is

$$G(s) = 7.759\,\frac{(s+0.058)}{s(s^2+2s+5)(s+3)(s+0.0015)}$$

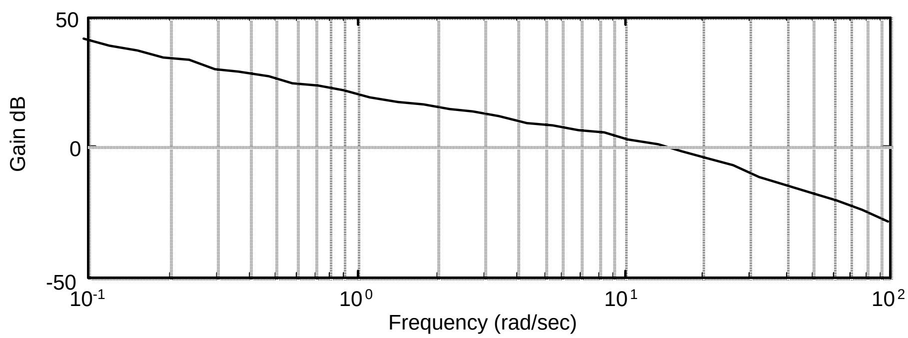
16.

a. Bode plots and specifications for gain compensated system are the same as Problem 13. Required phase margin and required bandwidth is the same as Problem 13. Select a phase margin frequency 7 rad/s higher than the bandwidth = 9 + 7 = 16 rad/s. The phase angle at the new phase-margin frequency is -201.6^0. The phase contribution required from the compensator is $-180^0 + 201.6^0 + 58.59^0 = 80.2^0$ at the phase-margin frequency. Using the geometry below:

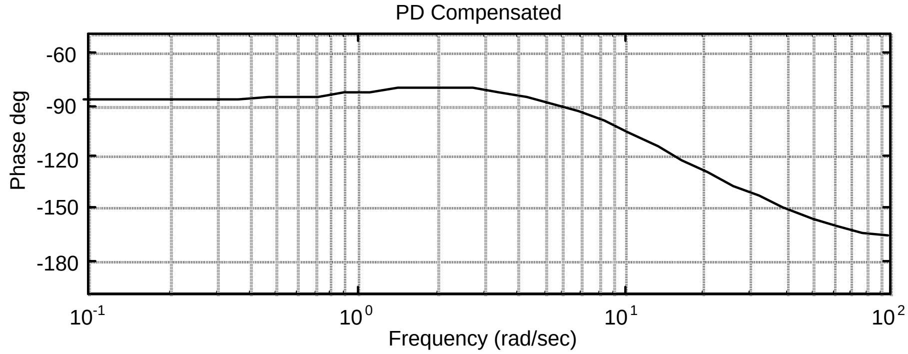


$\tan(80.2) = \frac{16}{z_c}$. Therefore, $z_c = 2.76$. Thus, $G_c(s) = \frac{1}{2.76}(s + 2.76)$.

The PD compensated Bode plots:



PD Compensated



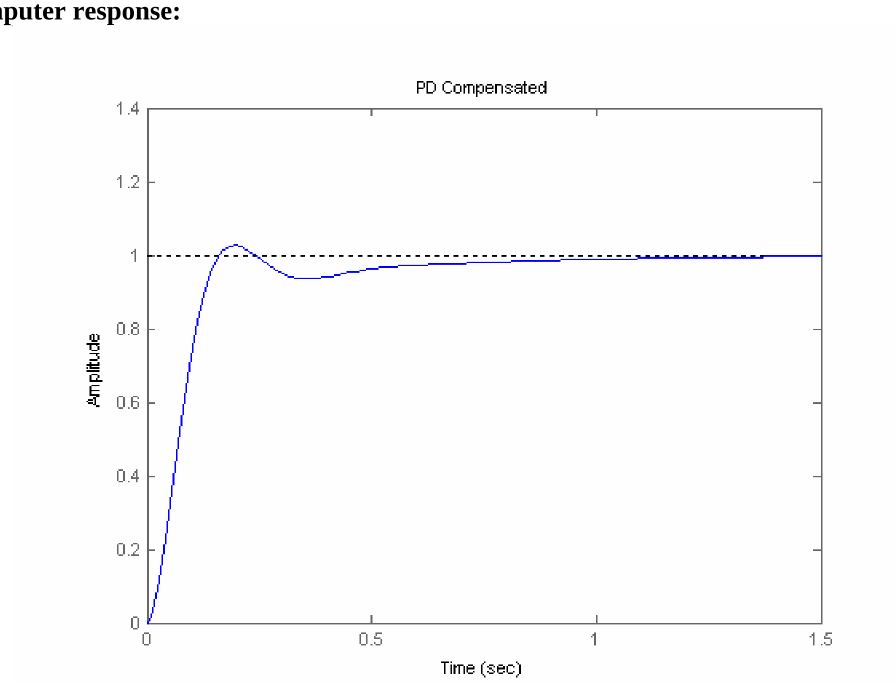
Compensated phase margin is 62.942^o.

b.

Program:

```
numg=1000;
deng=poly([0 -5 -20]);
G=tf(numg,deng);
numc=(1/2.76)*[1 2.76];
denc=1;
Gc=tf(numc,denc);
Ge=G*Gc;
T=feedback(Ge,1);
step(T)
title('PD Compensated')
```

Computer response:



17.

Program:

```
%Lead Compensator Design via Frequency Response
         %Input system
K=input('Type K to meet steady-state error  ');
numg=K*[1 1];
deng=poly([0 -2 -6]);
'Open-loop system'
'G(s)'
G=tf(numg,deng)
            %Generate uncompensated step response
T=feedback(G,1);
step(T)
title('Gain Compensated')

            %Input transient response specifications
Po=input('Type %OS  ');
%Ts=input('Type settling time   ');
Tp=input('Type peak time   ');
         %Determine required bandwidth
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
%wn=4/(z*Ts);
wn=pi/(Tp*sqrt(1-z^2));
wBW=wn*sqrt((1-2*z^2)+sqrt(4*z^4-4*z^2+2));

         %Make a Bode plot and get Bode data
%Get Bode data
bode(G)
title('Gain Compensated')

w=0.01:0.1:100;
[M,P]=bode(numg,deng,w);

         %Find current phase margin
[Gm,Pm,wcp,wcg]=margin(M,P,w);

         %Calculate required phase margin
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
Pmreq=atan(2*z/(sqrt(-2*z^2+sqrt(1+4*z^4))))*(180/pi);
```

```
         %Add a correction factor of 10 degrees
Pmreqc=Pmreq+10;

         %Calculate phase required from compensator
Pc=Pmreqc-Pm;

         %Design lead compensator
%Find compensator beta and peak compensator magnitude
beta=(1-sin(Pc*pi/180))/(1+sin(Pc*pi/180));
magpc=1/sqrt(beta);
%Find frequency at which uncompensated system has a magnitude of 1/magpc
%This frequency will be the new phase margin frequency
for i=1:1:length(M);
if M(i)-(1/magpc)<=0;
wmax=w(i);
break
end
end
%Calculate the lead compensator's break frequencies
zc=wmax*sqrt(beta);
pc=zc/beta;
Kc=1/beta;
numc=[1 zc];
denc=[1 pc];
'Gc(s)'
Gc=tf(numc,denc)
           %Display data
fprintf('\nK = %g',K)
fprintf('  Percent Overshoot = %g',Po)
fprintf(', Damping Ratio = %g',z)
%fprintf(', Settling Time = %g',Ts)
fprintf(', Peak Time = %g',Tp)
fprintf(', Current Phase Margin = %g',Pm)
fprintf(', Required Phase Margin = %g',Pmreq)
fprintf(', Required Phase Margin with Correction Factor = %g',Pmreqc)
fprintf(', Required Bandwidth = %g',wBW)
fprintf(', Required Phase Contribution from Compensator = %g',Pc)
fprintf(', Compensator Beta = %g',beta)
fprintf(', New phase margin frequency = %g',wmax)
fprintf('  Compensator gain, Kc = %g',Kc)
fprintf('  Compensator zero,= %g',-zc)
fprintf('  Compensator pole,= %g',-pc)
'G(s)Gc(s)'
Ge=G*Kc*Gc
pause
            %Generate compensated Bode plots
%Make a Bode plot and get Bode data
%Get Bode data
bode(Ge)
title('Lead Compensated')

w=0.01:0.1:1000;
[M,P]=bode(Ge,w);
%Find compensated phase margin
[Gm,Pm,wcp,wcg]=margin(M,P,w);
fprintf('\nCompensated Phase Margin,= %g',Pm)
pause
            %Generate step response
T=feedback(Ge,1);
step(T)
title('Lead Compensated')
```

Computer response:

```
Type K to meet steady-state error  360

ans =
```

```
Open-loop system

ans =

G(s)

Transfer function:
   360 s + 360
------------------
s^3 + 8 s^2 + 12 s

Type %OS  10
Type peak time   0.1

ans =

Gc(s)

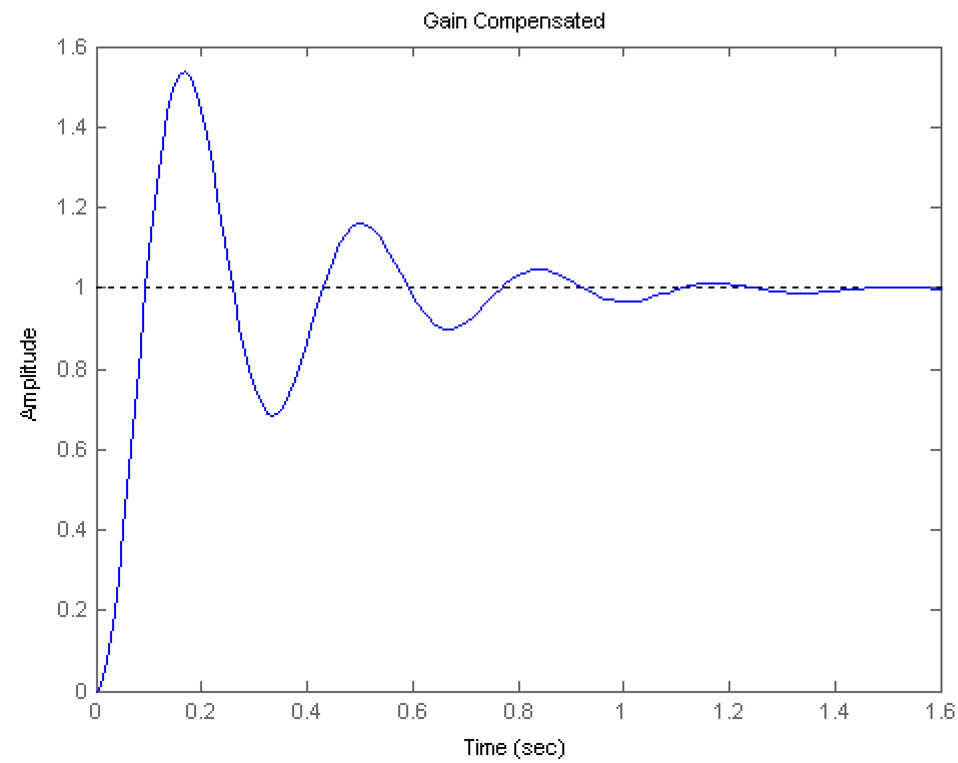
Transfer function:
s + 11.71
---------
s + 77.44

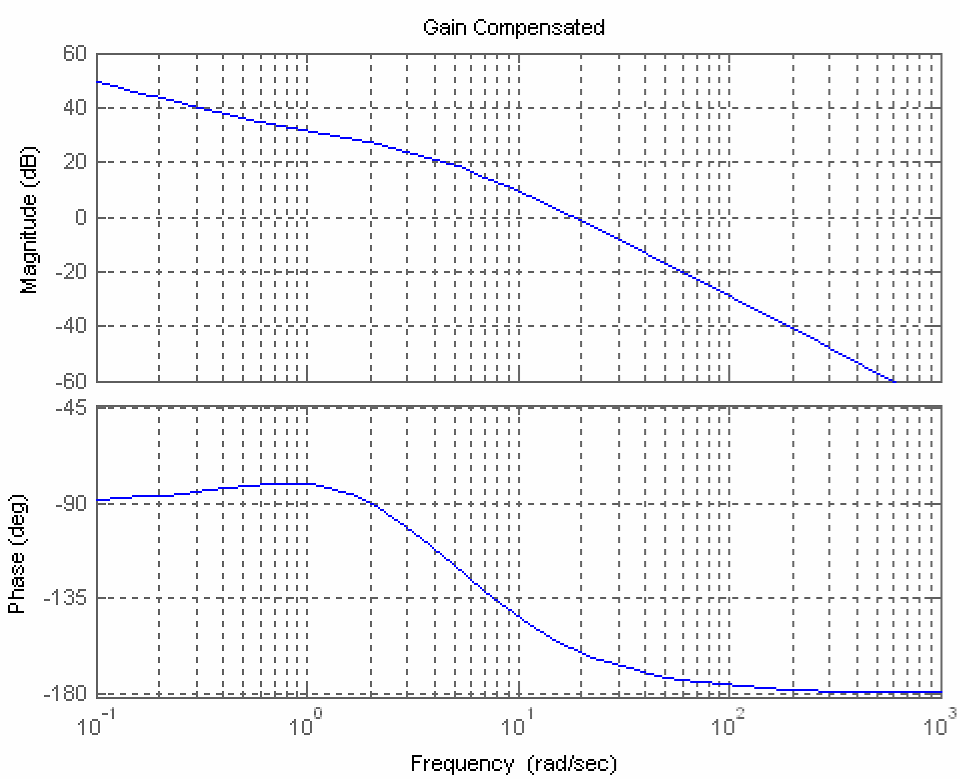
K = 360  Percent Overshoot = 10, Damping Ratio = 0.591155, Peak Time = 0.1,
Current Phase Margin = 21.0851, Required Phase Margin = 58.5931, Required
Phase Margin with Correction Factor = 68.5931, Required Bandwidth =
45.1795, Required Phase Contribution from Compensator = 47.508, Compensator
Beta = 0.151164, New phase margin frequency = 30.11  Compensator gain, Kc =
6.61532  Compensator zero,= -11.7067  Compensator pole,= -77.4437
ans =

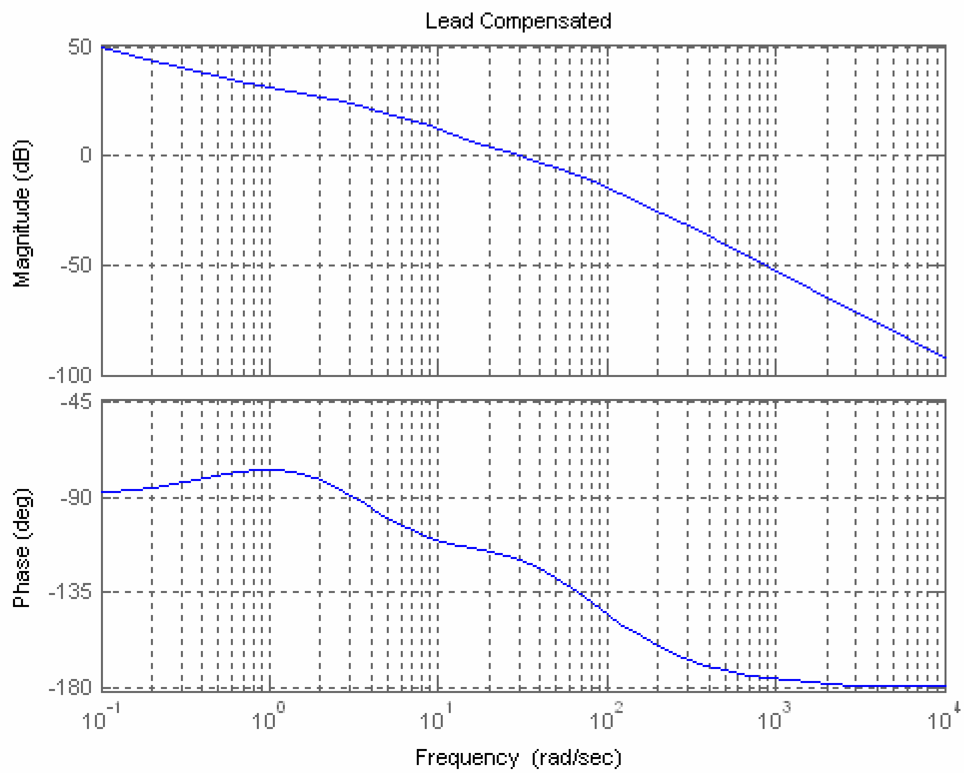
G(s)Gc(s)

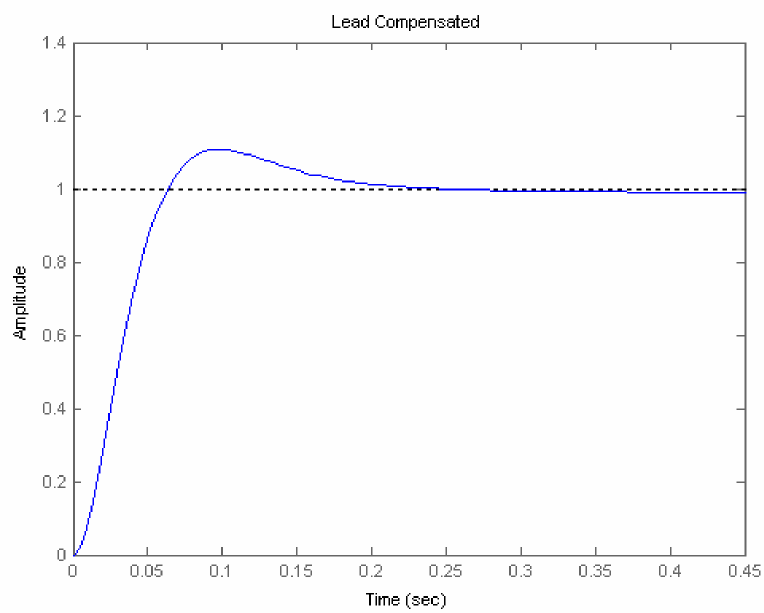
Transfer function:
 2382 s^2 + 3.026e004 s + 2.788e004
-------------------------------------
s^4 + 85.44 s^3 + 631.5 s^2 + 929.3 s

Compensated Phase Margin,= 60.676»
```


Gain Compensated
Amplitude
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
Time (sec)


Gain Compensated
Magnitude (dB)
60
40
20
0
-20
-40
-60
Phase (deg)
-45
-90
-135
-180
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)


Lead Compensated
Magnitude (dB)
50
0
-50
-100
Phase (deg)
-45
-90
-135
-180
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency (rad/sec)


Lead Compensated
Amplitude
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
Time (sec)

18.

Program:

```
%PD Compensator Design via Frequency Response
         %Input system
%Uncompensated system
K=input('Type K to meet steady-state error  ');
numg=K*[1 1];
deng=poly([0 -2 -6]);
G=tf(numg,deng);
T=feedback(G,1);
step(T)
title('Gain Compensated')
'Open-loop system'
'G(s)'
Gzpk=zpk(G)

         %Input transient response specifications
Po=input('Type %OS  ');
%Ts=input('Type settling time   ');
Tp=input('Type peak time   ');

         %Determine required bandwidth
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
%wn=4/(z*Ts);
wn=pi/(Tp*sqrt(1-z^2));
wBW=wn*sqrt((1-2*z^2)+sqrt(4*z^4-4*z^2+2));

         %Make a Bode plot and get Bode data
%Get Bode data
bode(G)
title('Gain Compensated')
w=0.01:0.1:100;
[M,P]=bode(G,w);

         %Find current phase margin
[Gm,Pm,wcp,wcg]=margin(M,P,w);

         %Calculate required phase margin
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
Pmreq=atan(2*z/(sqrt(-2*z^2+sqrt(1+4*z^4))))*(180/pi)+20;

         %Determine a phase margin frequency
wpm=wBW+7;
         %Find phase angle at new phase margin frequency and

         %calculate phase required from the compensator
for i=1:1:length(w);
if w(i)-wpm>=0;
wpm=w(i);
Pwpm=P(i);
break
end
end
         %Design PD compensator
Pc=Pmreq-(180+Pwpm);
zc=wpm/tan(Pc*pi/180);
Kc=1/zc;
numc=Kc*[1 zc];
denc=1;
'Gc(s)'
Gc=tf(numc,denc);
Gczpk=zpk(Gc)
           %Display data
fprintf('\nK = %g',K)
fprintf('  Percent Overshoot = %g',Po)
fprintf(', Damping Ratio = %g',z)
%fprintf(', Settling Time = %g',Ts)
```

```
fprintf(', Peak Time = %g',Tp)
fprintf(', Current Phase Margin = %g',Pm)
fprintf(', Required Phase Margin = %g',Pmreq)
fprintf(', Required Bandwidth = %g',wBW)
fprintf(', New phase margin frequency = %g',wpm)
fprintf(', Phase angle at new phase margin frequency = %g',Pwpm)
fprintf(', Required Phase Contribution from Compensator = %g',Pc)
fprintf('  Compensator gain, Kc = %g',Kc)
fprintf('  Compensator zero,= %g',-zc)

pause
         %Generate compensated Bode plots
%Make a Bode plot and get Bode data
%Get Bode data
'G(s)Gc(s)'
Ge=G*Gc;
Gezpk=zpk(Ge)
bode(Ge)
title('PD Compensated')
w=0.01:0.1:100;
[M,P]=bode(Ge,w);
%Find compensated phase margin
[Gm,Pm,wcp,wcg]=margin(M,P,w);
fprintf('\nCompensated Phase Margin,= %g',Pm)
pause
            %Generate step response
T=feedback(Ge,1);
step(T)
title('PD Compensated')
```

Computer response:

```
Type K to meet steady-state error  360

ans =

Open-loop system


ans =

G(s)


Zero/pole/gain:
  360 (s+1)
-------------
s (s+6) (s+2)

Type %OS  10
Type peak time   0.1

ans =

Gc(s)

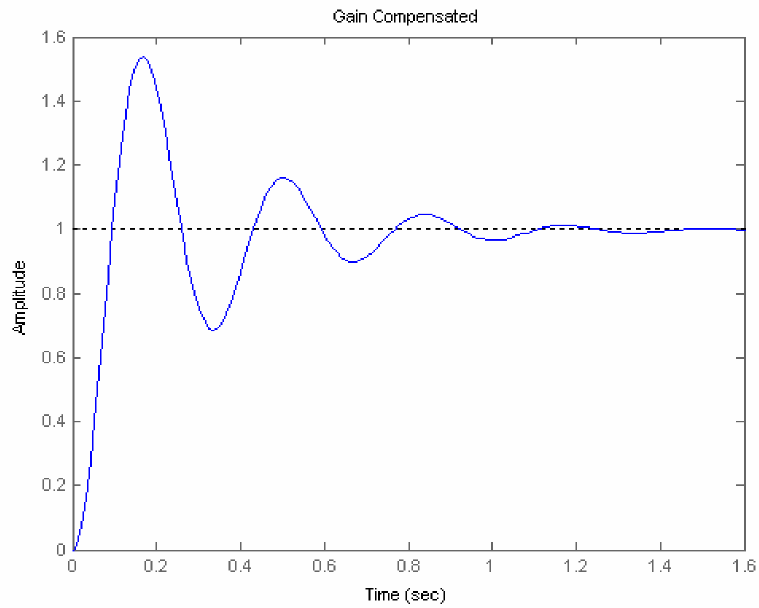

Zero/pole/gain:
0.05544 (s+18.04)

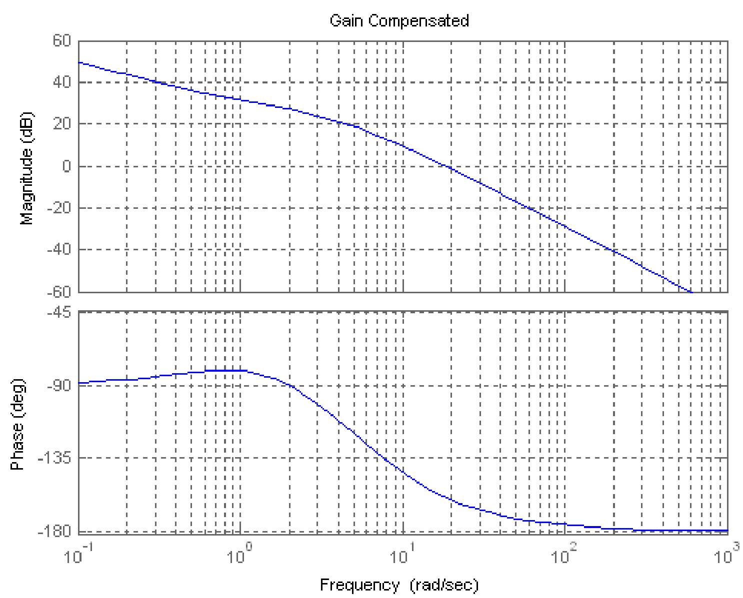

K = 360  Percent Overshoot = 10, Damping Ratio = 0.591155, Peak Time = 0.1,
Current Phase Margin = 21.0851, Required Phase Margin = 78.5931, Required
Bandwidth = 45.1795, New phase margin frequency = 52.21, Phase angle at new
phase margin frequency = -172.348, Required Phase Contribution from
Compensator = 70.9409  Compensator gain, Kc = 0.0554397  Compensator zero,=
-18.0376
ans =
```

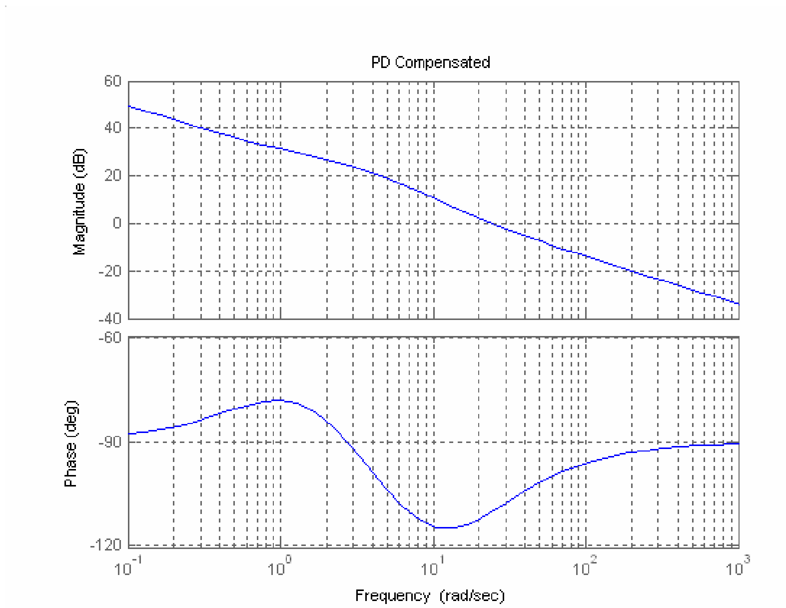
```
G(s)Gc(s)

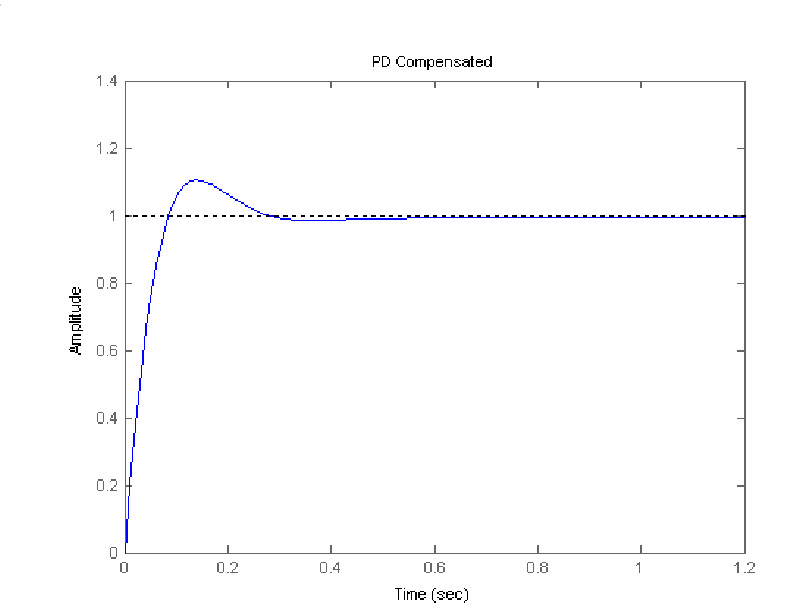
Zero/pole/gain:
19.9583 (s+18.04) (s+1)
-----------------------
     s (s+6) (s+2)

Compensated Phase Margin,= 69.546
```



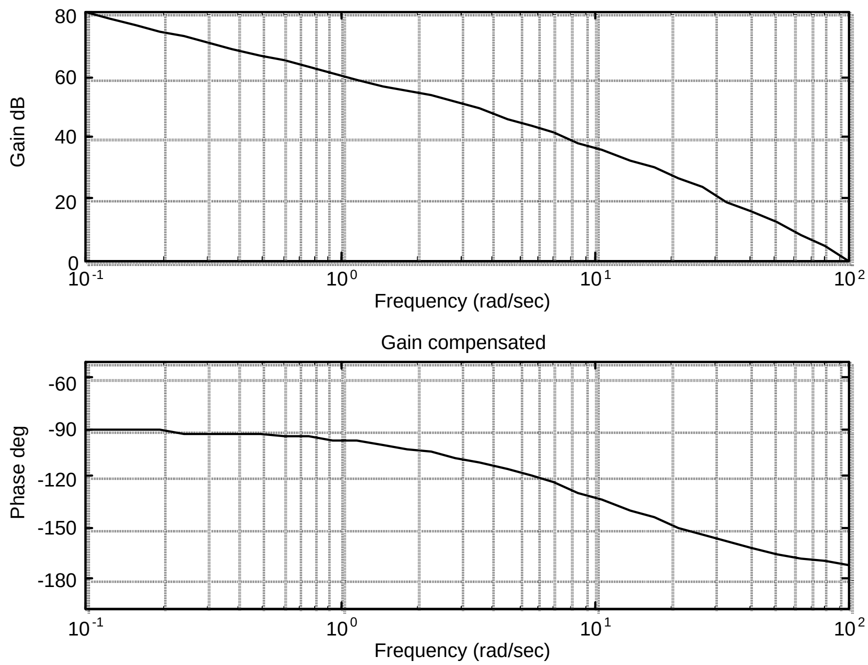






19.

K = 10714 for K_v = 1000. ζ = 0.517 for 15% overshoot using Eq. (4.39). Using Eq. (4.42), ω_n = 77.37. Using Eq. (10.54) the required bandwidth, ω_{BW} = 96.91. Using Eq. (10.73) with 5^o additional, Φ_m = 58.17^o. Choose the new phase-margin frequency ω_{Pm}= 0.8 w_{BW}= 77.53. Plotting the Bode plots for K = 10714,



At the new phase-margin frequency, the phase angle is -170.52. Thus, the contribution required from the lead is 58.17 - (180 -170.52) = 48.69^o. Using Eq. (11.11), β = 0.142.

Lag compensator design: $z_{clag} = \omega_{Pm}/10 = 77.53/10 = 7.753$. $p_{clag} = z_{clag}*\beta = 1.102$. $K_{clag} = p_{clag}/z_{clag} = 0.1421$. Thus, $G_{lag}(s) = 0.1421\,\frac{s+7.753}{s+1.102}$.

Lead compensator design: Using Eqs. (11.6), (11.9), and (11.12) $z_{lead} = 1/T = \omega_{Pm}*\sqrt{\beta} = 29.22$. $p_{lead} = z_{lead}/\beta = 205.74$. $K_{lead} = p_{lead}/z_{lead} = 7.04$. Thus, $G_{lead}(s) = 7.04\,\frac{s+29.22}{s+205.74}$.

20.

Program:

```
         %Lag-Lead Compensator Design via Frequency Response
         %Input system
K=input('Type K to meet steady-state error  ');
numg=K*[1 7];
deng=poly([0 -5 -15]);
G=tf(numg,deng);
'G(s)'
Gzpk=zpk(G)

         %Input transient response specifications
Po=input('Type %OS  ');
Ts=input('Type settling time   ');
%Tp=input('Type peak time   ');
T=feedback(G,1);
step(T)
title('Gain Compensated')
pause

        %Determine required bandwidth
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
wn=4/(z*Ts);
%wn=pi/(Tp*sqrt(1-z^2));
```

```
wBW=wn*sqrt((1-2*z^2)+sqrt(4*z^4-4*z^2+2));
%wBW=(4/(Ts*z))*sqrt((1-2*z^2)+sqrt(4*z^4-4*z^2+2));
%wBW=(pi/(Tp*sqrt(1-z^2)))*sqrt((1-2*z^2)+sqrt(4*z^4-4*z^2+2));

        %Determine required phase margin
Pmreq=atan(2*z/(sqrt(-2*z^2+sqrt(1+4*z^4))))*(180/pi)+5;

        %Choose new phase margin frequency
wpm=0.8*wBW;

        %Determine additional phase lead required at the
        %new phase margin frequency from the lead compensator
[M,P]=bode(G,wpm);
Pmreqc=Pmreq-(180+P);
beta=(1-sin(Pmreqc*pi/180))/(1+sin(Pmreqc*pi/180));
        %Display data
fprintf('\nPercent Overshoot = %g',Po)
fprintf(', Settling Time = %g',Ts)
%fprintf(', Peak Time = %g',Tp)
fprintf(', Damping Ratio = %g',z)
fprintf(', Required Phase Margin = %g',Pmreq)
fprintf(', Required Bandwidth = %g',wBW)
fprintf(', New Phase Margin Frequency = %g',wpm)
fprintf(', Required Phase from Lead Compensator = %g',Pmreqc)
fprintf(', Beta = %g',beta)
bode(numg,deng)
title('Gain compensated')
pause
          %Design lag compensator
zclag=wpm/10;
pclag=zclag*beta;
Kclag=beta;
'Lag compensator'
'Gclag'
Gclag=tf(Kclag*[1 zclag],[1 pclag]);
Gclagzpk=zpk(Gclag)
          %Design lead compensator
zclead=wpm*sqrt(beta);
pclead=zclead/beta;
Kclead=1/beta;
'Lead compensator'
'Gclead'
Gclead=tf(Kclead*[1 zclead],[1 pclead]);
Gcleadzpk=zpk(Gclead)
          %Create compensated forward path
'Gclag(s)Gclead(s)G(s)'
Ge=G*Gclag*Gclead;
Gezpk=zpk(Ge)
          %Test lag-lead compensator
T=feedback(Ge,1);
bode(Ge)
title('Lag-lead Compensated')
[M,P,w]=bode(Ge);
[Gm,Pm,wcp,wcg]=margin(M,P,w);
'Compensated System Results'
fprintf('\nResulting Phase Margin = %g',Pm)
fprintf(', Resulting Phase Margin Frequency = %g',wcg)
pause
step(T)
title('Lag-lead Compensated')
```

Computer response:

```
Type K to meet steady-state error  10714.29

ans =

G(s)
```

```
Zero/pole/gain:
10714.29 (s+7)
--------------
s (s+15) (s+5)

Type %OS  15
Type settling time   0.1

Percent Overshoot = 15, Settling Time = 0.1, Damping Ratio = 0.516931,
Required Phase Margin = 58.1718, Required Bandwidth = 96.9143, New Phase
Margin Frequency = 77.5314, Required Phase from Lead Compensator = 48.6912,
Beta = 0.142098
ans =

Lag compensator


ans =

Gclag


Zero/pole/gain:
0.1421 (s+7.753)
----------------
   (s+1.102)


ans =

Lead compensator


ans =

Gclead


Zero/pole/gain:
7.0374 (s+29.23)
----------------
   (s+205.7)


ans =

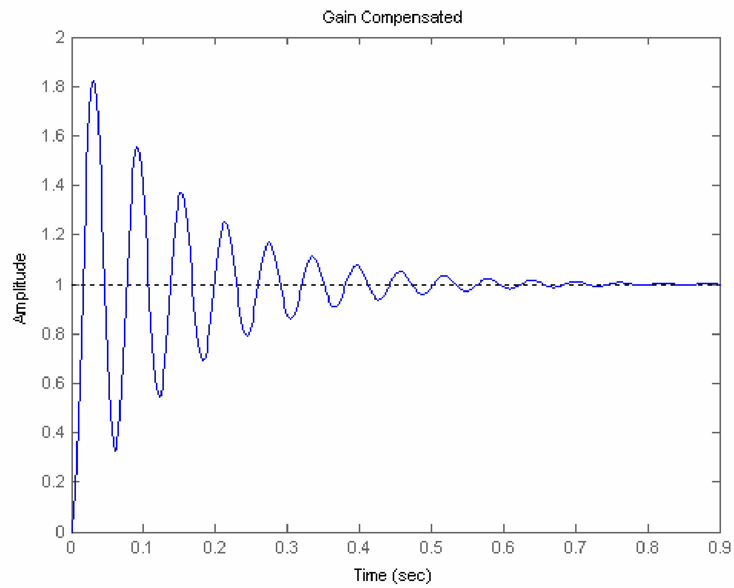
Gclag(s)Gclead(s)G(s)

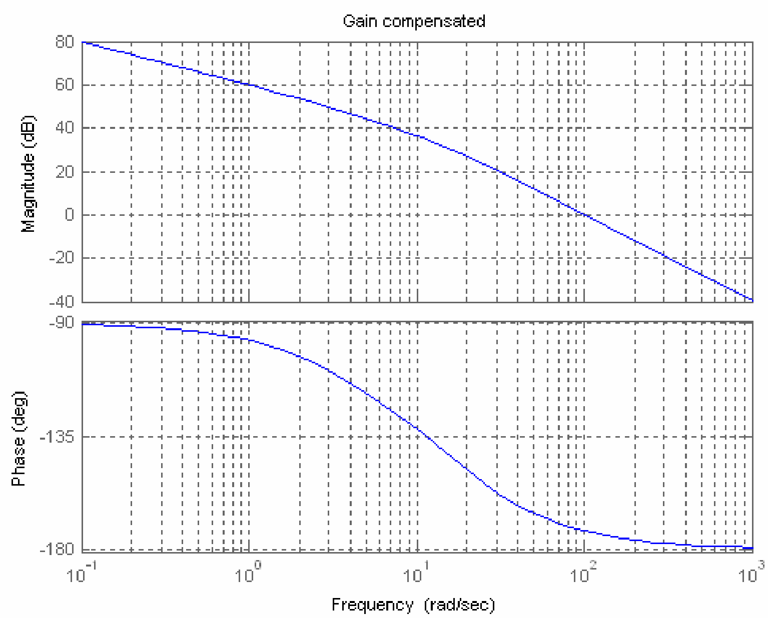

Zero/pole/gain:
10714.29 (s+29.23) (s+7.753) (s+7)
----------------------------------
s (s+205.7) (s+15) (s+5) (s+1.102)

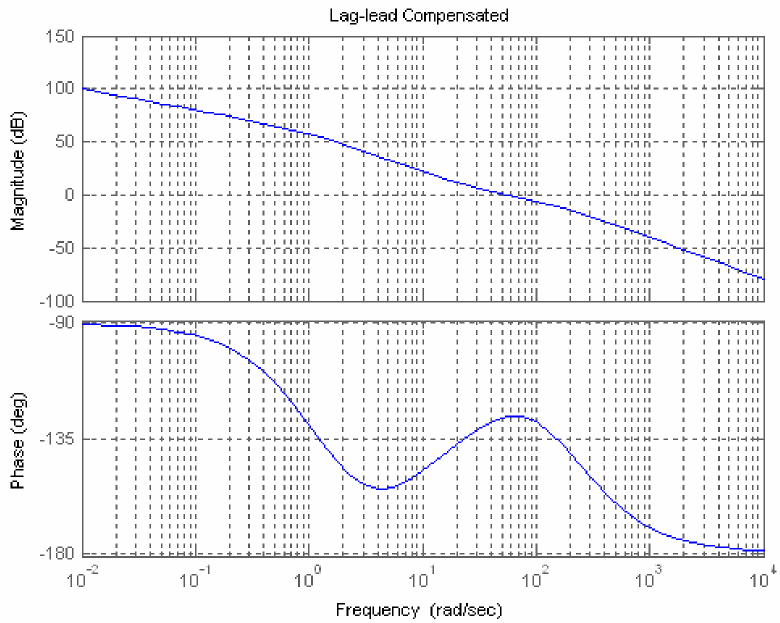

ans =

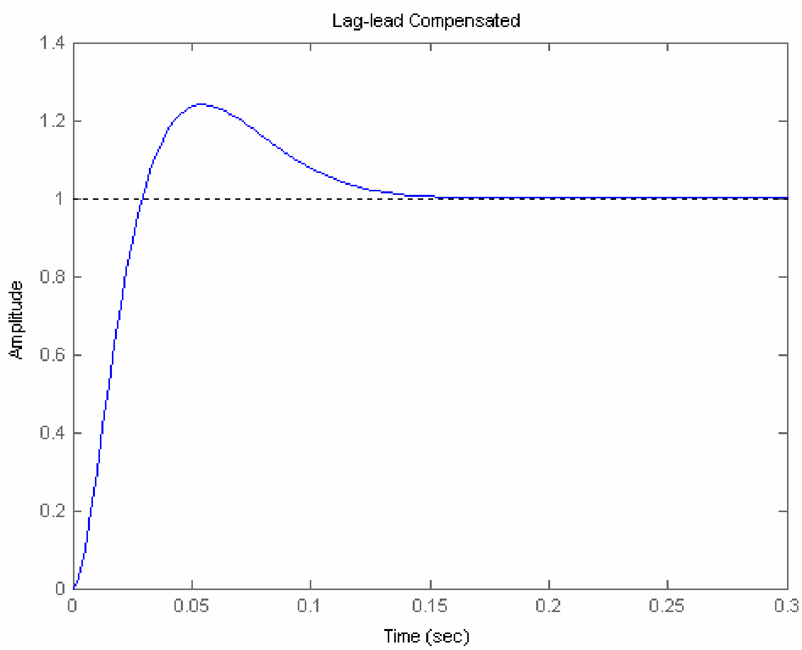
Compensated System Results


Resulting Phase Margin = 53.3994, Resulting Phase Margin Frequency =
55.5874
```


Gain Compensated
Amplitude
Time (sec)
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


Gain compensated
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
80
60
40
20
0
-20
-40
-90
-135
-180
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}

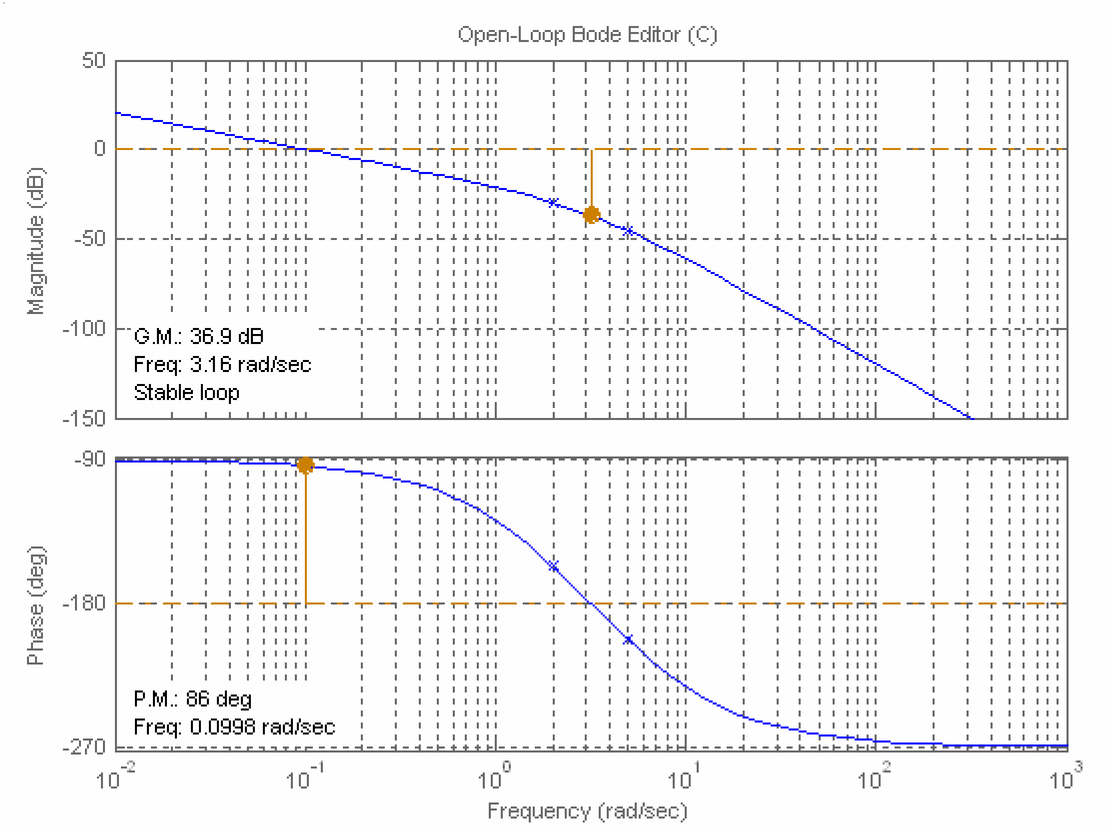




Percent overshoot exceeds requirements. Redesign if required.

21.

The required bandwidth for a peak time of 2 seconds and $\zeta = 0.456$ (i.e. 20% overshoot) is 2.3297 rad/s. Plotting the Bode diagrams for $K = 1$,



For 20% overshoot, Φ_M = 48.15º, or a phase angle of -180º + 48.15º = -131.85º. This angle occurs at 1.12 rad/s. If *K* = 13.1, the magnitude curve will intersect zero dB at 1.12 rad/s. Thus, the following function yields 20% overshoot: $G(s) = \dfrac{13.1}{s(s+2)(s+5)}$.

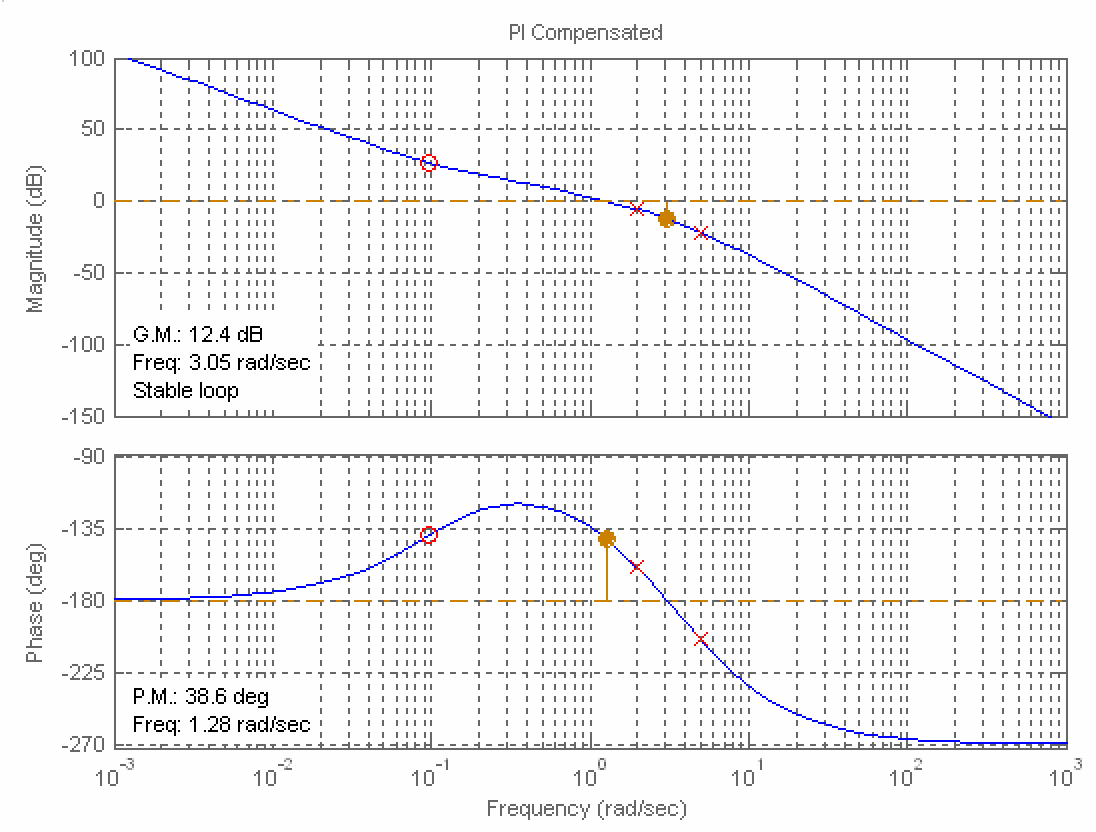
PI controller design: Allowing for a 5º margin, we want Φ_M = 48.15º + 5º = 53.15º, or a phase angle of -180º + 53.15º = -126.85º. This angle occurs at ω = 0.97 rad/s where the magnitude curve is 1.5321 dB. The controller should contribute - 1.5321 dB so that the magnitude curve passes through 0 dB at ω = 0.97 rad/s. Choosing the break frequency one decade below the phase margin frequency of 0.97 rad/s, and adjusting the controller's gain to yield -1.5321 dB at high frequencies, the ideal integral controller is

$$G_{cPI}(s) = \frac{1.198(s+0.097)}{s}$$

and the PI compensated forward path is

$$G_{PI}(s) = G(s)G_{cPI}(s) = \frac{15.694(s+0.097)}{s^2(s+2)(s+5)}$$

Plotting the Bode diagram for the PI compensated system yields,



This function is zero dB at ω = 1.28 rad/s. The phase at this frequency is – 141.4°. Thus, we have a phase margin of 38.6°.

PID controller design: Let us increase the phase margin frequency to 4 rad/s. At this frequency the phase is -193.48°. To obtain the required phase margin of 48.15° the phase curve must be raised an additional 61.63°. Assume the following form for the compensator : $G_{cPD}(s) = K'K_D(s+\frac{1}{K_D})$. The angle contributed by the compensator is $\phi_c = \tan^{-1}\frac{\omega}{1/K_D}$ = 61.63°. Letting ω = 4 rad/s, K_D = 0.463. Hence, the compensator is $G_{cPD}(s) = 0.463K'(s+2.16)$. The final PID compensated forward path is

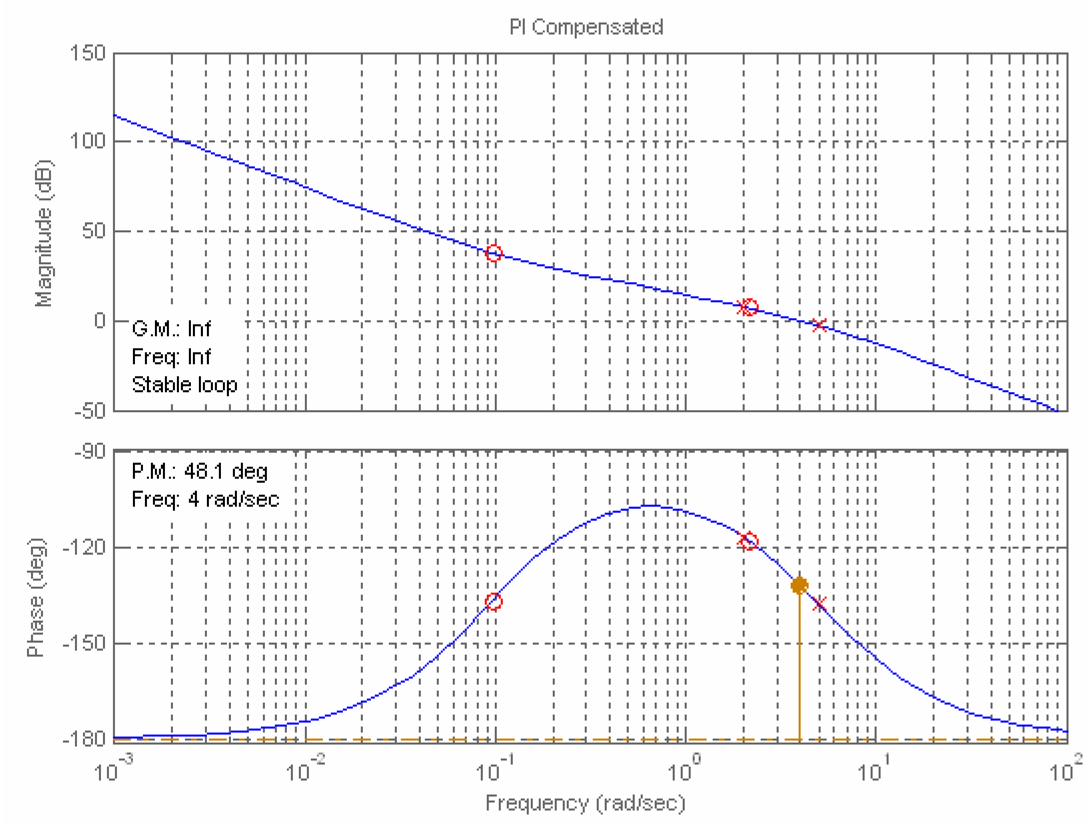
$$G_{PID}(s)=G_{PI}(s)G_{cPD}(s) = \frac{15.694(s+0.097)}{s^2(s+2)(s+5)} * 0.463K'(s+2.16) = \frac{2.266K'(s+0.097)(s+2.16)}{s^2(s+2)(s+5)}$$

Letting K' = 1 the magnitude of this function at 4 rad/s is -20.92 dB. Thus, K' must be adjusted to bring the magnitude to zero dB. Hence, K' = 11.12 (20.92 dB).

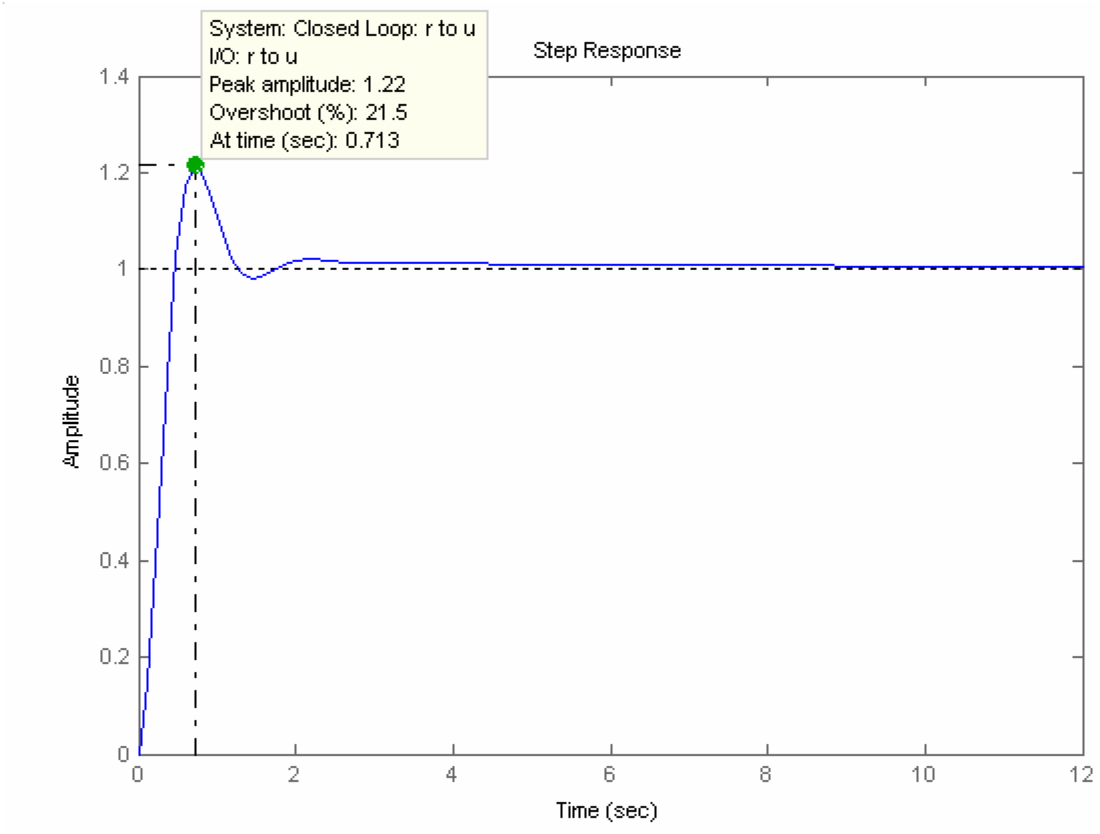
Thus,

$$G_{PID}(s) = \frac{25.2(s+0.097)(s+2.16)}{s^2(s+2)(s+5)}$$

The PID compensated Bode plot follows:



The PID compensated time response is shown below:



22.

Program:

```
%Input system
numg1=1;
deng1=poly([0 -3 -6]);
G1=tf(numg1,deng1);
[numg2,deng2]=pade(0.5,5);
G2=tf(numg2,deng2);
'G(s)=G1(s)G2(s)'
G=G1*G2;
Gzpk=zpk(G)
Tu=feedback(G,1);
step(Tu)
title('K = 1')
%Percent Overshoot to Damping Ratio to Phase Margin
Po=input('Type %OS  ');
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
Pm=atan(2*z/(sqrt(-2*z^2+sqrt(1+4*z^4))))*(180/pi);
fprintf('\nPercent Overshoot = %g',Po)
fprintf(', Damping Ratio = %g',z)
fprintf(', Phase Margin = %g',Pm)
%Get Bode data
bode(G)
title('K = 1')
pause
w=0.1:0.01:100;
[M,P]=bode(G,w);
Ph=-180+Pm;
for i=1:1:length(P);
if P(i)-Ph<=0;
M=M(i);
K=1/M;
```

```
fprintf(', Frequency  = %g',w(i))
fprintf(', Phase  = %g',P(i))
fprintf(', Magnitude = %g',M)
fprintf(', Magnitude (dB) = %g',20*log10(M))
fprintf(', K = %g',K)
break
end
end
T=feedback(K*G,1);
step(T)
title('Gain Compensated')
```

Computer response:

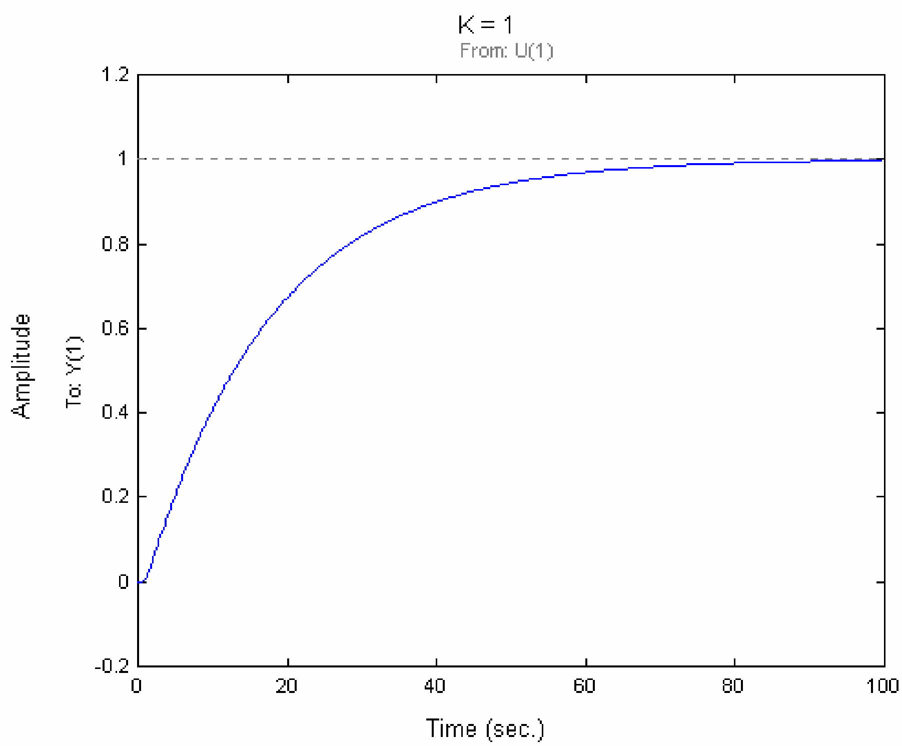
```
ans =

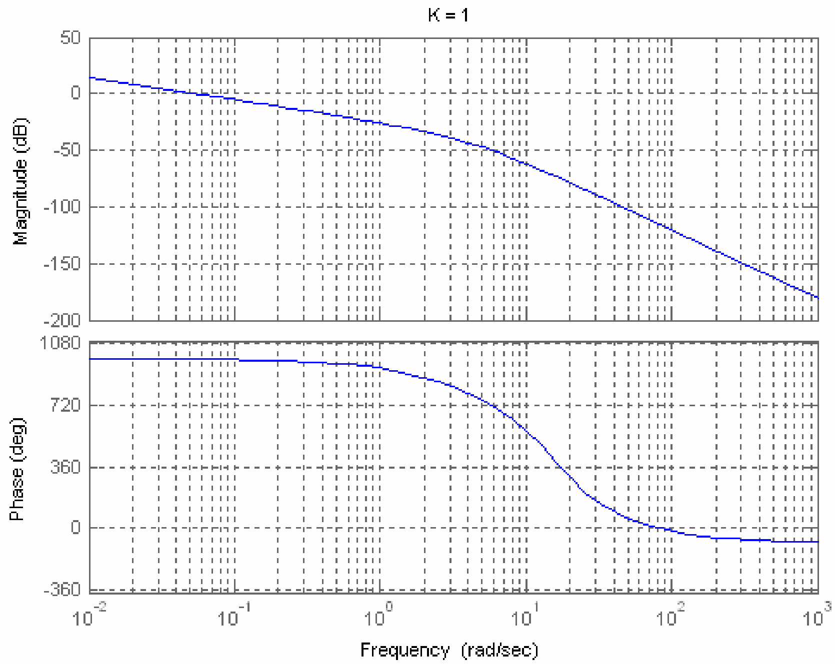
G(s)=G1(s)G2(s)

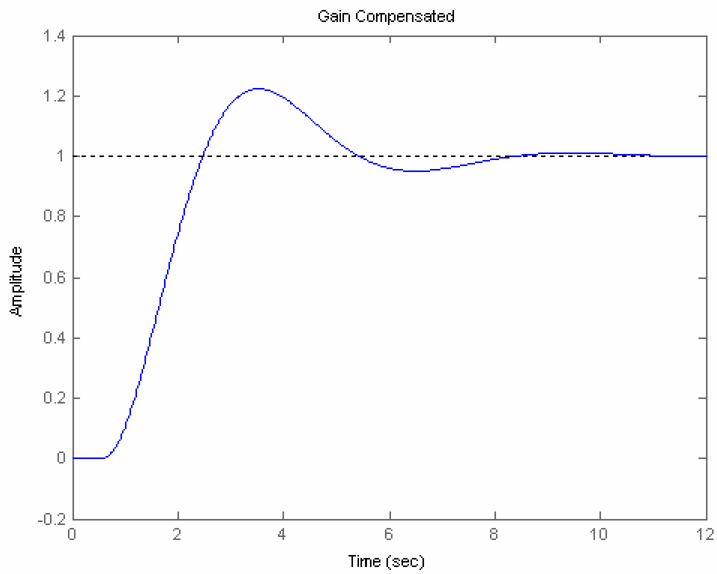
Zero/pole/gain:
      - (s-14.59) (s^2 - 26.82s + 228.4) (s^2 - 18.6s + 290.5)
--------------------------------------------------------------------
s (s+14.59) (s+6) (s+3) (s^2 + 26.82s + 228.4) (s^2 + 18.6s + 290.5)

Type %OS  20

Percent Overshoot = 20, Damping Ratio = 0.45595, Phase Margin = 48.1477,
Frequency  = 0.74, Phase  = -132.087, Magnitude = 0.0723422, Magnitude (dB)
= -22.8122, K = 13.8232»
```





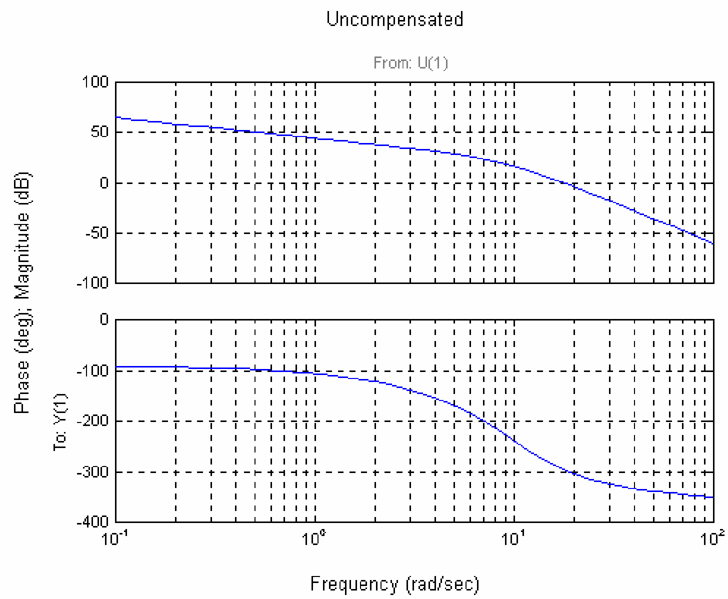


Second-order approximation not valid.

SOLUTIONS TO DESIGN PROBLEMS

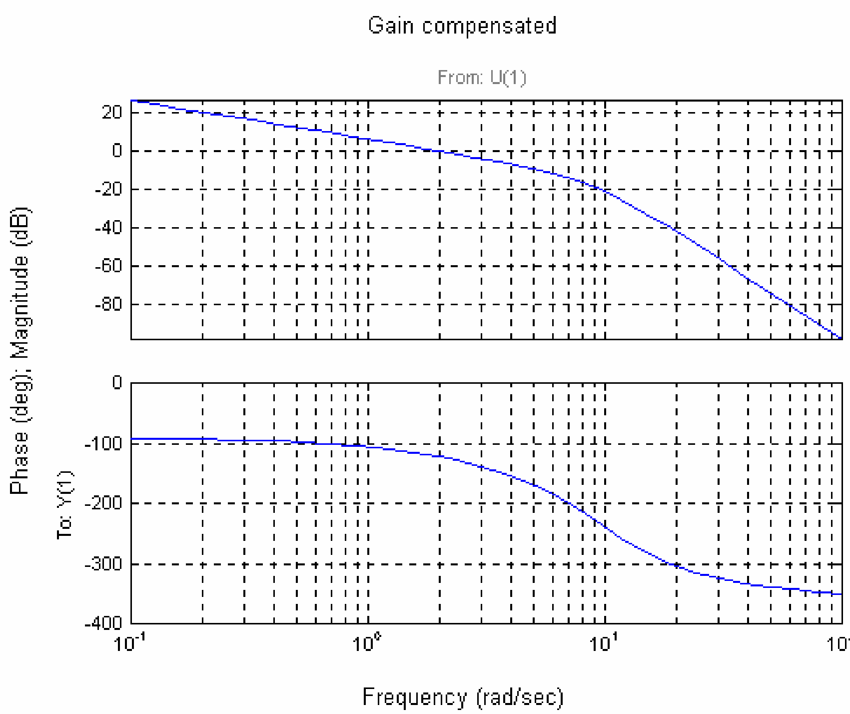
23.

a. Plot the Bode plot for $K = 1$.



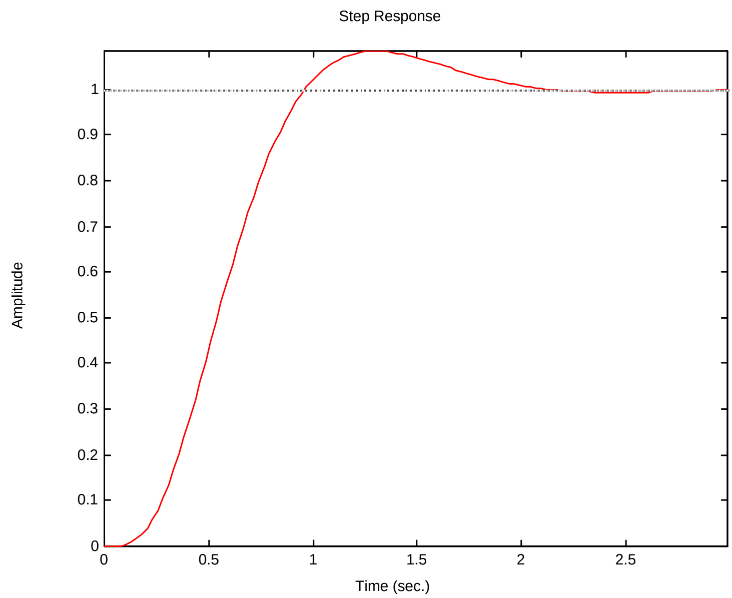
Using Eqs. (4.39) and (10.73) a percent overshoot = 10 is equivalent to a $\zeta = 0.591$ and $\phi_M = 58.59^0$. The phase-margin frequency = 1.933 rad/s where the phase is $58.59^0 - 180^0 = -121.41^0$. The magnitude = 38.37 dB, or 82.85. Hence K = 1/ 82.85 = 0.01207.

b. Plot the gain-compensated Bode plot (K = 0.01207).



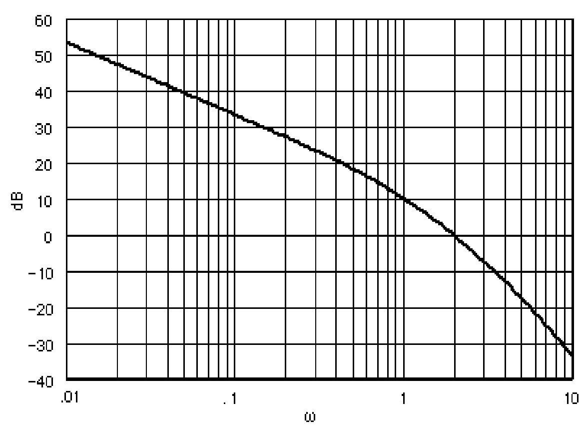
The bandwidth, ω_{BW}, is the frequency at which the magnitude is –7dB. From the compensated plots, this frequency is 3.9 rad/s. Eq. (10.55), T_s = 2.01 s. Using Eq. (10.57), T_p = 1.16 s.

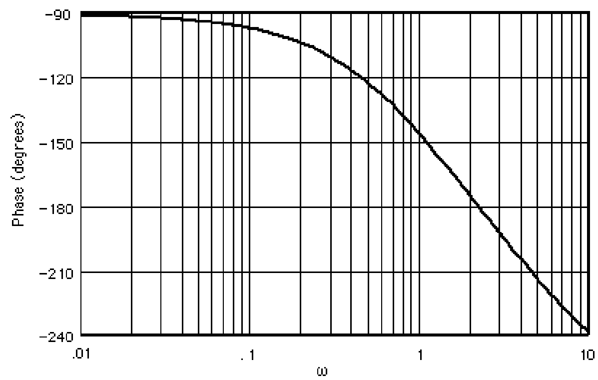
c.



24.

$G(s) = \frac{10K}{s(s+1)(s+5)}$. For K_v = 5, K = 2.5. Plot the Bode diagrams for this value of gain.





The uncompensated system has unity gain at ω = 2.04 rad/s. The phase is - 176.08° at this frequency yielding a phase margin of 3.92°. We want a 60° phase margin plus, after trial and error, a correction factor of 20°, or a total of 80°. Thus, the lead compensator must contribute 80° - 3.92° = 76.08°. Using Eqs. (11.11), and (11.12),

The value of beta is:	0.01490254
The |G(jwmax)| for the compensator is:	8.1916207
or in db:	18.2673967

The magnitude curve has a gain of - 18.27 dB at ω = 5.27 rad/s. Therefore, choose this frequency as the new phase margin frequency. Using Eqs. (11.9) and (11.6), the compensator transfer function has the following specifications:

T	1.55438723
zero	-0.6433403
pole	-43.169841
gain	67.1026497

The compensated forward path is

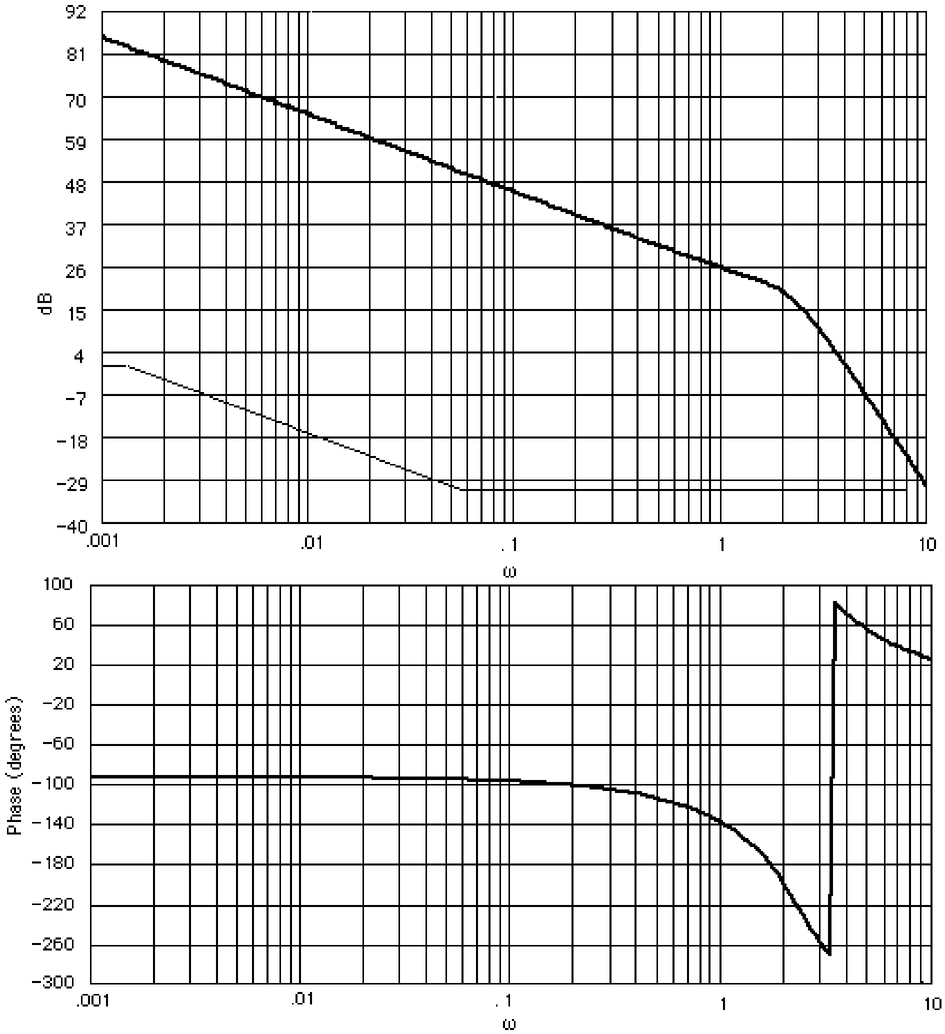
$$G(s) = \frac{25*67.1(s+0.64)}{s(s+1)(s+5)(s+43.17)} = \frac{1677.5(s+0.64)}{s(s+1)(s+5)(s+43.17)}$$

25.

$G(s) = \dfrac{10}{s(s^2+2s+5)(s+3)}$. Therefore, $K_{vo} = \dfrac{2}{3}$. We want $K_{vn} = 30K_{vo} = 20$. Increasing K by 30 times yields

$$G(s) = \frac{300}{s(s^2+2s+5)(s+3)}$$

Plotting the Bode diagrams,



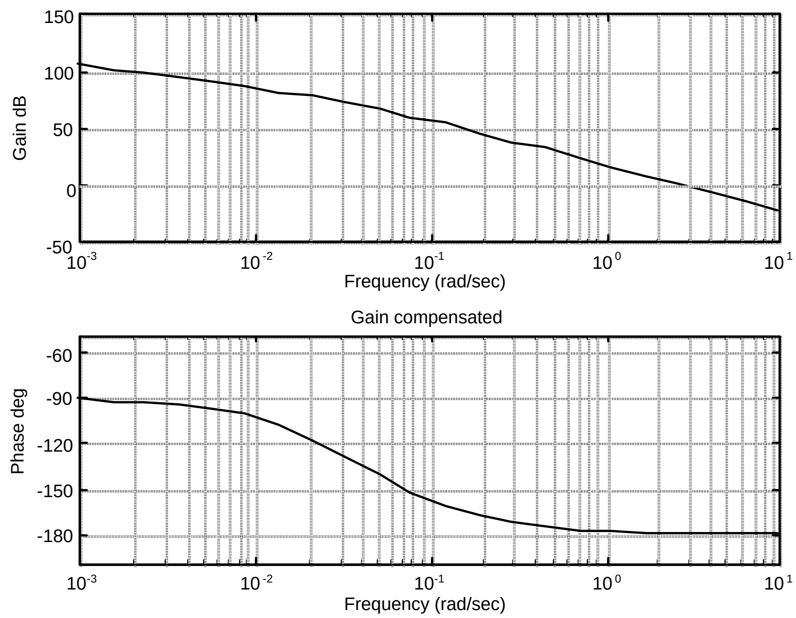
For 11% overshoot, the phase margin should be 57.48°. Adding a correction, we will use a 65° phase margin, or a phase angle of 115°, which occurs at ω = 0.58 rad/s. The magnitude curve is 30.93 dB. Thus the high-frequency asymptote of the lag compensator is - 30.93 dB. Drawing the lag-compensator curve as shown on the magnitude curve, the break frequencies are found and the compensator's transfer function is evaluated as

$$G_c(s) = 25.86 \times 10^{-3} \frac{s+0.058}{s+0.0015}$$

26.

a. The equivalent forward transfer function is $G_e(s) = \dfrac{4.514e\text{-}06K}{s(s+0.04348)}$.

$K_v = 200 = \dfrac{4.514e\text{-}06K}{0.04348}$ or K = 1926500. Using Eq. (4.39), ζ = 0.456. Using Eq. (10.55), ω_{BW} = 1.16. Using Eq. (10.73) with 15° additional, the required phase margin, ϕ_{req} = 63.15°. Select a new phase-margin frequency, $\omega_{Pm} = 0.8\ \omega_{BW}$ = 0.93. Plot the Bode plots for K = 1926500.



At ω_{Pm} = 0.93, the phase, ϕ = -177.3^o. Hence, the phase required from the compensator, $\phi_C = \phi_{req}$ - $(180+\phi)$ = 63.15 - (180 - 177.3) = 60.45^o. Using Eq. (11.11), β = 0.07.

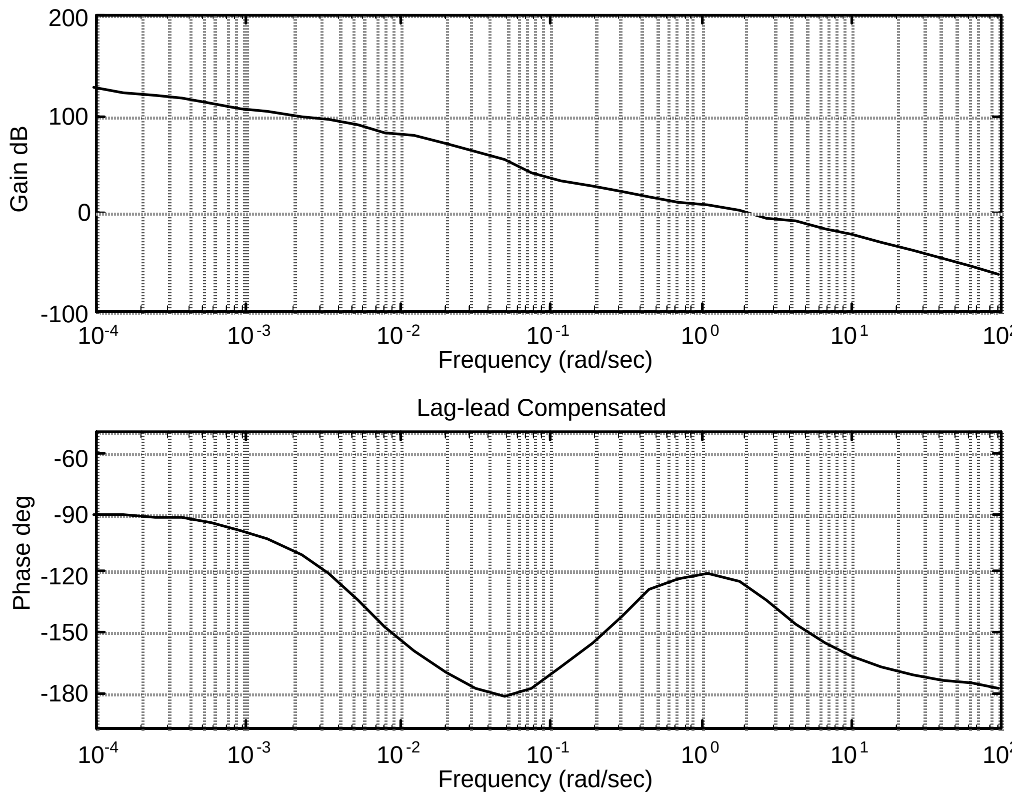
Design lag: $z_{clag} = \omega_{pm}/10 = 0.093$; $p_{clag} = z_{clag}*\ \beta = 0.0065$; $K_{clag} = \beta = 0.07$. Thus,

$$G_{clag}(s) = 0.07\,\frac{s+0.093}{s+0.0065}\ .$$

Design lead compensator: $z_{clead}=\omega_{Pm}*\sqrt{\beta}\ = 0.25$; $p_{clead}=z_{clead}/\beta = 3.57$; $K_{clead}=1/\beta = 14.29$. Thus,

$$G_{clead}(s) = 14.29\,\frac{s+0.25}{s+3.57}\ .$$

The lag-lead compensated Bode plot:

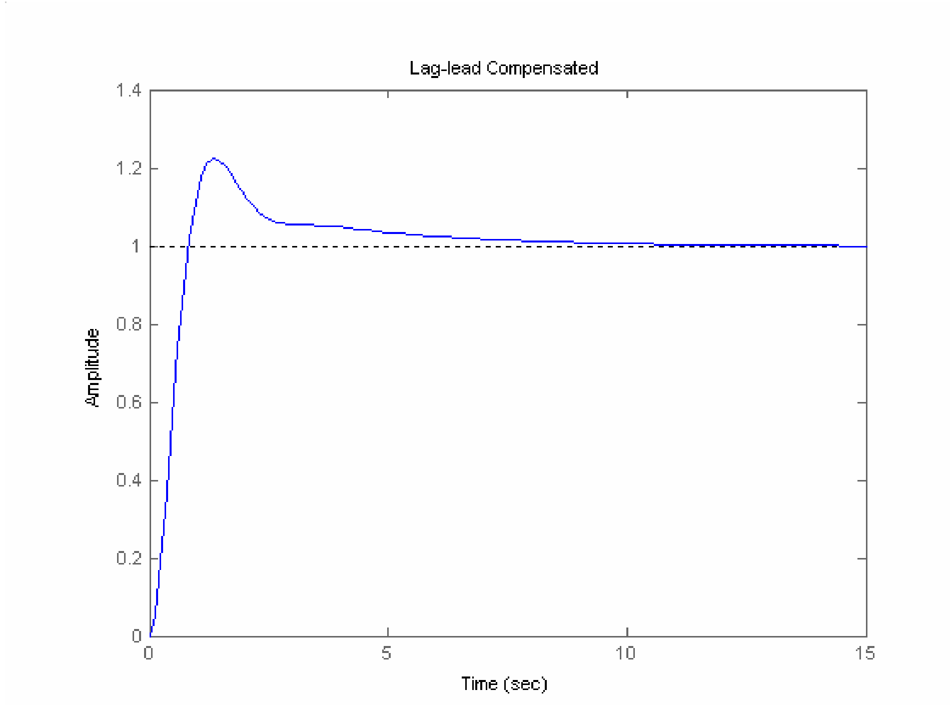


b.

Program:

```
K=1926500;
numg=4.514e-6;
deng=[1 0.04348 0];
G=tf(numg,deng);
numgclag=0.07*[1 0.093];
dengclag=[1 0.0065];
Gclag=tf(numgclag,dengclag);
numgclead=14.29*[1  0.25];
dengclead=[1 3.57];
Gclead=tf(numgclead,dengclead);
Ge=K*G*Gclag*Gclead;
T=feedback(Ge,1);
step(T)
title('Lag-lead Compensated')
```

Computer response:



27.

From Chapter 8,

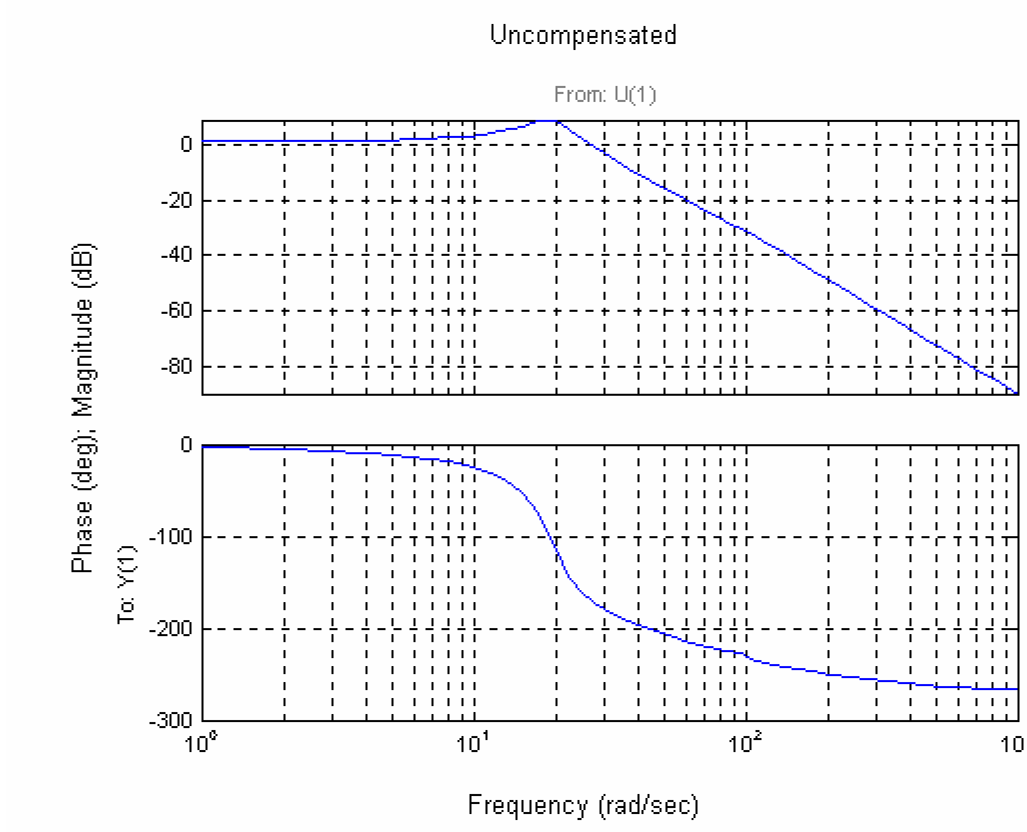
$$G_e(s) = \frac{0.6488K\,(s+53.85)}{(s^2 + 8.119s + 376.3)\,(s^\wedge 2 + 15.47s + 9283)}$$

Cascading the notch filter,

$$G_{et}(s) = \frac{0.6488K\,(s+53.85)(s^2 + 16s + 9200)}{(s^2 + 8.119s + 376.3)\,(s^\wedge 2 + 15.47s + 9283)(s+60)^2}$$

Since $e_{step}(\infty) = \dfrac{1}{1+K_p}$, $K_p = 9$ yields 10% error. Thus, $K_p = \dfrac{K_e * 53.85 * 9200}{376.3 * 9283 * 60^2} = 9$. Thus,

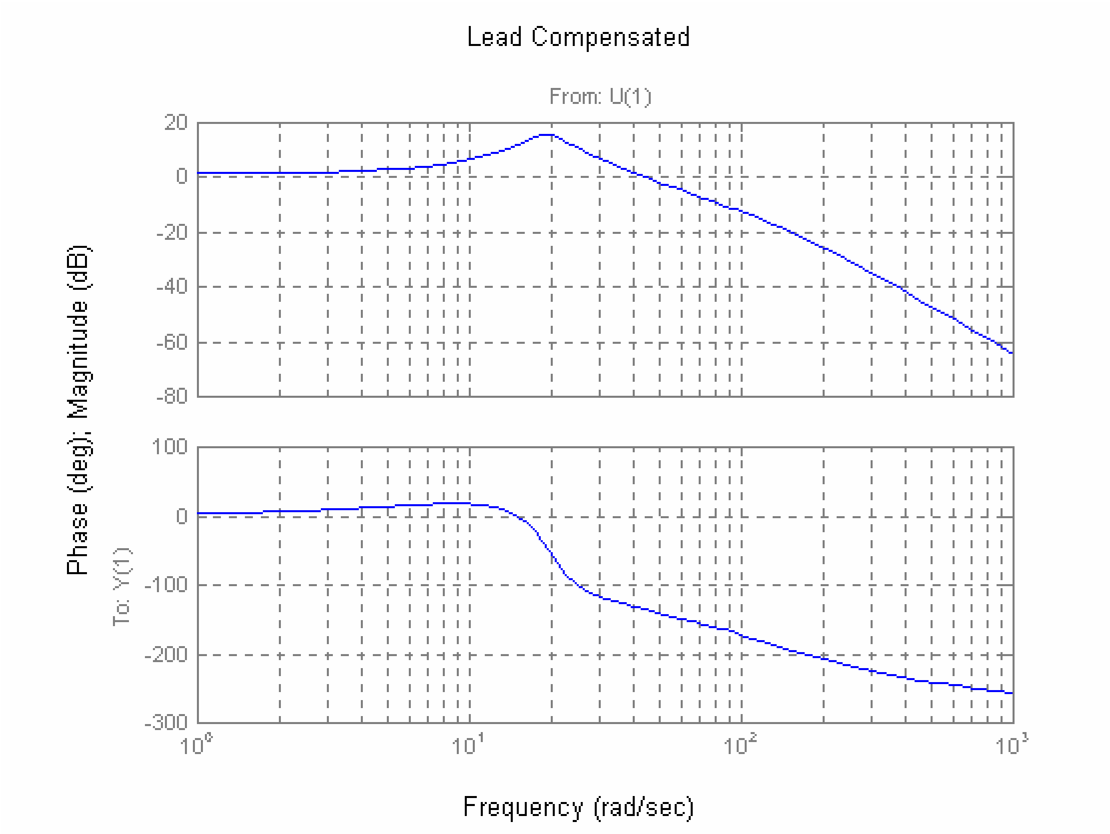
$K_e = 0.6488K = 228452$. Let us use $K_e = 30{,}000$ in designing the lead portion and we'll make up the rest with the lag. Plotting the Bode plot for $K_e = 30{,}000$,



Design lead: The uncompensated phase margin = 10.29^0. Assume a required phase margin of 45^0. The required phase margin, assuming a 30^0 correction is 75^0. The phase contribution required from the compensator is $75^0 - 10.29^0 = 64.71^0$. Using the inverse of Eq. (11.11), the compensator's β = 0.05033. Using Eq. (11.12), $|G_c(j\omega_{\max})| = \frac{1}{\sqrt{\beta}} = 4.457 = 12.98 \text{ dB}$. The new phase margin frequency is where the uncompensated system has a magnitude of –12.98 dB, or ω_{max} = 44.65 rad/s. Using Eqs. (11.6) and (11.9), the compensator is $G_{lead}(s) = \frac{19.87(s+10.02)}{(s+199)}$. The plant is

$$G(s) = \frac{228452(s+53.85)(s^2+16s+9200)}{(s+60)^2(s^2+8.119s+376.3)(s^2+15.47s+9283)}$$

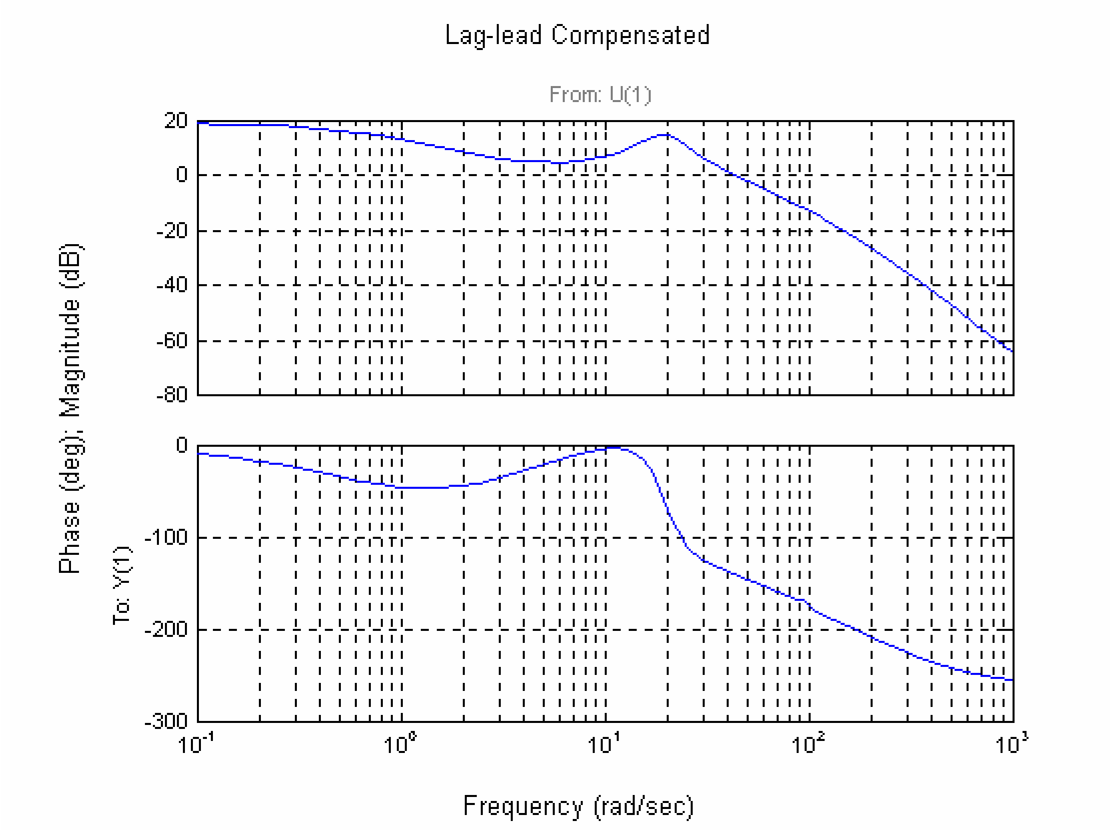
. Draw the lead-compensated Bode plot.



Design lag: The phase-margin frequency occurs where the phase is -135^0, or at the required 45^0 phase margin. From the lead-compensated Bode plots, this phase margin occurs at 43.64 rad/s. Let the upper break of the lag compensator be one decade lower, or 4.364. Since the magnitude is 17.97 dB at the new phase-margin frequency, set the high-frequency asymptote of the lag compensator at –17.97 dB. Draw a -20 dB/dec slope starting at 0.4364 rad/s and –17.96 dB and moving toward 0 dB. At 0 dB locate the lag compensator's low-frequency break, or 0.551. Thus,

$$G_{lag}(s) = \frac{0.551}{4.364}\frac{(s+4.364)}{(s+0.551)} = 0.126\frac{(s+4.364)}{(s+0.551)}$$

Check bandwidth: Draw the lag-lead compensated Bode diagram for $G(s)G_{lag}(s)G_{lead}(s)$.



From the open-loop plot, the magnitude is at –7 dB at 70 rad/s. Hence, the bandwidth is sufficient. Also, the lag-lead compensated Bode plot shows a phase margin of 40^0 . The transfer function, $G(s) = G(s)G_{lag}(s)G_{lead}(s)$ shows $K_p = 9$, or an error of 0.1. Thus all requirements have been met.

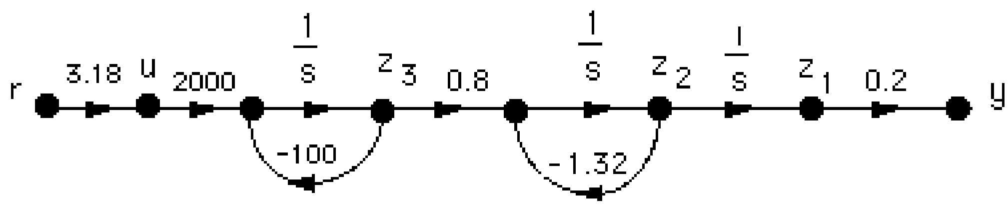
T W E L V E

Design via State Space

SOLUTION TO CASE STUDY CHALLENGE

Antenna Control: Design of Controller and Observer

a. We first draw the signal-flow diagram of the plant using the physical variables of the system as state variables.



Writing the state equations for the physical variables shown in the signal-flow diagram, we obtain

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1.32 & 0.8 \\ 0 & 0 & -100 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 2000 \end{bmatrix} u \; ; \quad y = \begin{bmatrix} 0.2 & 0 & 0 \end{bmatrix} \mathbf{z}$$

The characteristic polynomial for this system is $s^3 + 101.32s^2 + 132s + 0$. Hence, the **A** and **B** matrices of the phase-variable form are

Ax			Bx
0	1	0	0
0	0	1	0
0	-132	-101.32	1

Writing the controllability matrices and their determinants for both systems yields

CMz	Controllability Matrix of z		CMx	Controllability Matrix of x	
0	0	1600	0	0	1
0	1600	-162112	0	1	-101.32
2000	-200000	20000000	1	-101.32	10133.7424
Det(CMz)	-5.12E+09		**Det(CMx)**	-1	

where the system is controllable. Using Eq. (12.39), we find the transformation matrix and its inverse to be

P	Transformation Matrix z=Px		PINV		
1600	0	0	0.000625	0	0
0	1600	0	0	0.000625	0
0	2640	2000	0	-0.000825	0.0005

The characteristic polynomial of the phase-variable system with state feedback is

$$s^3 + (k_3 + 101.32)s^2 + (k_2 + 132)s + (k_1 + 0)$$

For 15% overshoot, T_s = 2 seconds, and a third pole 10 times further from the imaginary axis than the dominant poles, the characteristic polynomial is

$$(s + 20)(s^2 + 4s + 14.969) = s^3 + 24s^2 + 94.969s + 299.38$$

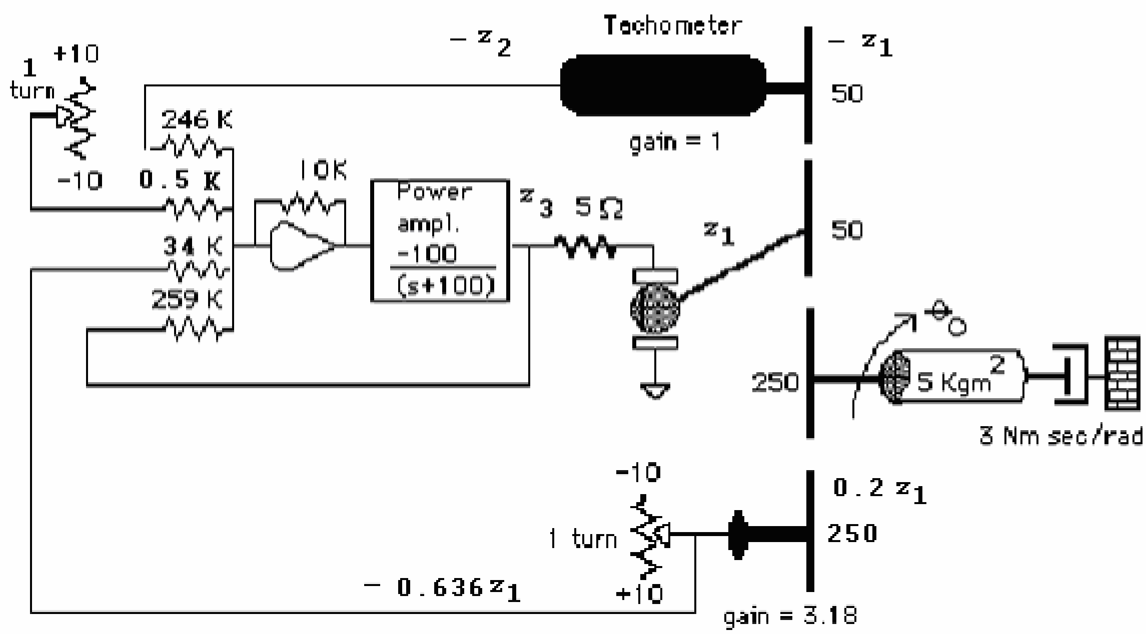
Equating coefficients, the controller for the phase-variable system is

Kx	Controller for x	
299.38	-37.031	-77.32

Using Eq. (12.42), the controller for the original system is

Kz	Controller for z	
0.1871125	0.04064463	-0.03866

b. Using $\mathbf{K_z}$, gain from θ_m = - 0.1871125 (including gear train, pot, and operational amplifier); gain from tachometer = - 0.04064463; and gain from power amplifier output = 0.03866.



c. Using the original system from part (a) and its characteristic polynomial, we find the observer canonical form which has the following **A** and **C** matrices:

Ax		
-101.32	1	0
-132	0	1
0	0	0

Cx		
1	0	0

Writing the observability matrices and their determinants for both systems yields

OMz	Observability Matrix of z		**OMx**	Observability Matrix of x	
0.2	0	0	1	0	0
0	0.2	0	-101.32	1	0
0	-0.264	0.16	10133.7424	-101.32	1
Det(OMz)	0.0064		**Det(OMx)**	1	

where the system is observable. Using Eq. (12.89), we find the transformation matrix and its inverse to be

P	Transformation Matrix z=Px		**PINV**		
5	0	0	0.20	0.00	0.00
-506.6	5	0	20.26	0.20	0.00
62500	-625	6.25	26.40	20.00	0.16

The characteristic polynomial of the dual phase-variable system with state feedback is

$$s^3 + (l_1 + 101.32)s^2 + (l_2 + 132)s + (l_3 + 0)$$

For 10% overshoot, $\omega_n = 10\sqrt{14.969} = 38.69$ rad/s, and a third pole 10 times further from the imaginary axis than the dominant observer poles, the characteristic polynomial is

$$(s + 228.72)(s^2 + 45.743s + 1496.916) = s^3 + 274.46s^2 + 11959s + 3.4237x10^5$$

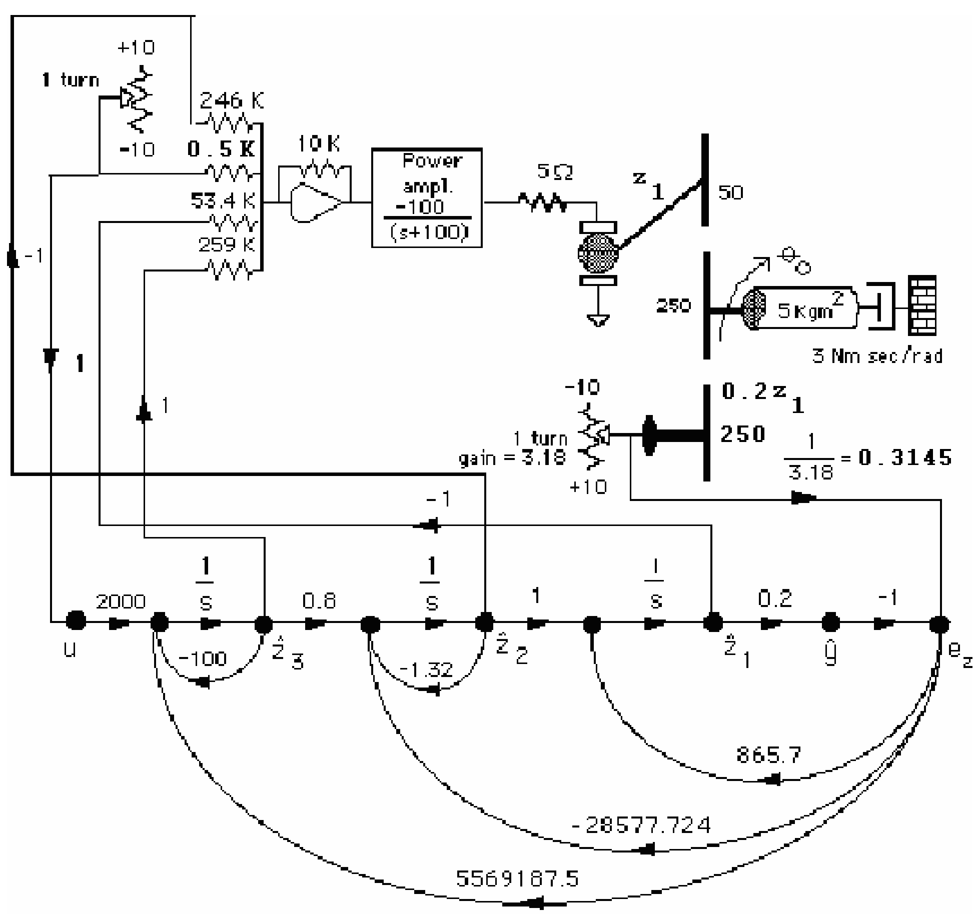
Equating coefficients, the observer for the observer canonical system is

Lx	Observer for x
173.14	
11827	
342370	

Using Eq. (12.92), the observer for the original system is

Lz	Observer for z
865.7	
-28577.724	
5569187.5	

d.



e.

Program:

```
'Controller'
A=[0 1 0;0 -1.32 0.8;0 0 -100];
B=[0;0;2000];
C=[0.2 0 0];
D=0;
pos=input('Type desired %OS  ');
Ts=input('Type desired settling time   ');
z=(-log(pos/100))/(sqrt(pi^2+log(pos/100)^2));
wn=4/(z*Ts);                      %Calculate required natural
                                  %frequency.
[num,den]=ord2(wn,z);             %Produce a second-order system that
                                  %meets the transient response
                                  %requirements.
r=roots(den);                     %Use denominator to specify dominant
                                  %poles.
poles=[r(1) r(2) 10*real(r(1))];
                                  %Specify pole placement for all
                                  %poles.
K=acker(A,B,poles)
'Observer'
pos=input('Type desired %OS  ');
z=(-log(pos/100))/(sqrt(pi^2+log(pos/100)^2));
wn=10*wn                           %Calculate required natural
                                   %frequency.
[num,den]=ord2(wn,z);              %Produce a second-order system that
                                   %meets the transient response
                                   %requirements.
r=roots(den);                      %Use denominator to specify dominant
                                   %poles.
poles=[r(1) r(2) 10*real(r(1))];%Specify pole placement for all
                                   %poles.
l=acker(A',C',poles)'
```

Computer response:

```
ans =

Controller

Type desired %OS  15
Type desired settling time   2

K =

  0.1871    0.0406   -0.0387

ans =

Observer

Type desired %OS  10

wn =

  38.6899

l =

  1.0e+006 *

    0.0009
   -0.0286
    5.5691
```

ANSWERS TO REVIEW QUESTIONS

1. Both dominant and non-dominant poles can be specified with state-space design techniques.

2. Feedback all state variables to the plant's input through a variable gain for each. Decide upon a closed-loop characteristic equation that has a pole configuration to yield a desired response. Write the characteristic equation of the actual system. Match coefficients and solve for the values of the variable gains.

3. Phase-variable form

4. The control signal developed by the controller must be able to affect every state variable.

5. If the signal-flow diagram is in the parallel form, which leads to a diagonal system matrix, controllability can be determined by inspection by seeing that all state variables are fed by the control signal.

6. The system is controllable if the determinant of the controllability matrix is non-zero.

7. An observer is a system that estimates the state variables using information from the output of the actual plant.

8. If the plant's state-variables are not accessible, or too expensive to monitor

9. An observer is a copy of the plant. The difference between the plant's output and the observer's output is fed back to each of the derivatives of the observer's state variables through separate variable gains.

10. Dual phase-variable

11. The characteristic equation of the observer is derived and compared to a desired characteristic equation whose roots are poles that represent the desired transient response. The variable gains of each feedback path are evaluated to make the coefficients of the observer's characteristic equation equal the coefficients of the desired characteristic equation.

12. Typically, the transient response of the observer is designed to be much faster than that of the controller. Since the observer emulates the plant, we want the observer to estimate the plant's states rapidly.

13. Det[**A**-**BK**], where **A** is the system matrix, **B** is the input coupling matrix, and **K** is the controller.

14. Det[**A**-**LC**], where **A** is the system matrix, **C** is the output coupling matrix, and **L** is the observer.

15. The output signal of the system must be controlled by every state variable.

16. If the signal-flow diagram is in the parallel form, which leads to a diagonal system matrix, observability can be determined by inspection by seeing that all state variables feed the output.

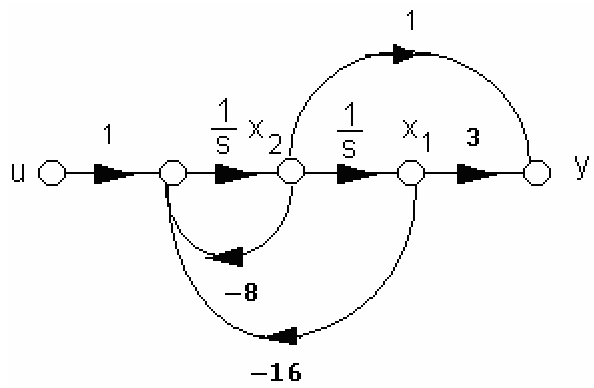
17. The system is observable if the determinant of the observability matrix is non-zero.

SOLUTIONS TO PROBLEMS

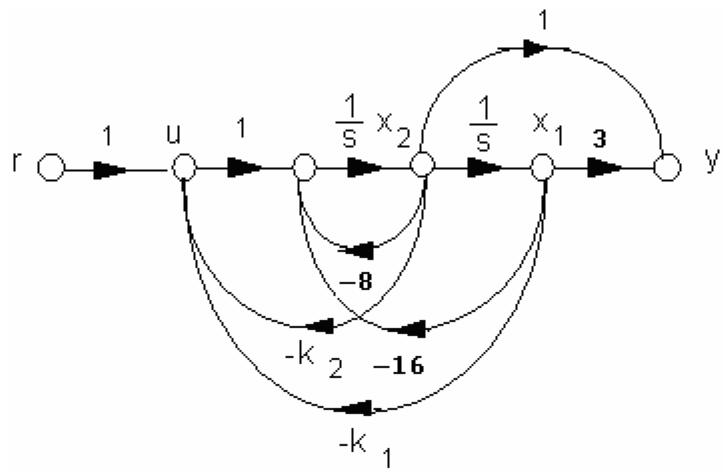
1.

i. $G(s) = \dfrac{(s+3)}{(s+4)^2} = \dfrac{1}{s^2+8s+16} * (s+3)$

a.



b.



c.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -(k_1+16) & -(k_2+8) \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \;\; ; \;\; y = \begin{bmatrix} 3 & 1 \end{bmatrix}\mathbf{x}$$

d.

$$T(s) = \frac{s+3}{s^2 + (k_2+8)s + (k_1+16)}$$

$$\dot{\mathbf{x}}_1 = (-20 - 71.25k_1)x_1 - 71.25k_2x_2 - 71.25k_3x_3 + 71.25r$$

$$\dot{\mathbf{x}}_2 = 27.5k_1x_1 + (-10x_2 + 27.5k_2)x_2 + 27.5k_3x_3 - 27.5r$$

$$\dot{\mathbf{x}}_3 = -6.25k_1x_1 - 6.25k_2x_2 - 6.25k_3x_3 + 6.25r$$

$$\mathbf{A} = \begin{bmatrix} (-20-71.25k_1) & -71.25k_2 & -71.25k_3 \\ 27.5k_1 & (-10x_2+27.5k_2) & 27.5k_3 \\ -6.25k_1 & -6.25k_2 & -6.25k_3 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 71.25 \\ -27.5 \\ 6.25 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

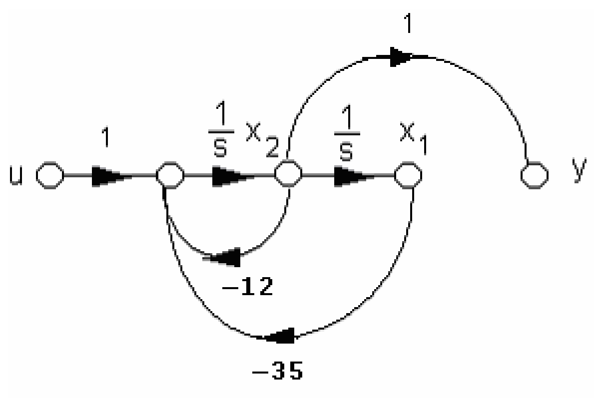
$$T(s) = \frac{200(s^2+7s+25)}{4s^3 + (120+285k_1-110k_2+25k_3)s^2 + (800+2850k_1-2200k_2+750k_3)s + 5000k_3}$$

e.

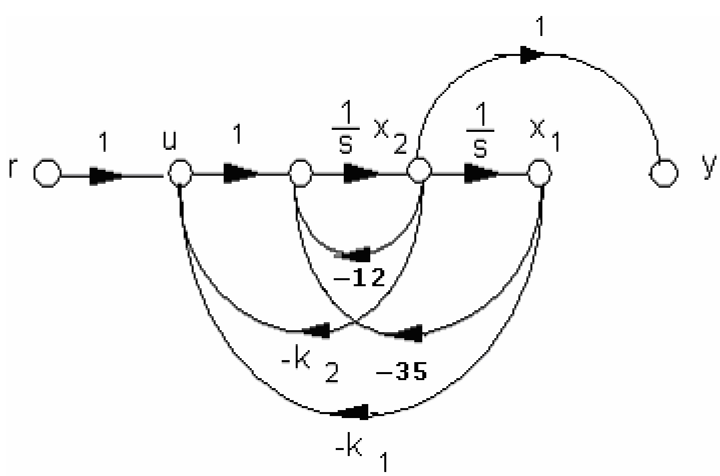
Part d. yields same result as i(d).

ii. $G(s) = \dfrac{s}{(s+5)(s+7)} = \dfrac{1}{s^2+12s+35} * s$

a.



b.



c.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -(k_1+35) & -(k_2+12) \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \;\; ; \;\; y = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{x}$$

d.

$$T(s) = \frac{s}{s^2 + (k_2+12)s + (k_1+35)}$$

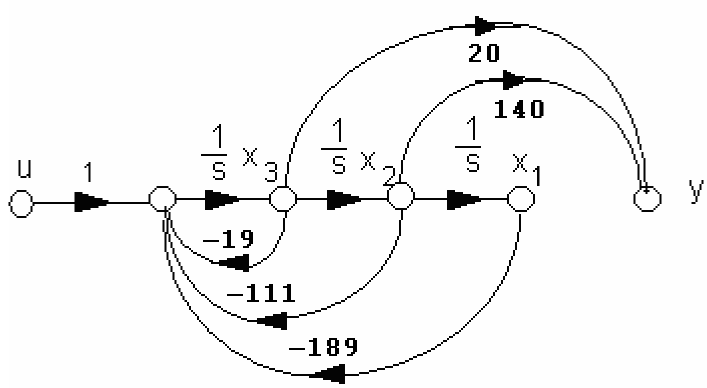
e.

$$T(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B};\;\; \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -(k_1+35) & -(k_2+12) \end{bmatrix};\;\; \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\;\; \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

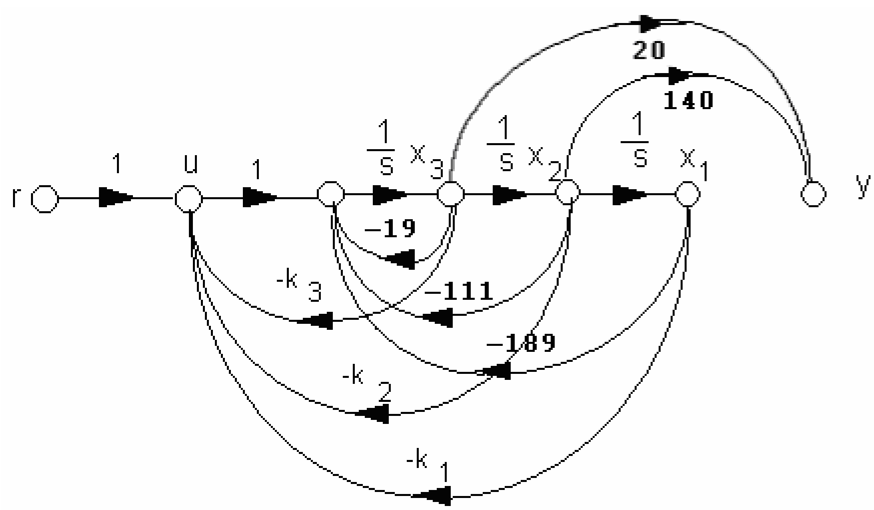
which yields the same result as ii(d).

iii. $G(s) = \dfrac{20s(s+7)}{(s+3)(s+7)(s+9)} = \dfrac{1}{s^3 + 19s^2 + 111s + 189} * \;(20s^2 + 140s)$

a.



b.



c.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(k_1+189) & -(k_2+111) & -(k_3+19) \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \ ; \ y = \begin{bmatrix} 0 & 140 & 20 \end{bmatrix}\mathbf{x}$$

d.

$$T(s) = \frac{20s(s+7)}{s^3 + (k_3+19)s^2 + (k_2+111)s + (k_1+189)}$$

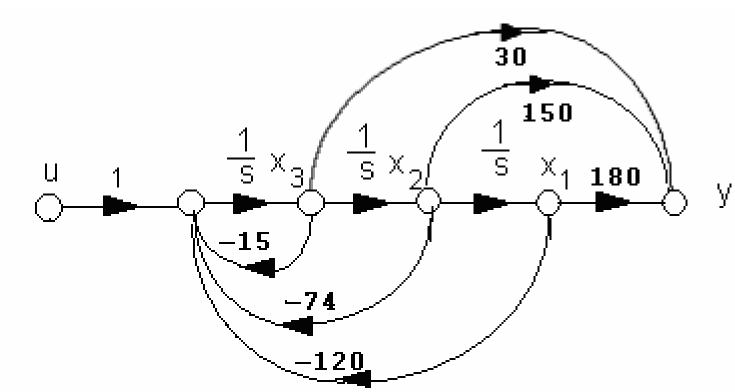
e.

$$T(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B};\ \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(k_1+189) & -(k_2+111) & -(k_3+19) \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};\ \mathbf{C} = \begin{bmatrix} 0 & 140 & 20 \end{bmatrix}$$

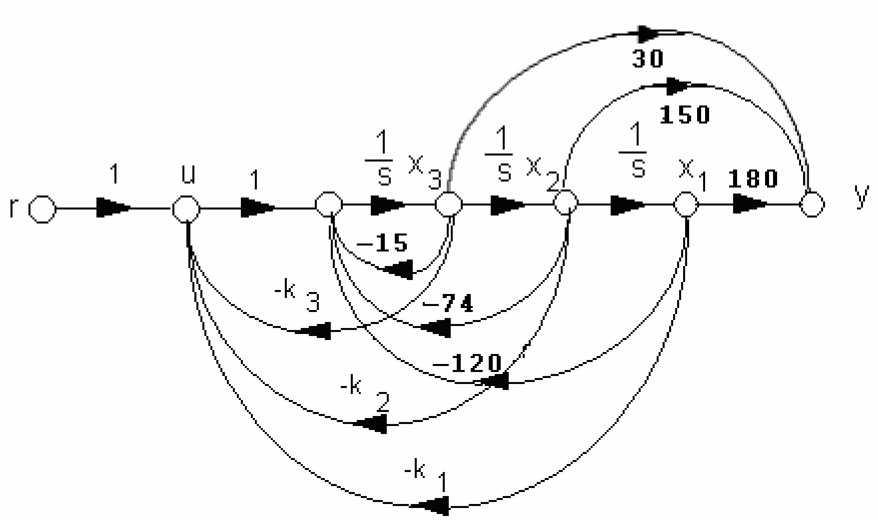
which yields the same result as iii(d).

iv. $G(s) = \dfrac{30(s+2)(s+3)}{(s+4)(s+5)(s+6)} = \dfrac{1}{s^3 + 15s^2 + 74s + 120} * \ (30s^2 + 150s + 180)$

a.



b.



c.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(k_1+120) & -(k_2+74) & -(k_3+15) \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \ ; \ y = \begin{bmatrix} 180 & 150 & 30 \end{bmatrix}\mathbf{x}$$

d.

$$T(s) = \frac{30s^2 + 150s + 180}{s^3 + (k_3+15)s^2 + (k_2+74)s + (k_1+120)}$$

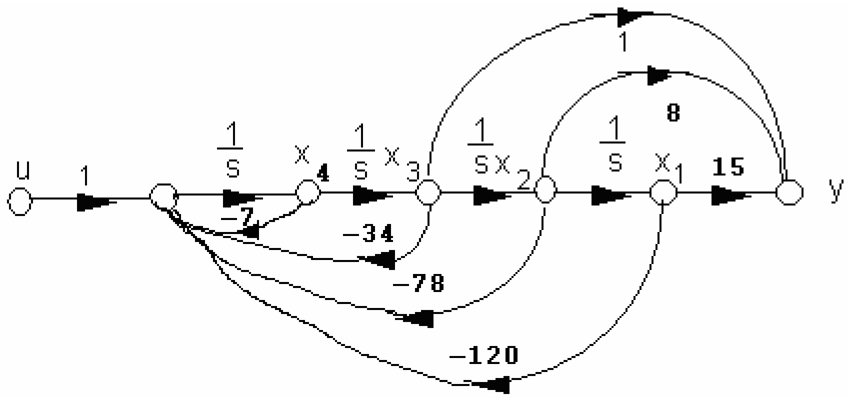
e.

$$T(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B};\ \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(k_1+120) & -(k_2+74) & -(k_3+15) \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};\ \mathbf{C} = \begin{bmatrix} 180 & 150 & 30 \end{bmatrix}$$

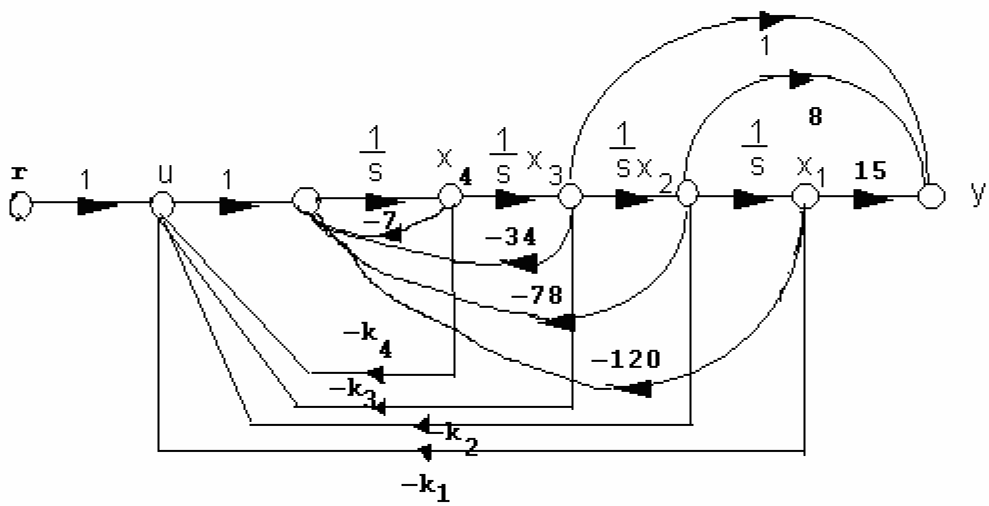
which yields the same result as iv(d).

v. $G(s) = \dfrac{s^2 + 8s + 15}{(s^2 + 4s + 10)(s^2 + 3s + 12)} = \dfrac{1}{s^4 + 7s^3 + 34s^2 + 78s + 120} * (s^2 + 8s + 15)$

a.



b.



c.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+120) & -(k_2+78) & -(k_3+34) & -(k_4+7) \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r \ ; \ y = \begin{bmatrix} 15 & 8 & 1 & 0 \end{bmatrix}\mathbf{x}$$

d.

$$T(s) = \frac{s^2+8s+15}{s^4+(k_4+7)s^3+(k_3+34)s^2+(k_2+78)s+(k_1+120)}$$

e.

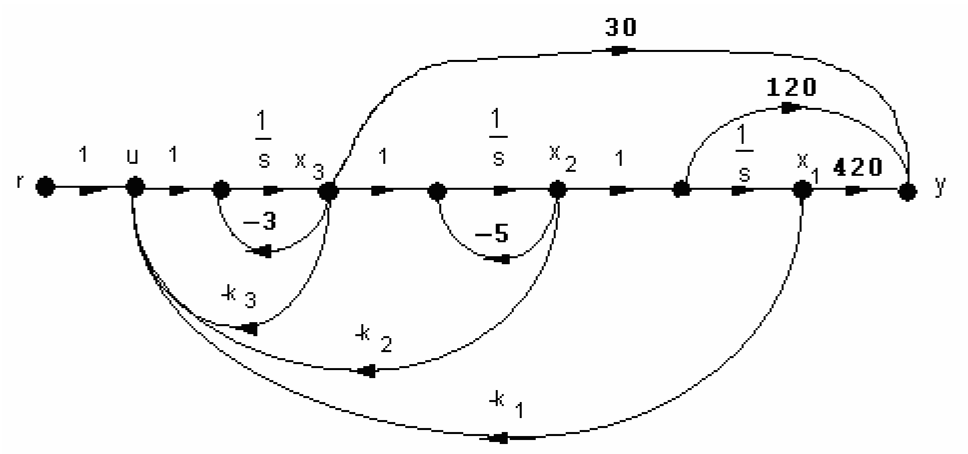
$$T(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B};\ \mathbf{A}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-(k_1+120)&-(k_2+78)&-(k_3+34)&-(k_4+7)\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix};\ \mathbf{C}=\begin{bmatrix}15&8&1&0\end{bmatrix}$$

which yields the same result as v(d).

2.

i

a. The output is



Since,

$$y=(30s^2+270s+420)x_1=30\ddot{x}_1+270\dot{x}_1+420x_1=30\dot{x}_2+270x_2+420x_1$$
$$=30(-5x_2+x_3)+270x_2+420x_1=420x_1+120x_2+30x_3$$

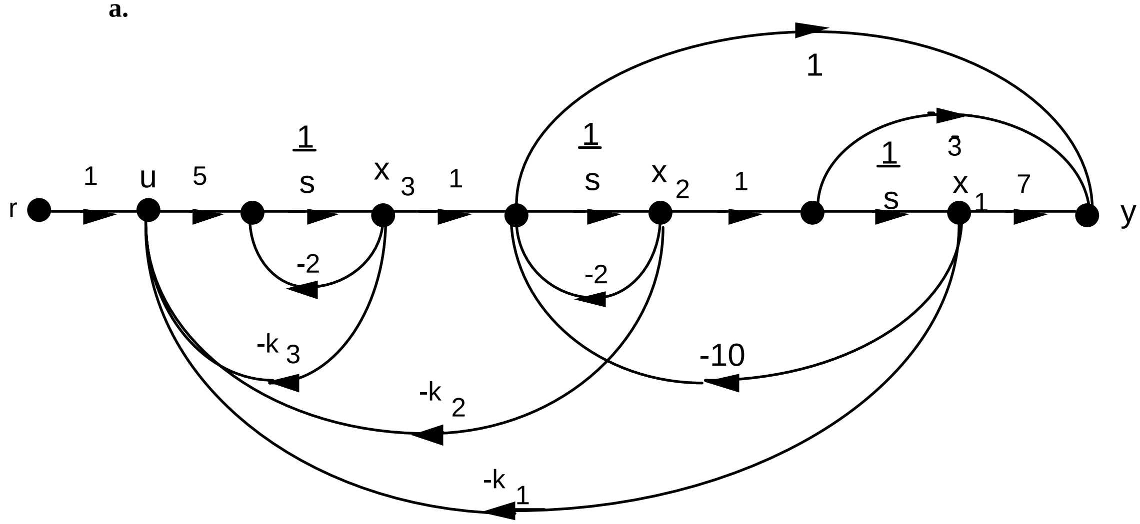
b.

$$T(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B};\ \mathbf{A}=\begin{bmatrix}0&1&0\\0&-5&1\\-k_1&-k_2&-(k_3+3)\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}0\\0\\1\end{bmatrix};\ \mathbf{C}=\begin{bmatrix}420&120&30\end{bmatrix}$$

$$T(s)=\frac{30(s+2)(s+7)}{s^3(k_3+8)s^2+(5k_3+k_2+15)s+k_1}$$

ii

a.



b.

$$T(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B};\ \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 & -(5k_3+2) \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix};\ \mathbf{C} = \begin{bmatrix} -3 & 1 & 1 \end{bmatrix}$$

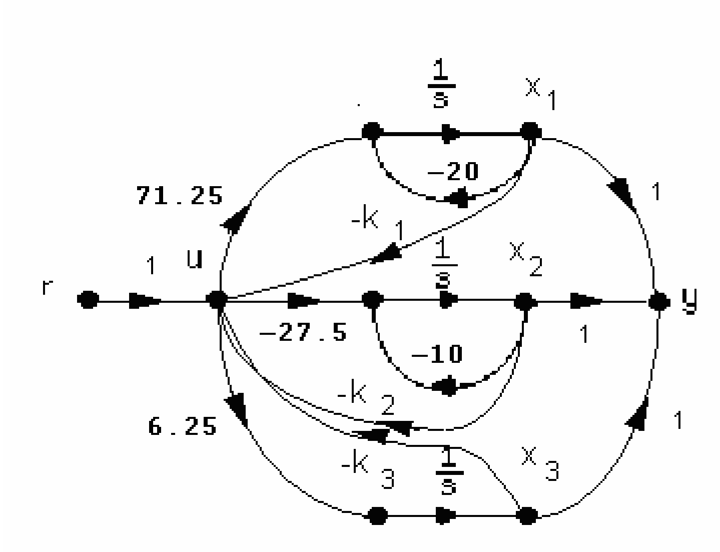
$$= \frac{5(s^2+3s+7)}{s^3+(5k_3+4)s^2+(10k_3+k_2+14)s+(50k_3+k_1+20)}$$

3.

i

a.

$$G(s) = \frac{50(s^2+7s+25)}{s(s+10)(s+20)} = \frac{6.25}{s} - \frac{27.5}{s+10} + \frac{71.25}{s+20}$$



b. Writing the state equations:

$$\dot{\mathbf{x}}_1 = -20x_1 + 71.25u$$

$$\dot{\mathbf{x}}_2 = -10x_2 - 27.5u$$

$$\dot{\mathbf{x}}_3 = 6.25u$$

But, $u = -k_1x_1 - k_2x_2 - k_3x_3 + r$. Substituting into the state equations,

$$\dot{\mathbf{x}}_1 = (-20 - 71.25k_1)x_1 - 71.25k_2x_2 - 71.25k_3x_3 + 71.25r$$

$$\dot{\mathbf{x}}_2 = 27.5k_1x_1 + (-10x_2 + 27.5k_2)x_2 + 27.5k_3x_3 - 27.5r$$

$$\dot{\mathbf{x}}_3 = -6.25k_1x_1 - 6.25k_2x_2 - 6.25k_3x_3 + 6.25r$$

Therefore, $T(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, where

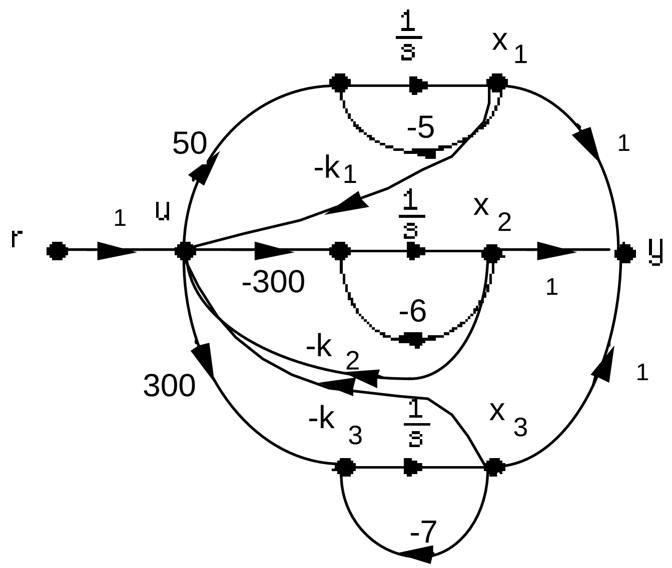
$$\mathbf{A} = \begin{bmatrix} (-20 - 71.25k_1) & -71.25k_2 & -71.25k_3 \\ 27.5k_1 & (-10x_2 + 27.5k_2) & 27.5k_3 \\ -6.25k_1 & -6.25k_2 & -6.25k_3 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 71.25 \\ -27.5 \\ 6.25 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Hence,

$$T(s) = \frac{200(s^2 + 7s + 25)}{4s^3 + (120 + 285k_1 - 110k_2 + 25k_3)s^2 + (800 + 2850k_1 - 2200k_2 + 750k_3)s + 5000k_3}$$

ii
a.

$$G(s)=\frac{50(s+3)(s+4)}{(s+5)(s+6)(s+7)}=\frac{50}{s+5}-\frac{300}{s+6}+\frac{300}{s+7}$$



b. Writing the state equations:

$$\dot{x}_1=-5x_1+50u$$

$$\dot{x}_2=-6x_2-300u$$

$$\dot{x}_3=-7x_3+300u$$

But,

$$u=-k_1x_1-k_2x_2-k_3x_3+r$$

Substituting into the state equations, collecting terms, and converting to vector-matrix form yields

$$\dot{\mathbf{x}}=\begin{bmatrix}-(5+k_1) & -50k_2 & -50k_3\\ 300k_1 & (300k_2-6) & 300k_3\\ -(300k_1+7) & -300k_2 & 300k_3\end{bmatrix}\mathbf{x}+\begin{bmatrix}50\\-300\\300\end{bmatrix}r$$

$$y=\begin{bmatrix}1 & 1 & 1\end{bmatrix}\mathbf{x}$$

Therefore, $T(s) = \boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}$, or

$$T(s)=\frac{50s^2+1750s+(6900-88200k_1)}{s^3+(300k_3-300k_2+k_1+11)s^2+2(1475k_3-750k_2+3k_1-7350k_3k_1+7350k_2k_1+15)s+300k_3(23-294k_1)}$$

4.

The plant is given by

$$G(s)=\frac{20}{(s+1)(s+3)(s+7)}=\frac{20}{s^3+11s^2+31s+21}$$

The characteristic polynomial for the plant with phase-variable state feedback is

$$s^3+(k_3+11)s^2+(k_2+31)s+(k_1+21)=0$$

The desired characteristic equation is

$$(s+53.33)(s^2+10.67s+106.45)=s^3+64s^2+675.48s+5676.98$$

based upon 10% overshoot, T_s = 0.5 second, and a third pole ten times further from the imaginary axis than the dominant poles. Comparing the two characteristic equations,

$$k_1=5655.98,\ k_2=644.48, \text{ and } k_3=53.$$

5.

a. The system in controller canonical form is:

$$\mathbf{A}=\begin{bmatrix}-a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0\\ 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix};\mathbf{B}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix};\mathbf{C}=\begin{bmatrix}c_1 & c_2 & c_3 & c_4\end{bmatrix}$$

The characteristic equation of the plant is:

$$s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$

Forming the closed-loop system by feeding back each state variable and the input to u forming

$$u = -\mathbf{Kx} + r$$

where

$$\mathbf{K} = [k_1 \ \ k_2 \ \ \ldots \ \ k_n]$$

and substituting u into the state equation, we obtain

$$\dot{\mathbf{x}} = \mathbf{Ax}+\mathbf{B}u=(\mathbf{A}-\mathbf{BK})\mathbf{x}+\mathbf{B}r$$

Forming **A - BK**:

$$\mathbf{A}-\mathbf{BK}=\begin{bmatrix}-(a_{n-1}+k_1) & -(a_{n-2}+k_2) & \cdots & -(a_1+k_{n-1}) & -(a_0+k_n)\\ 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}$$

The characteristic equation is:

$$s^n + (a_{n-1}+k_1)s^{n-1} + (a_{n-2}+k_2)\ldots + (a_1+k_{n-1})s + (a_0+k_n) = 0$$

Assuming a desired characteristic equation,

$$s_n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2}+ \ldots + d_2 s^2 + d_1 s + d_0 = 0$$

Equating coefficients,

$$d_i = a_i + k_{n-i}\ ;\ i = 0, 1, 2, \ldots n-1$$

from which

$$k_{n-i} = d_i - a_i \qquad (1)$$

b. The desired characteristic equation is

$$s^3 + 15.9s^2 + 136.08s + 413.1 = 0$$

the characteristic equation of the plant is

$$s^3 + 5s^2 + 4s + 0 = 0$$

Using Eq. (1) above, $k_{3-i} = d_i - a_i$. Therefore, $k_3 = d_0 - a_0 = 413.1 - 0 = 413.1$; $k_2 = d_1 - a_1 = 136.08 - 4 = 132.08$; $k_1 = d_2 - a_2 = 15.9 - 5 = 10.9$. Hence,

$$\mathbf{K} = \begin{bmatrix} 10.9 & 132.08 & 413.1 \end{bmatrix}$$

6.

Using Eqs. (4.39) and (4.34) to find $\zeta = 0.5169$ and $\omega_n = 7.3399$, respectively. Factoring the denominator of Eq. (4.22), the required poles are $-3.7942 \pm j6.2832$. We place the third pole at -2 to cancel the open loop zero. Multiplying the three closed-loop pole terms yields the desired characteristic equation:

$s^3 + 9.5885s^2 + 69.0516s + 107.7493 = 0$. Since $G(s) = \dfrac{100s^2 + 2200s + 4000}{s^3 + 8s^2 + 19s + 12}$, the controller canonical form is $\mathbf{A} = \begin{bmatrix} -8 & -19 & -12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\mathbf{C} = [100 \quad 2200 \quad 4000]$. The first row of $\mathbf{A}$ contains the coefficients of the characteristic equation. Thus comparing the first row of $\mathbf{A}$ to the desired characteristic equation and using the results of Problem 5, $k_1 = -(9.5885 - 8) = 1.5885$; $k_2 = -(69.0516 - 19) = 50.0516$; and $k_3 = -(107.7493 - 12) = 95.7493$.

7.

The plant is given by

$$G(s) = \frac{20(s+2)}{s(s+4)(s+6)} = \frac{20s+40}{s^3+10s^2+24s+0}$$

The characteristic polynomial for the plant with phase-variable state feedback is

$$s^3 + (k_3 + 10)s^2 + (k_2 + 24)s + (k_3 + 0)$$

The desired characteristic equation is

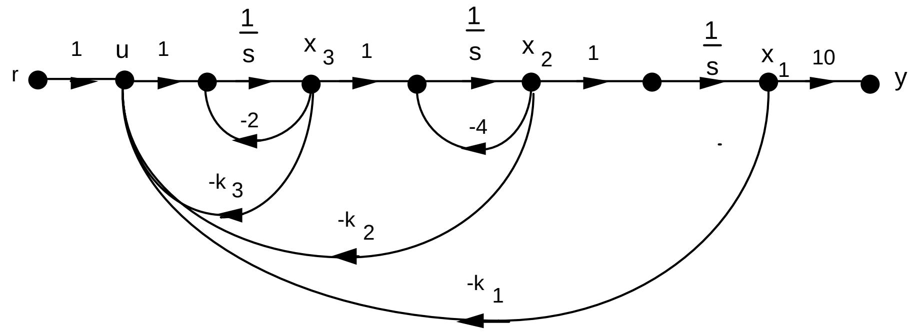
$$(s+20)(s^2 + 4s + 11.45) = s^3 + 24s^2 + 91.45s + 229$$

based upon 10% overshoot, T_s =2 seconds, and a third pole ten times further from the imaginary axis than the dominant poles. Comparing the two characteristic equations,

$$k_1 = 229,\ k_2 = 67.45, \text{and } k_3 = 14\,.$$

8.

Drawing the signal-flow diagram,



Writing the state equations yields the following **A** matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -4 & 1 \\ -k_1 & -k_2 & -[2+k_3] \end{pmatrix}$$

from which,

$$|sI - A| = s^3 + (k_3 + 6)s^2 + (4k_3 + k_2 + 8)s + k_1$$

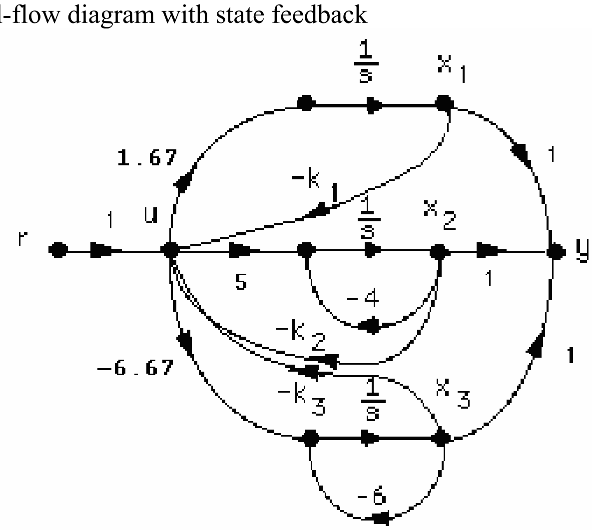
The desired characteristic equation is $(s + 80)(s^2 + 16s + 183.137) = s^3 + 96s^2 + 1463.1s + 14651$ based upon 10% overshoot, $T_s = 0.5$ second, and a third pole ten times further from the imaginary axis than the dominant poles. Comparing the two characteristic equations, , $k_1 = 14651$, $k_2 = 1095.1$, and $k_3 = 90$.

9.

Expand G(s) by partial fractions and obtain

$$G(s) = \frac{20}{s(s+4)(s+6)} = \frac{1.67}{s} + \frac{5}{s+4} - \frac{6.67}{s+6}$$

Drawing the signal-flow diagram with state feedback



Writing the state equations yields the following system matrix:

$$\mathbf{A} = \begin{bmatrix} -1.67k_1 & -1.67k_2 & -1.67k_3 \\ -5k_1 & -(5k_2+4) & -5k_3 \\ 6.67k_1 & 6.67k_2 & (6.67k_3-6) \end{bmatrix}$$

Evaluating the characteristic polynomial yields,

$$|s\mathbf{I}-\mathbf{A}| = (-6.67k_3 + 5k_2 + 1.67k_1 + 10)s^2 + (-26.68k_3 + 30k_2 + 16.7k_1 + 24)s + 40.08k_1$$

From Problem 7, the desired characteristic polynomial is

$$s^3 + 24s^2 + 91.45s + 229.$$

Equating coefficients and solving simultaneously yields

$$k_1 = 5.71,\ k_2 = -4.58, \text{ and } k_3 = -4.10.$$

10.

Writing the state equation and the controllability matrix for the system yields

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & 1 \\ -1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}u\ ;\ \mathbf{C_M} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} b_1 & -5b_1+b_2 \\ b_2 & -b_1-3b_2 \end{bmatrix}$$

The controllability matrix has a zero determinant if $b_2 = b_1$.

11.

The controllability matrix is given by Eq. (12.26) for each of the following solutions:

a.

$$\mathbf{A} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};\ \mathbf{C_M} = \begin{bmatrix} 0 & 1 & -5 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix};\ \det \mathbf{C_M} = 0\ ;\ \text{system is uncontrollable}$$

b.

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};\ \mathbf{C_M} = \begin{bmatrix} 0 & 1 & -4 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix};\ \det \mathbf{C_M} = -1;\ \text{system is controllable}$$

c.

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix};\ \mathbf{C_M} = \begin{bmatrix} 0 & 2 & -7 \\ 2 & 1 & -3 \\ 1 & -3 & 9 \end{bmatrix};\ \det \mathbf{C_M} = 7;\ \text{system is controllable}$$

d.

$$\mathbf{A}=\begin{bmatrix}-4 & 1 & 0\\ 0 & 0 & 1\\ -5 & 0 & -3\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}1\\0\\1\end{bmatrix};\ \mathbf{C_M}=\begin{bmatrix}1 & -4 & 17\\ 0 & 1 & -8\\ 1 & -8 & 44\end{bmatrix};\ \det\ \mathbf{C_M}=-5;\ \text{system is controllable}$$

e.

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}1\\-2\end{bmatrix};\ \mathbf{C_M}=\begin{bmatrix}1 & -2\\ -2 & 4\end{bmatrix};\ \det\ \mathbf{C_M}=0;\ \text{system is uncontrollable}$$

f.

$$\mathbf{A}=\begin{bmatrix}-4 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -6\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}1\\0\\1\end{bmatrix};\ \mathbf{C_M}=\begin{bmatrix}1 & -4 & 16\\ 0 & 0 & 0\\ 1 & -6 & 36\end{bmatrix};\ \det\ \mathbf{C_M}=0;\ \text{system is uncontrollable}$$

This system can also be determined uncontrollable by inspection.

12.

Program:

```
'(d)'
A=[-4 1 0;0 0 1;-5 0 -3]
B=[1;0;1]
Cm=ctrb(A,B)
Rank=rank(Cm)
pause
'(f)'
A=[-4 0 0;0 -5 0;0 0 -6]
B=[1;0;1]
Cm=ctrb(A,B)
Rank=rank(Cm)
```

Computer response:

```
ans =

(d)

A =

    -4     1     0
     0     0     1
    -5     0    -3

B =

     1
     0
     1

Cm =

     1    -4    17
     0     1    -8
     1    -8    44

Rank =

     3
```

```
ans =

(f)

A =

    -4     0     0
     0    -5     0
     0     0    -6
B =

     1
     0
     1

Cm =

     1    -4    16
     0     0     0
     1    -6    36

Rank =

     2
```

13.

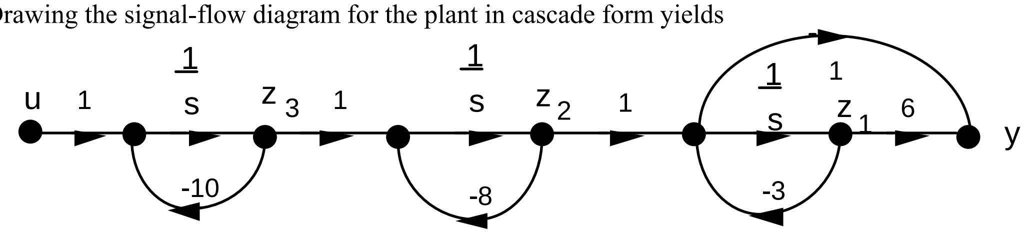
From Eq. (12.46) we write the controller canonical form: $\mathbf{A_{cc}} = \begin{bmatrix} -8 & -17 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$; $\mathbf{B_{cc}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

The controllability matrices are found using Eq. (12.35). For the original system of Eq. (12.44),

$\mathbf{C_{Mz}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$. For the controller canonical form, $\mathbf{C_{Mcc}} = \begin{bmatrix} 1 & -8 & 47 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix}$. The transformation matrix is, $\mathbf{P} = \mathbf{C_{Mz}C_{Mcc}}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 5 \\ 1 & 7 & 10 \end{bmatrix}$. Comparing the first row of $\mathbf{A_{cc}}$ with the desired characteristic equation, Eq. (12.50), $\mathbf{K_{cc}} = [\,-2 \quad -4 \quad 10\,]$. Transforming back to the original system, $\mathbf{K_z} = \mathbf{K_{cc}P}^{-1} = [-20 \quad 10 \quad -2]$.

14.

Drawing the signal-flow diagram for the plant in cascade form yields



Writing the **A** and **B** matrices for the z system,

Az			**Bz**
-3	1	0	0

0	-8	1	0
0	0	-10	1

Writing the **A** and **B** matrices for the x (phase-variable) system,

Ax			**Bx**	
0	1	0	0	
0	0	1	0	Phase-Variable Form
-240	-134	-21	1	

From the phase variable from, the characteristic polynomial is $s^3 + 21s^2 + 134s + 240$.

Finding the controllability matrices and their determinants for the z and x systems shows that there is controllability,

CMz	Controllability Matrix of z		**CMx**	Controllability Matrix of x	
0	0	1	0	0	1
0	1	-18	0	1	-21
1	-10	100	1	-21	307
Det(CMz)	-1		**Det(CMx)**	-1	

Using Eq. (12.39), the transformation matrix **P** and its inverse are found to be

P	Transformation Matrix z=Px		**PINV**		
1	0	0	1.00	0.00	0.00
3	1	0	-3.00	1.00	0.00
24	11	1	9.00	-11.00	1.00

Using the given transient requirements, and placing the third closed-loop pole over the zero at -6 yields the following desired closed-loop characteristic polynomial:

$$(s^2 + 8s + 45.78)(s + 6) = s^3 + 14s^2 + 93.78s + 274.68$$

Using the phase-variable system with state feedback the characteristic polynomial is

$$s^3 + (k_3 + 21)s^2 + (k_2 + 134)s + (k_1 + 240)$$

Equating the two characteristic polynomials yields the state feedback vector for the x system as

Kx	Controller for x	
34.68	-40.22	-7

Using Eq. (12.42),

Kz	Controller for z	
92.34	36.78	-7

15.

Program:

```
A=[-3 1 0;0 -8 1;0 0 -10];        %Generate system matrix A
B=[0;0;1];                        %Generate input coupling matrix B
```

```
C=[3 1 0];                          %Generate output coupling matrix C
D=0;                                %Generate matrix D
Po=10;                              %Input desired percent overshoot
Ts=1;                               %Input desired settling time
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
                                    %Calculate required damping ratio
wn=4/(z*Ts);                        %Calculate required natural
                                    %frequency
[num,den]=ord2(wn,z);               %Produce a second-order system that
                                    %meets transient requirements
r=roots(den);                       %Use denominator to specify
                                    %dominant poles
poles=[r(1) r(2) -6];               %Specify pole placement for all
                                    %poles.
                                    %A few tries at the the third-pole
                                    %value shows T(s) with a closed-
                                    %loop zero at -7.
                                    %Thus, choose the third pole to
                                    %cancel this zero.
K=acker(A,B,poles)                  %Calculate controller gains in z-
                                    %system
Anew=A-B*K;                         %Form compensated A matrix
Bnew=B;                             %Form compensated B matrix
Cnew=C;                             %Form compensated C matrix
Dnew=D;                             %Form compensated D matrix
[numt,dent]=ss2tf(Anew,Bnew,Cnew,Dnew);
                                    %Form T(s)
'T(s)'                              %Display label
T=tf(numt,dent)                     %Display T(s)
poles=pole(T)                       %Display poles of T(s)
```

Computer response:

```
K =

   92.3531   36.7844   -7.0000

ans =

T(s)

Transfer function:
   -3.553e-015 s^2 + s + 6
------------------------------
s^3 + 14 s^2 + 93.78 s + 274.7

poles =

  -4.0000 + 5.4575i
  -4.0000 - 5.4575i
  -6.0000
```

16.

Expanding by partial fractions,

$$G(s)=\frac{(s+6)}{(s+3)(s+8)(s+10)}=\frac{0.085714}{(s+3)}-\frac{0.2}{(s+8)}-\frac{0.28571}{(s+10)}$$

Writing the **A** and **B** matrices for the z system with k_i's set to zero,

Az			**Bz**
-3	0	0	0.085714

0	-8	0	0.2
0	0	-10	-0.28571

Writing the **A** and **B** matrices for the x (phase-variable) system,

Ax			**Bx**	
0	1	0	0	
0	0	1	0	Phase-Variable Form
-240	-134	-21	1	

From the phase variable from, the characteristic polynomial is $s^3 + 21s^2 + 134s + 240$.

Finding the controllability matrices and their determinants for the z and x systems shows that there is controllability,

CMz	Controllability Matrix of z		**CMx**	Controllability Matrix of x	
0.085714	-0.257142	0.771426	0	0	1
0.2	-1.6	12.8	0	1	-21
-0.28571	2.8571	-28.571	1	-21	307
Det(CMz)	0.342850857		**Det(CMx)**	-1	

Using Eq. (12.39), the transformation matrix **P** and its inverse are found to be

P	Transformation Matrix z=Px		**PINV**		
6.85712	1.542852	0.085714	0.33	-0.50	-0.25
6	2.6	0.2	-1.00	4.00	2.50
-6.85704	-3.14281	-0.28571	3.00	-32.00	-25.00

Using the given transient requirements, and placing the third closed-loop pole over the zero at -6 yields the following desired closed-loop characteristic polynomial:

$$(s^2 + 8s + 45.78)(s + 6) = s^3 + 14s^2 + 93.78s + 274.68$$

Using the phase-variable system with state feedback the characteristic polynomial is

$$s^3 + (k_3 + 21)s^2 + (k_2 + 134)s + (k_1 + 240)$$

Equating the two characteristic polynomials yields the state feedback vector for the x system as

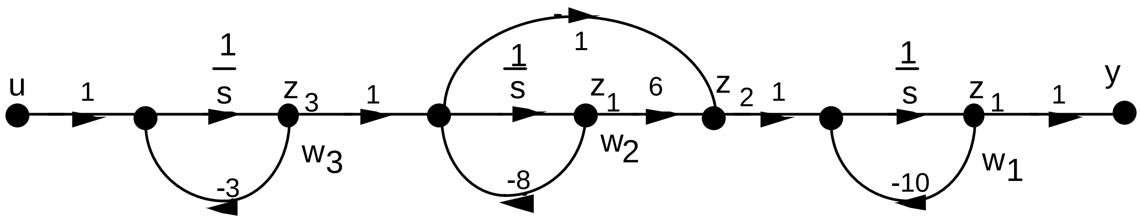
Kx	Controller for x	
34.7062	-40.2156	-7

Using Eq. (12.42),

Kz	Controller for z	
30.78443595	45.7845	65.78543678

17.

Draw signal-flow diagram showing state variables, **z**, at the output of each subsystem and the state variables, **w**, at the output of the integrators.



Recognizing that $z_2 = 6w_2 - 8w_2 + w_3 = -2w_2 + w_3$, we can write the state equations for **w** as

$$\dot{\mathbf{w}} = \begin{bmatrix} -10 & -2 & 1 \\ 0 & -8 & 1 \\ 0 & 0 & -3 \end{bmatrix}\mathbf{w} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{w}$$

Writing the relationship between **z** and **w** yileds

$$\mathbf{z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{w} = \mathbf{P}^{-1}\mathbf{w}$$

Thus

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

Converting the state equations in **w** to state equations in **z**, we use Eqs. (5.87) and obtain the **A** matrix and **B** vector as

Az			**Bz**
-10	1	0	0
0	-8	3	1
0	0	-3	1

Writing the **A** and **B** matrices for the x (phase-variable) system,

Ax			**Bx**	
0	1	0	0	
0	0	1	0	Phase-Variable Form
-240	-134	-21	1	

From the phase variable from, the characteristic polynomial is $s^3 + 21s^2 + 134s + 240$

Finding the controllability matrices and their determinants for the z and x systems shows that there is controllability,

CMz	Controllability Matrix of z		**CMx**	Controllability Matrix of x	
0	1	-15	0	0	1
1	-5	31	0	1	-21
1	-3	9	1	-21	307
Det(CMz)	-8		**Det(CMx)**	-1	

Using Eq. (12.39), the transformation matrix **P** and its inverse are found to be

P	Transformation Matrix z=Px		**PINV**		
6	1	0	-0.25	-0.13	0.13
60	16	1	2.50	0.75	-0.75
80	18	1	-25.00	-3.50	4.50

Using the given transient requirements, and placing the third closed-loop pole over the zero at -6 yields the following desired closed-loop characteristic polynomial:

$$(s^2 + 8s + 45.78)(s + 6) = s^3 + 14s^2 + 93.78s + 274.68$$

Using the phase-variable system with state feedback the characteristic polynomial is

$$s^3 + (k_3 + 21)s^2 + (k_2 + 134)s + (k_1 + 240)$$

Equating the two characteristic polynomials yields the state feedback vector for the x system as

Kx	Controller for x	
34.68	-40.22	-7

Using Eq. (12.42),

Kz	Controller for z	
65.78	-10	3

18.

Using Eqs. (4.39) and (4.34) to find $\zeta = 0.5169$ and $\omega_n = 18.3498$ respectively. Factoring the denominator of Eq. (4.22), the required poles are $-9.4856 \pm j15.708$. We place the third pole 10 times further at -94.856. Multiplying the three closed-loop pole terms yields the desired characteristic equation: $s^3 + 114s^2 + 2136s + 31940 = 0$. Representing the plant in parallel form:

$$\mathbf{A_{par}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix};\ \mathbf{B_{par}} = \begin{bmatrix} 3.125 \\ -6.25 \\ 3.125 \end{bmatrix};\ \mathbf{C_{par}} = [1 \quad 1 \quad 1].$$ Using Eq. (12.26),

$$\mathbf{C_{Mpar}} = \begin{bmatrix} 3.125 & 0 & 0 \\ -6.25 & 25 & -100 \\ 3.125 & -25 & 200 \end{bmatrix}$$, which is controllable since the determinant is 7812.5. Since

$$G(s) = \frac{100}{s^3 + 12s^2 + 32s}$$, the controller canonical form is $\mathbf{A_{cc}} = \begin{bmatrix} -12 & -32 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$; $\mathbf{B_{cc}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$;

C = [0 0 100]. Using Eq. (12.26), $\mathbf{C_{Mcc}} = \begin{bmatrix} 1 & -12 & 112 \\ 0 & 1 & -12 \\ 0 & 0 & 1 \end{bmatrix}$, which is controllable since the determinant is 1. The first row of $\mathbf{A_{cc}}$ contains the coefficients of the characteristic equation. Comparing the first row of $\mathbf{A_{cc}}$ to the desired characteristic equation and using the results of Problem 5, $(12 + k_1) = 114$; $(32 + k_2)= 2136$; and $(0 + k_3)= 31940$. Hence $\mathbf{K_{cc}}$ =[31940 2104 102]. The transformation matrix is, $\mathbf{P} = \mathbf{C_{Mpar}}\mathbf{C_{Mcc}}^{-1} = \begin{bmatrix} 100 & 37.5 & 3.125 \\ 0 & -50 & -6.25 \\ 0 & 12.5 & 3.125 \end{bmatrix}$. Transforming back to the original system, $\mathbf{K_{par}} = \mathbf{K_{cc}}\mathbf{P}^{-1}$ = [319.396 251.5184 216.2255].

19.

$$G(s) = \frac{1}{s(s+3)(s+7)} = \frac{1}{s^3+10s^2+21s+0}$$

Writing the **A** and **C** matrices for the observer canonical system,

Az

-10	1	0
-21	0	1
0	0	0

Cz

1	0	0

The characteristic polynomial is $s^3 + 10s^2 + 21s + 0$.
Now check observability by calculating the observability matrix and its determinant.

OMz	Observability Matrix of z	
1	0	0
-10	1	0
79	-10	1

Det(OMz) 1

Using the given transient requirements, and placing the third closed-loop pole 10 times further from the imaginary axis than the dominant poles yields the following desired characteristic polynomial:

$$(s+300)(s^2+60s+5625) = s^3+360s^2+23625s+1687500$$

Equating this polynomial to Eq. (12.67), yields the observer gains as:

Lz	Observer for z
350	
23604	
1687500	

20.

Using Eqs. (4.39) and (4.34) to find $\zeta = 0.5912$ and $\omega_n = 19.4753$ respectively. Factoring the denominator of Eq. (4.22), the required poles are -11.513 ± j15.708. We place the third pole 20 times further at –230.26. Multiplying the three closed-loop pole terms yields the desired characteristic equation: $s^3 + 253.28s^2 + 5681.19s + 87334.19 = 0$.

Representing the plant in observer canonical form: $\mathbf{A_{oc}} = \begin{bmatrix} -20 & 1 & 0 \\ -108 & 0 & 1 \\ -144 & 0 & 0 \end{bmatrix}$; $\mathbf{B_{oc}} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$;

$\mathbf{C_{oc}} = [1 \quad 0 \quad 0]$. The first column of $\mathbf{A_{oc}}$ contains the coefficients of the characteristic equation. Comparing the first column of $\mathbf{A_{oc}}$ to the desired characteristic equation and using Eq. (12.67), $l_1 = 253.28 - 20 = 233.28$; $l_2 = 5681.19 - 108 = 5573.19$; and $l_3 = 87334.19 - 144 = 87190.19$. Hence, $\mathbf{L_{oc}} = [233.28 \;\; 5573.19 \;\; 87190.19]^T$.

21.

The **A**, **L**, and **C** matrices for the phase-variable system are:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -21 & -10 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$$

Hence,

$$|\lambda - (\mathbf{A} - \mathbf{LC})| = \begin{vmatrix} \lambda + l_1 & -1 & 0 \\ l_2 & \lambda & -1 \\ l_3 & 21 & \lambda + 10 \end{vmatrix}$$

or

$$|\lambda - (\mathbf{A} - \mathbf{LC})| = \lambda^3 + (10 + l_1)\lambda^2 + (21 + 10l_1 + l_2)\lambda + (21l_1 + 10l_2 + l_3)$$

From Problem 19, the desired characteristic polynomial is $\lambda^3 + 360\lambda^2 + 23625\lambda + 1687500$. Equating coefficients yields:

$$10 + l_1 = 360;\ (21 + 10l_1 + l_2) = 23625;\ (21l_1 + 10l_2 + l_3) = 1687500$$

Solving successively,

$$l_1 = 350;\ l_2 = 20104;\ l_3 = 1479110$$

22.

The **A**, **L**, and **C** matrices for the phase-variable system are:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -45 & -14 \end{bmatrix}; \ \mathbf{C} = \begin{bmatrix} 2 & 1 \end{bmatrix}; \ \mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

Hence,

$$|\lambda - (\mathbf{A} - \mathbf{LC})| = \left| \begin{bmatrix} \lambda + 2l_1 & l_1 - 1 \\ 2l_2 + 45 & l_2 + \lambda + 14 \end{bmatrix} \right|$$

or

$$\lambda^2 + (2l_1 + l_2 + 14)\lambda + (2l_2 - 17l_1 + 45)$$

From the problem statement, the desired characteristic polynomial is $\lambda^2 + 144\lambda + 14400$.

Equating coefficients yields,

$$(2l_1 + l_2 + 14) = 144; \ (2l_2 - 17l_1 + 45) = 14400$$

Solving simultaneously,

$$l_1 = -671.2; \ l_2 = 1472.4$$

23.

The **A** matrix for each part is given in the solution to Problem 11. Each observability matrix is calculated from Eq. (12.79).

a.

$$A = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}; \ C = (5,5,5); \ O_M = \begin{pmatrix} 5 & 5 & 5 \\ -10 & -10 & -10 \\ 20 & 20 & 20 \end{pmatrix}; \ |O_M| = 0; \text{ unobservable}$$

b.

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}; \ C = (5,0,5); \ O_M = \begin{pmatrix} 5 & 0 & 5 \\ -10 & 5 & -15 \\ 20 & -20 & 45 \end{pmatrix}; \ |O_M| = 125; \text{ observable}$$

c.

$$A = \begin{pmatrix} -4 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix}; \ C = (1,0,0); \ O_M = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 16 & -4 & 1 \end{pmatrix}; \ |O_M| = 1; \text{ observable}$$

d.

$$A = \begin{pmatrix} -4 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & 0 & -3 \end{pmatrix}; \ C = (1,0,0); \ O_M = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 16 & -4 & 1 \end{pmatrix}; \ |O_M| = 1; \text{ observable}$$

e.

$$A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}; \ C = (1,0); \ O_M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \ |O_M| = 1; \text{ observable}$$

f.

$$A = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{pmatrix}; \ C = (1,1,1); \ O_M = \begin{pmatrix} 1 & 1 & 1 \\ -4 & -5 & -6 \\ 16 & 25 & 36 \end{pmatrix}; \ |O_M| = -2; \text{ observable}$$

24.

Program:

```
'(a)'
A=[-2 0 1;0 -2 0;0 0 -3]           %Form compensated A matrix
C=[5 5 5]                        %Form compensated C matrix
Om=obsv(A,C)                     %Form observability matrix
Rank=rank(Om)                    %Find rank of observability
                                 %matrix
'(f)'
A=[-4 0 0;0 -5 0;0 0 -6]         %Form compensated A matrix
```

```
C=[1 1 1]                       %Form compensated C matrix
Om=obsv(A,C)                    %Form observability matrix
Rank=rank(Om)                   %Find rank of observability
```

Computer response:

```
ans =

(a)

A =

    -2     0     1
     0    -2     0
     0     0    -3

C =

     5     5     5

Om =

     5     5     5
   -10   -10   -10
    20    20    20

Rank =

     1

ans =

(f)

A =

    -4     0     0
     0    -5     0
     0     0    -6

C =

     1     1     1

Om =

     1     1     1
    -4    -5    -6
    16    25    36

Rank =

     3
```

25.

Representing the system in state space yields

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \; ; \quad y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \mathbf{x}$$

Using Eq. (12.79),

$$\mathbf{O_M} = \begin{bmatrix} c_1 & c_2 \\ -c_2 & (c_1 - 2c_2) \end{bmatrix} \text{ and det } \mathbf{O_M} = c_1^2 - 2c_1 c_2 + c_2^2$$

Thus, the system is unobservable if $c_1 = c_2$.

26.

The **A** and **C** matrices for the system represented in cascade form is

Az		
-20	1	0
0	-13	1
0	0	-5

Cz		
1	0	0

The characteristic polynomial found from the transfer function of the plant is

$$s^3 + 38s^2 + 425s + 1300$$

From this characteristic polynomial, we can write observer canonical form of the state equations. The **A** and **C** matrices of the observer canonical form are given below as

Ax		
-38	1	0
-425	0	1
-1300	0	0

Cx		
1	0	0

To test observability, we write the observability matrices for both systems and show that both observability matrices have non zero determinants. Using Eq. (12.79),

OMz	Observability Matrix of z		**OMx**	Observability Matrix of x	
1	0	0	1	0	0
-20	1	0	-38	1	0
400	-33	1	1019	-38	1
Det(OMz)	1		**Det(OMx)**	1	

Using Eq. (12.89), we obtain the transformation matrix, **P**, and its inverse as

P	Transformation Matrix z=Px		**PINV**		
1	0	0	1.00	0.00	0.00
-18	1	0	18.00	1.00	0.00
25	-5	1	65.00	5.00	1.00

Using the characteristic polynomial given in the problem statement, the plant's characteristic equation, and Eq. (12.67), the observer for the observer canonical system is

Lx	Observer for x
562	
39575	
1498700	

Using Eq. (12.92), the observer for the cascade system is found to be

Lz	Observer for z
562	
29459	
1314875	

27.

Program:
```
A=[-20 1 0;0 -13 1;0 0 -5]
B=[0;0;1]
C=[1 0 0]
D=0
poles=roots([1 600 40000 1500000])
L=acker(A',C',poles);
'L'
L'
```

Computer response:
```
A =

   -20     1     0
     0   -13     1
     0     0    -5

B =

     0
     0
     1

C =

     1     0     0

D =

     0

poles =

  1.0e+002 *

  -5.2985
  -0.3508 + 0.4001i
  -0.3508 - 0.4001i


ans =

L


ans =

  1.0e+006 *

    0.0006
    0.0295
    1.3149
```

28.

Expanding the plant by partial fractions, we obtain

$$G(s) = \frac{1}{(s+5)(s+13)(s+20)} = \frac{0.008333}{(s+5)} - \frac{0.017857}{(s+13)} + \frac{0.0095238}{(s+20)}$$

The **A** and **C** matrices for the system represented in parallel form is

Az		
-5	0	0
0	-13	0
0	0	-20

Cz		
1	1	1

The characteristic polynomial found from the transfer function of the plant is

$$s^3 + 38s^2 + 425s + 1300$$

From this characteristic polynomial, we can write the observer canonical form of the state equations. The **A** and **C** matrices of the observer canonical form are given below as

Ax		
-38	1	0
-425	0	1
-1300	0	0

Cx		
1	0	0

To test observability, we write the observability matrices for both systems and show that both observability matrices have non zero determinants. Using Eq. (12.79),

OMz	Observability Matrix of z		**OMx**	Observability Matrix of x	
1	1	1	1	0	0
-5	-13	-20	-38	1	0
25	169	400	1019	-38	1
Det(OMz)	-840		**Det(OMx)**	1	

Using Eq. (12.89), we obtain the transformation matrix, **P**, and its inverse as

P	Transformation Matrix z=Px		**PINV**		
0.2083333	-0.04166667	0.008333333	1.00	1.00	1.00
-3.017857	0.232142857	-0.01785714	33.00	25.00	18.00
3.8095238	-0.19047619	0.00952381	260.00	100.00	65.00

Using the characteristic polynomial given in the problem statement, the plant's characteristic equation, and Eq. (12.67), the observer for the observer canonical system is

Lx	Observer for x
562	

39575
1498700

Using Eq. (12.92), the observer for the parallel system is found to be

Lz	Observer for z
10957.29167	
-19271.4821	
8876.190476	

29.

Use Eqs. (4.39) and (4.42) to find $\zeta = 0.5912$ and $\omega_n = 135.328$ respectively. Factoring the denominator of Eq. (4.22), the required poles are $-80 \pm j109.15$. We place the third pole 10 times further at -800. Multiplying the three closed-loop pole terms yields the desired characteristic equation:

$s^3 + 960s^2 + 146313.746s + 14650996.915 = 0$.

Since $G(s) = \dfrac{50}{s^3 + 18s^2 + 99s + 162}$, the plant in observer canonical form is: $\mathbf{A_{oc}} = \begin{bmatrix} -18 & 1 & 0 \\ -99 & 0 & 1 \\ -162 & 0 & 0 \end{bmatrix}$;

$\mathbf{B_{oc}} = \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}$; $\mathbf{C_{oc}} = [1 \quad 0 \quad 0]$. Using Eq. (12.79), $\mathbf{O_{Moc}} = \begin{bmatrix} 1 & 0 & 0 \\ -18 & 1 & 0 \\ 225 & -18 & 1 \end{bmatrix}$, which is observable since the determinant is 1. Since $G(s) = \dfrac{50}{s^3 + 18s^2 + 99s + 162}$, the phase-variable form is

$\mathbf{A_{pv}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -162 & -99 & -18 \end{bmatrix}$; $\mathbf{B_{pv}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; $\mathbf{C} = [50 \quad 0 \quad 0]$. Using Eq. (12.79),

$\mathbf{O_{Mpv}} = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix}$, which is observable since the determinant is 125000. The first column of $\mathbf{A_{oc}}$ contains the negatives values of the coefficients of the characteristic equation. Comparing the first column of $\mathbf{A_{oc}}$ to the desired characteristic equation and using Eq. (12.67), $l_1 = 960-18 = 942$; $l_2 = 146313.746-99 = 146214.746$; and $l_3 = 14650996.915-162 = 14650834.915$. Hence, $\mathbf{L_{oc}} = [942 \quad 146214.746 \quad 14650834.915]$. The transformation matrix is,

$$\mathbf{P} = \mathbf{O_{Mpv}}^{-1}\mathbf{O_{Moc}} = \begin{bmatrix} 0.02 & 0 & 0 \\ -0.36 & 0.02 & 0 \\ 4.5 & -0.36 & 0.02 \end{bmatrix}$$

Transforming back to the original system, $\mathbf{L_{pv}} = \mathbf{PL_{oc}} = [18.84 \quad 2585.175 \quad 244618.39]^T$.

30.

The open-loop transfer function of the plant is $T(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \frac{s+2}{s^2-s-2}$.

Using Eqs. (12.115), the closed-loop state equations with integral control is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_N \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -k_1 & -k_2+2 & k_e \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_N \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r \quad ; \quad y = (1, 1, 0) \begin{pmatrix} x_1 \\ x_2 \\ x_N \end{pmatrix}$$

The characteristic polynomial is

$$s^3 + (k_2-1)s^2 + (k_2 + k_1 + k_e - 2)s + 2k_e$$

The desired characteristic polynomial is calculated from the desired transient response stated in the problem. Also, the third pole will be placed to cancel the zero at -2. Hence, the desired characteristic polynomial is

$$(s + 2)(s^2 + 16s + 183.137) = s^3 + 18s^2 + 215.14s + 366.27$$

Equating coefficients of the characteristic polynomials yields,

$$k_e = 183.135,\ k_2 = 19,\ k_1 = 15.005$$

31.

The open-loop transfer function of the plant is $T(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \frac{s+3}{s^2+7s+10}$.

Using Eqs. (12.115), the closed-loop state equations with integral control is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_N \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ -k_1 & -(5+k_2) & k_e \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_N \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r \quad ; \quad y = (1, 1, 0) \begin{pmatrix} x_1 \\ x_2 \\ x_N \end{pmatrix}$$

The characteristic polynomial is

$$s^3 + (k_2 + 7)s^2 + (2k_2 + k_1 + k_e + 10)s + 3k_e$$

The desired characteristic polynomial is calculated from the desired transient response stated in the problem. Also, the third pole will be placed to cancel the zero at -3. Hence, the desired characteristic polynomial is

$$(s + 3)(s^2 + 16s + 183.14) = s^3 + 19s^2 + 231.14s + 549.41$$

Equating coefficients of the characteristic polynomials yields,

$$k_e = 183.137,\ k_2 = 12,\ k_1 = 14.003$$

SOLUTIONS TO DESIGN PROBLEMS

32.

Writing the **A** and **B** matrices for (G(s) represented in phase-variables form,

A			**B**
0	1	0	0
0	0	1	0
1.30E+06	4551	-286	10

From the phase-variable from, the characteristic polynomial is $s^3 + 286s^2 - 4551s - 1301586$.
Finding the controllability matrix and it's determinant shows that there is controllability,

CM		
0	0	10
0	10	-2860
10	-2860	863470
Det(CM)	-1000	

Using the given transient requirements, and arbitrarily placing the third closed-loop pole more than 5 times further than the dominant pair at -50 yields the following desired closed-loop characteristic polynomial:

$$(s^2 + 16s + 134.384)(s + 50) = s^3 + 66s^2 + 934.4s + 6719.2$$

Using the phase-variable system with state feedback the characteristic polynomial is

$$s^3 + (k_3 + 286)s^2 + (k_2 - 4551)s + (k_1 - 1301586)$$

Equating the two characteristic polynomials yields the state feedback vector for the phase-variable system as

K		
1308305.2	5485.4	-220

33.

Controller design:

The transfer function for the plant is

$$G(s) = \frac{5}{(s+0.4)(s+0.8)(s+5)} = \frac{5}{s^3+6.2s^2+6.32s+1.6}$$

The characteristic polynomial for the plant with phase-variable state feedback is

$$s^3 + (6.2 + k_3)s^2 + (6.32 + k_2)s + (1.6 + k_1)$$

Using the given transient response of 5% overshoot and $T_s = 10$ minutes, and placing the third pole ten times further from the imaginary axis than the dominant pair, the desired characteristic equation is

$$(s + 4)(s^2 + 0.8s + 0.336) = s^3 + 4.8s^2 + 3.536s + 1.344.$$

Comparing the two characteristic equations, $k_1 = -0.256$, $k_2 = -2.784$, and $k_3 = -1.4$.

Observer design:

The **A** and **C** matrices for the system represented in phase-variable form is

Az

0	1	0
0	0	1
-1.6	-6.32	-6.2

Cz

5	0	0

The characteristic polynomial found from the transfer function of the plant is

$$s^3 + 6.2s^2 + 6.32s + 1.6$$

From this characteristic polynomial, we can write the dual phase-variable form of the state equations. The **A** and **C** matrices of the dual phase-variable form are given below as

Ax

-6.2	1	0
-6.32	0	1
-1.6	0	0

Cx

1	0	0

To test observability, we write the observability matrices for both systems and show that both observability matrices have nonzero determinants. Using Eq. (12.79),

OMz	Observability Matrix of z		**OMx**	Observability Matrix of x	
5	0	0	1	0	0
0	5	0	-6.2	1	0
0	0	5	32.12	-6.2	1
Det(OMz)	125		**Det(OMx)**	1	

Using Eq. (12.89), we obtain the transformation matrix, **P**, and its inverse as

P	Transformation Matrix z=Px		**PINV**		
0.2	0	0	5.00	0.00	0.00
-1.24	0.2	0	31.00	5.00	0.00
6.424	-1.24	0.2	31.60	31.00	5.00

Using the characteristic polynomial given in the problem statement, the observer for the dual phase-variable system is

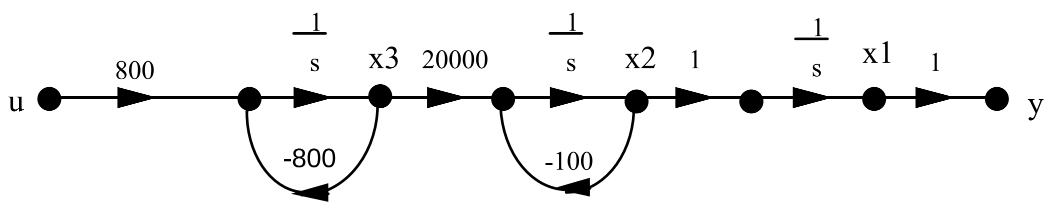
Lx
41.8
347.28
1342.4

Using Eq. (12.92), the observer for the cascade system is found to be

Lz
8.36
17.624
106.376

34.

a. Using the following signal-flow graph,



the plant is represented in state space with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -100 & 20000 \\ 0 & 0 & -800 \end{bmatrix}; \ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 800 \end{bmatrix}; \text{ and } \mathbf{C} = [1 \quad 0 \quad 0].$$

Using Eq. (12.26),

$$\mathbf{C_M} = \begin{bmatrix} 0 & 0 & 1.6\text{E}07 \\ 0 & 1.6\text{E}07 & -1.44E10 \\ 800 & -6.4\text{E}05 & 5.12\text{E}08 \end{bmatrix}$$

The system is controllable since the determinant of $\mathbf{C_M}$ = -2.04e^{17}. Use Eqs. (4.39) and (4.42) to find ζ = 0.5912 and ω_n = 135.3283 respectively. Factoring the denominator of Eq. (4.22), the required poles are -80 ± j109.15. Place the third pole 10 times farther at = 800. Multiplying the three closed-loop pole terms yields the desired characteristic equation

$$s^3 + 960s^2 + \ 1.463\text{E}05s + 1.4651\text{E}07 = 0.$$

Since the plant's characteristic equation is $s^3 + 900s^2 + 80000s$, we write the plant in controller canonical form as

$$\mathbf{A_{cc}} = \begin{bmatrix} -900 & -80000 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \ \mathbf{B_{cc}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \text{ and } \mathbf{C_{cc}} = [0 \quad 0 \quad 1.6\text{E}07]$$

The controllability matrix for controllable canonical form is

$$\mathbf{C_{Mcc}} = \begin{bmatrix} 1 & -900 & 730000 \\ 0 & 1 & -900 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing the first row of $\mathbf{A_{cc}}$ to the desired characteristic equation and using the results of Problem 5, k_1 = -(900 - 960) = 60; k_2 = -(80000 - 1.463E05) = 66300; and k_3 = -(0 - 1.465E07) = 1.465E07. Hence.

$$\mathbf{K_{cc}} = [60 \quad 66300 \quad 1.465E07]$$

The transformation matrix is,

$$\mathbf{P} = \mathbf{C_M C_{Mcc}}^{-1} = \begin{bmatrix} 0 & 150 & 1.6E07 \\ 0 & 1.6E07 & 0 \\ 800 & 8E04 & 0 \end{bmatrix}$$

Transforming back to the original system,

$$\mathbf{K} = \mathbf{K_{cc}P}^{-1} = [9.1569E\text{-}01 \quad 3.7696E\text{-}03 \quad 7.5E\text{-}02]$$

The controller compensated system is

$$\mathbf{A - BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -100 & 20000 \\ -732.55 & -3.0157 & -860 \end{bmatrix}; \ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 800 \end{bmatrix}; \ \mathbf{C} = [1 \quad 0 \quad 0]$$

b. To evaluate the steady-state error, use Eq. (7.89) where

$$\mathbf{A - BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -100 & 20000 \\ -732.55 & -3.0157 & -860 \end{bmatrix}$$

is the system matrix. Thus,

$$(sI - [A - BK])^{-1} =$$

$$\frac{1}{s^3 + 960s^2 + 1.4631\times 10^5 s + 14651040} \begin{pmatrix} s^2 + 960s + 1.4631\times 10^5 & s + 860 & 20000 \\ -14651040 & s^2 + 860s & 20000s \\ -732.55s - 73255 & -3.0157s - 732.55 & s^2 + 100s \end{pmatrix}$$

The steady-state error is given by

$$sR(s)[1 - C(sI - [A - BK])^{-1}B] \text{ as } s\text{->}0$$

For a step input, R(s) = 1/s. Since

$$1 - C(sI - [A - BK])^{-1}B = 1 - \frac{1}{s^3 + 960s^2 + 1.4631\times 10^5 s + 14651040} \cdot 16000000$$

for a step input e(∞) = -0.092073. Using Eqs. 12.115, the system with integral control is:

$$A_I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -100 & 20000 & 0 \\ -800K_1 & -800K_2 & -800K_3 - 800 & 800K_e \\ -1 & 0 & 0 & 0 \end{pmatrix}; \ B_I = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

$$C_I = (1, 0, 0, 0)$$

Assume the following desired characteristic equation:

$$(s^3 + 960s^2 + 1.463E05s + 1.4651E07)(s + 1000) =$$

$$s^4 + 1960s^3 + 1.1063\times 10^6 s^2 + 1.6096\times 10^8 s + 1.4651\times 10^{10} = 0,$$

which is the desired characteristic equation from part (a) plus an additional pole at -1000. But the integral controlled system characteristic equation is

$|s\mathbf{I} - \mathbf{A_I}| = s^4 + 100\,(8K_3+9)\,s^3 + 80000\,(K_3+200K_2+1)\,s^2 + 16000000\,K_1\,s + 16000000\,K_e$

Equating coefficients to the desired characteristic equation

$100\,(8K_3+9) = 1960$; $80000\,(K_3+200K_2+1) = 1.1063\times10^6$; $16000000K_1 = 1.6096\times10^8$;

$16000000\,K_e = 1.4651\times10^{10}$

Solving for the controller gains: $K_e = 915.69$; $K_1 = 10.06$; $K_2 = 0.05752$; and $K_3 = 1.325$.

Substituting into $\mathbf{A_I}$ yields the integral controlled system.

$$A_I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -100 & 20000 & 0 \\ -8048.2 & -46.016 & -1860 & 7.3255\times10^5 \\ -1 & 0 & 0 & 0 \end{pmatrix}; B_I = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; C_I = (1, 0, 0, 0)$$

Finding the characteristic equation as a check yields

$$s^4 + 1960s^3 + 1.1063\times10^6 s^2 + 1.6096\times10^8 s + 1.4651\times10^{10}$$

which checks with the desired characteristic equation. Now check the steady-state error using Eq. (7.89) using the integral controlled system. We find the error is zero.

c.

Program:

```
 'Controller Compensated'
A=[0 1 0;0 -100 20000;-732.55 -3.0157 -860];
B=[0;0;800];
C=[1 0 0];
D=0;
S=ss(A,B,C,D)
step(S)
title('Controller Compensated')
pause
'Integral Controller'
A=[0 1 0 0;0 -100 20000 0;-8048.2 -46.016 -1860 7.3255e05;-1 0 0 0];
B=[0;0;0;1];
C=[1 0 0 0];
D=0;
S=ss(A,B,C,D)
step(S)
title('Integral Controller')
```

Computer response:

ans =

Controller Compensated

a =

	x1	x2	x3
x1	0	1	0
x2	0	-100	2e+004
x3	-732.5	-3.016	-860

```
b =
        u1
   x1    0
   x2    0
   x3  800


c =
       x1  x2  x3
   y1   1   0   0


d =
       u1
   y1   0
```

Continuous-time model.

ans =

Integral Controller

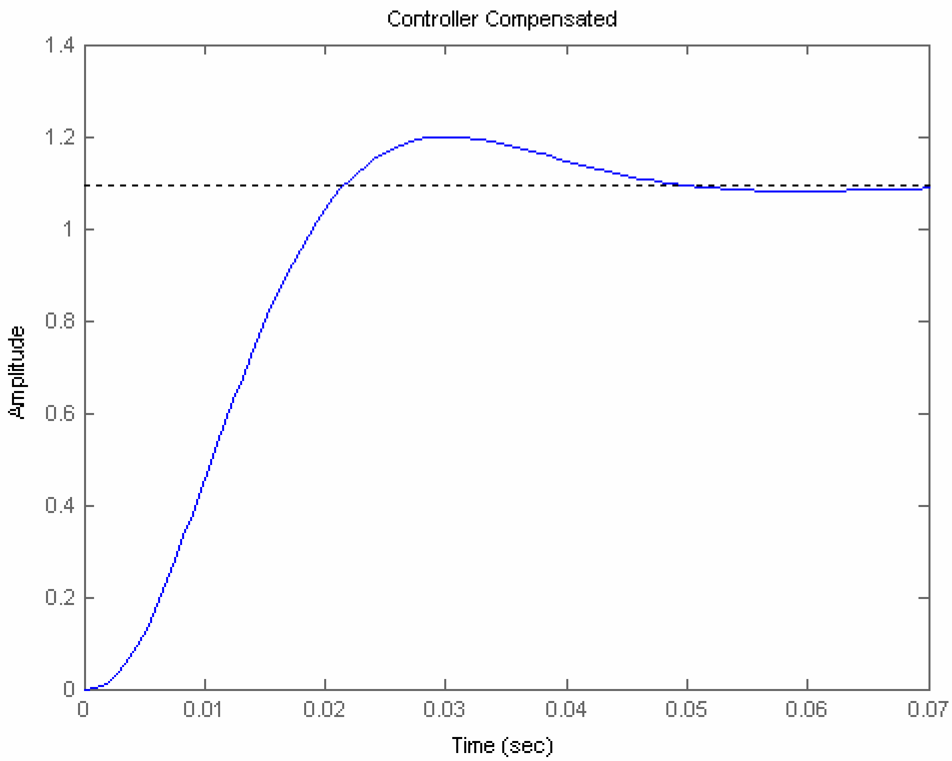
```
a =
              x1          x2          x3          x4
   x1          0           1           0           0
   x2          0        -100      2e+004           0
   x3      -8048      -46.02       -1860  7.326e+005
   x4         -1           0           0           0

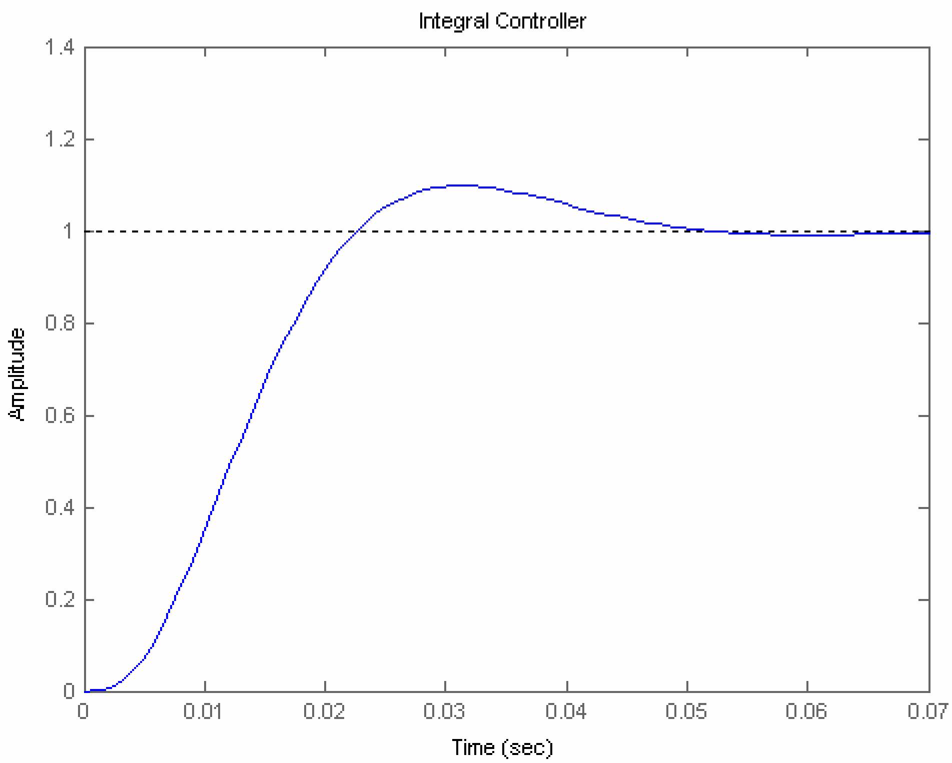

b =
       u1
   x1   0
   x2   0
   x3   0
   x4   1


c =
       x1  x2  x3  x4
   y1   1   0   0   0


d =
       u1
   y1   0
```

Continuous-time model.


Controller Compensated
Amplitude
Time (sec)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0.01
0.02
0.03
0.04
0.05
0.06
0.07


Integral Controller
Amplitude
Time (sec)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0.01
0.02
0.03
0.04
0.05
0.06
0.07

35.

Program:

```
%Enter G(s)
numg=0.072*conv([1 23],[1 0.05 0.04]);
deng=conv([1 0.08 0.04],poly([0.7 -1.7]));
'G(s)'
G=tf(numg,deng)
'Plant Zeros'
plantzeros=roots(numg)
%Input transient response specifications
Po=input('Type %OS  ');
Ts=input('Type settling time   ');

%Determine pole location
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
wn=4/(z*Ts);
%wn=pi/(Tp*sqrt(1-z^2));
[num,den]=ord2(wn,z);
r=roots(den);
poles=[r(1) r(2) plantzeros(2) plantzeros(3)]
characteristiceqdesired=poly(poles)


%Find controller canonical form of state-space representation of G(s)
'Controller Canonical Form'
[Ac Bc Cc Dc]=tf2ss(numg,deng)

%Design controller gains
Kc=acker(Ac,Bc,poles)
Acnew=Ac-Bc*Kc
Bcnew=Bc
Ccnew=Cc
Dcnew=Dc
characteristiceqcontroller=poly(eig(Acnew))

%Transform to phase-variable form
P=[0 0 0 1;0 0 1 0;0 1 0 0;1 0 0 0];
'Phase-variable form'
Ap=inv(P)*Ac*P
Bp=inv(P)*Bc
Cp=Cc*P
Dp=Dc
Kp=acker(Ap,Bp,poles)
Apnew=Ap-Bp*Kp
Bpnew=Bp
Cpnew=Cp
Dpnew=Dp
characteristiceqphase=poly(eig(Apnew))
[numt,dent]=ss2tf(Apnew,Bpnew,Cpnew,Dpnew);
T=tf(numt,dent);
'T(s)'
T=minreal(T)
step(T)
'T(s)'
Tzpk=zpk(T)
'Poles of T(s)'
p=pole(T
```

Computer response:

```
ans =
```

```
G(s)

Transfer function:
 0.072 s^3 + 1.66 s^2 + 0.08568 s + 0.06624
---------------------------------------------
s^4 + 1.08 s^3 - 1.07 s^2 - 0.0552 s - 0.0476


ans =

Plant Zeros


plantzeros =

 -23.0000
  -0.0250 + 0.1984i
  -0.0250 - 0.1984i

Type %OS  10
Type settling time   0.5

poles =

  -8.0000 +10.9150i  -8.0000 -10.9150i  -0.0250 + 0.1984i  -0.0250 -
0.1984i


characteristiceqdesired =

    1.0000   16.0500  183.9775    9.7969    7.3255


ans =

Controller Canonical Form


Ac =

   -1.0800    1.0700    0.0552    0.0476
    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0


Bc =

     1
     0
     0
     0


Cc =

    0.0720    1.6596    0.0857    0.0662


Dc =

     0


Kc =
```

```
   14.9700  185.0475    9.8521    7.3731


Acnew =

  -16.0500 -183.9775   -9.7969   -7.3255
    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0


Bcnew =

     1
     0
     0
     0


Ccnew =

    0.0720    1.6596    0.0857    0.0662


Dcnew =

     0


characteristiceqcontroller =

    1.0000   16.0500  183.9775    9.7969    7.3255


ans =

Phase-variable form


Ap =

         0    1.0000         0         0
         0         0    1.0000         0
         0         0         0    1.0000
    0.0476    0.0552    1.0700   -1.0800


Bp =

     0
     0
     0
     1


Cp =

    0.0662    0.0857    1.6596    0.0720


Dp =

     0

Kp =
```

```
    7.3731    9.8521  185.0475   14.9700


Apnew =

         0    1.0000         0         0
         0         0    1.0000         0
         0         0         0    1.0000
   -7.3255   -9.7969 -183.9775  -16.0500


Bpnew =

     0
     0
     0
     1


Cpnew =

    0.0662    0.0857    1.6596    0.0720


Dpnew =

     0


characteristiceqphase =

    1.0000   16.0500  183.9775    9.7969    7.3255


ans =

T(s)


Transfer function:
 0.072 s + 1.656
------------------
s^2 + 16 s + 183.1


ans =

T(s)

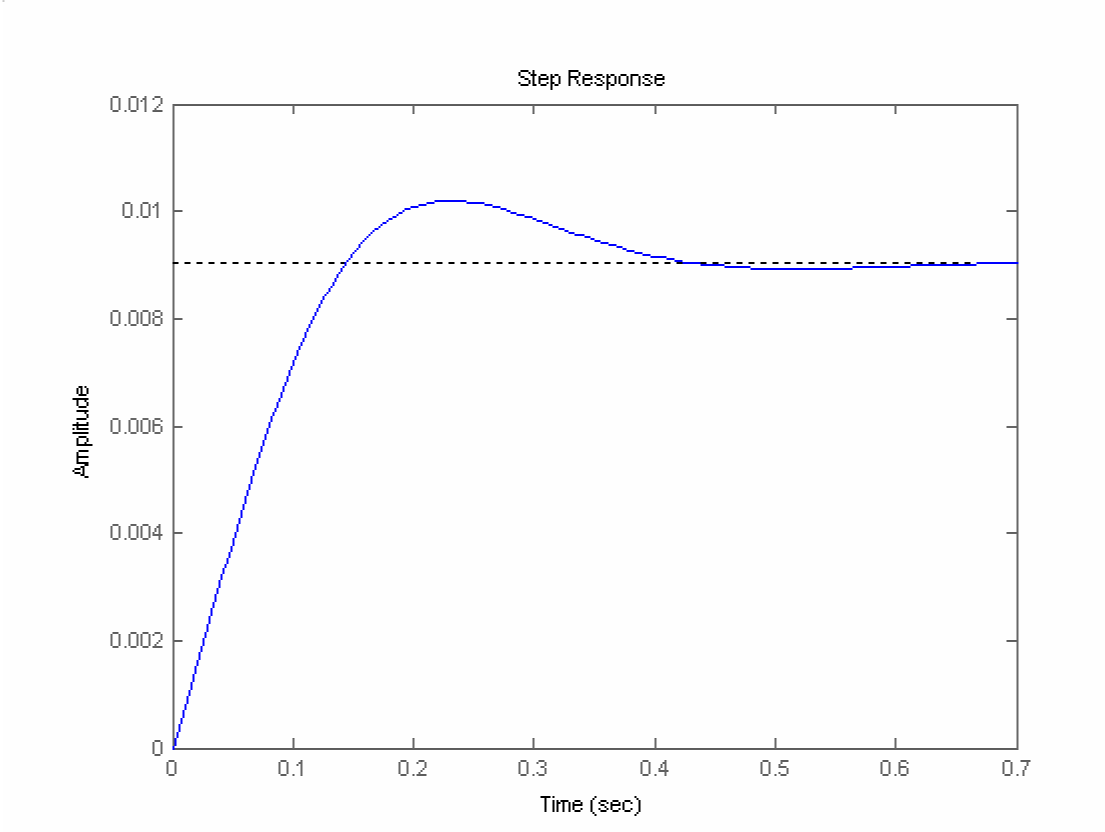

Zero/pole/gain:
    0.072 (s+23)
--------------------
(s^2  + 16s + 183.1)


ans =

Poles of T(s)


p =

  -8.0000 +10.9150i
  -8.0000 -10.9150i
```



36.

Program:

```
%Enter G(s)
numg=0.072*conv([1 23],[1 0.05 0.04]);
deng=conv([1 0.08 0.04],poly([0.7 -1.7]));
'Uncompensated Plant Transfer Function'
'G(s)'
G=tf(numg,deng)
'Uncompensated Plant Zeros'
plantzeros=roots(numg)
%Input transient response specifications
Po=input('Type %OS  ');
Ts=input('Type settling time   ');

%Determine pole location
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
wn=4/(z*Ts);
%wn=pi/(Tp*sqrt(1-z^2));
[num,den]=ord2(wn,z);
r=roots(den);
'Desired Observer Poles'
poles=[r(1) r(2) plantzeros(2) plantzeros(3)]'
'Desired Characteristic Equation of Observer'
poly(poles)


%Find phase variable form of state-space representation of Estimated Plant
%Find controller canonical form
[Ac Bc Cc Dc]=tf2ss(numg,deng);

%Transform to phase-variable form of Uncompensated Plant
P=[0 0 0 1;0 0 1 0;0 1 0 0;1 0 0 0];
'Phase-variable form of Estimated Plant'
Ap=inv(P)*Ac*P
```

```
Bp=inv(P)*Bc
Cp=Cc*P
Dp=Dc

%Design observer gains for phase variables
'Observer gains'
Lp=acker(Ap',Cp',poles)'
 'Error System Matrix'
Aep=Ap-Lp*Cp
'Error System Eigenvalues'
eig(Aep)
'Error Characteristic Polynomial'
poly(eig(Aep))
```

Computer response:

```
 ans =

Uncompensated Plant Transfer Function


ans =

G(s)


Transfer function:
 0.072 s^3 + 1.66 s^2 + 0.08568 s + 0.06624
---------------------------------------------
s^4 + 1.08 s^3 - 1.07 s^2 - 0.0552 s - 0.0476


ans =

Uncompensated Plant Zeros


plantzeros =

 -23.0000
  -0.0250 + 0.1984i
  -0.0250 - 0.1984i

Type %OS  10
Type settling time   0.5/15

ans =

Desired Observer Poles


poles =

  1.0e+002 *

  -1.2000 - 1.6373i
  -1.2000 + 1.6373i
  -0.0003 - 0.0020i
  -0.0003 + 0.0020i


ans =

Desired Characteristic Equation of Observer
```

```
ans =

  1.0e+004 *

    0.0001    0.0240    4.1218    0.2070    0.1648


ans =

Phase-variable form of Estimated Plant


Ap =

         0    1.0000         0         0
         0         0    1.0000         0
         0         0         0    1.0000
    0.0476    0.0552    1.0700   -1.0800


Bp =

     0
     0
     0
     1


Cp =

    0.0662    0.0857    1.6596    0.0720


Dp =

     0


ans =

Observer gains


Lp =

  1.0e+004 *

   -0.0002
    0.0043
   -0.0986
    2.5994


ans =

Error System Matrix


Aep =

  1.0e+004 *

    0.0000    0.0001    0.0003    0.0000
   -0.0003   -0.0004   -0.0071   -0.0003
    0.0065    0.0084    0.1636    0.0072
   -0.1722   -0.2227   -4.3139   -0.1873
```

```
ans =

Error System Eigenvalues


ans =

  1.0e+002 *

  -1.2000 + 1.6373i
  -1.2000 - 1.6373i
  -0.0003 + 0.0020i
  -0.0003 - 0.0020i


ans =

Error Characteristic Polynomial


ans =

  1.0e+004 *

    0.0001    0.0240    4.1218    0.2070    0.1648
```

37.

a. Using Eqs. (12.115), the system with integral control is:

$$A_I = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -K_1+0.0476 & -K_2+0.0552 & -K_3+1.07 & -K_4-1.08 & K_e \\ -0.06624 & -0.08568 & -1.6596 & -0.072 & 0 \end{pmatrix}; B_I = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

$$C_I = (1, 0, 0, 0, 0)$$

Assume the following desired characteristic equation,

$$(s+8+10.915i)(s+8-10.915i)(s+0.025+0.1984i)(s+0.025-0.1984i)(s+23) =$$
$$s^5+39.05s^4+553.13s^3+4241.3s^2+232.65s+168.43$$

which is the desired characteristic equation from Problem 35 plus an additional pole at -23, the closed-loop zero. But the integral controlled system characteristic equation is |s**I** - **A_I**| =

$$s^5+(K_4+1.08)s^4+(K_3+0.072K_e-1.07)s^3+(K_2+1.6596K_e-0.0552)s^2+(K_1+0.08568K_e-0.0476)s$$
$$+0.06624K_e$$

Equating coefficients to the desired characteristic equation

$$K_4+1.08 = 39.05;\ K_3+0.072K_e-1.07 = 553.13;\ K_2+1.6596K_e-0.0552 = 4241.3;$$
$$K_1+0.08568K_e-0.0476 = 232.65;\ \text{and}\ 0.06624K_e = 168.43$$

Solving for the controller gains

$$K_1 = 14.829;\ K_2 = 21.328;\ K_3 = 371.12;\ K_4 = 37.97\ \text{and}\ K_e = 2542.8$$

Substituting into $\mathbf{A_I}$ yields the integral controlled system,

$$A_I = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -14.781 & -21.272 & -370.05 & -39.05 & 2542.8 \\ -0.06624 & -0.08568 & -1.6596 & -0.072 & 0 \end{pmatrix}; B_I = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

$$C_I = (0.06624, 0.08568, 1.6596, 0.072, 0)$$

Finding the characteristic equation as a check yields

$$s^5 + 39.05 s^4 + 553.13 s^3 + 4241.3 s^2 + 232.65 s + 168.43$$

which checks with the desired. Now check the steady-state error using Eq. (7.89) using the integral controlled system. We find the error is zero.

b.

Program:

```
%Design with Integral Control
'State-Space Representation of System with Integral Control'
AI=[0 1  0 0 0;0 0 1 0 0;0 0 0 1 0;...
-14.781 -21.272 -370.05 -39.05 2542.8;...
-0.06624 -0.08568 -1.6596 -0.072 0]
BI=[0;0;0;0;1]
CI=[0.06624 0.08568 1.6596 0.072 0]
DI=0

%Convert to transfer function
[numt,dent]=ss2tf(AI,BI,CI,DI);
'Integral Control Transfer Function'
'T(s)'
T=tf(numt,dent)
'Integral Control Transfer Function Zeros'
roots(numt)
'Integral Control Transfer Function Poles'
roots(dent)
step(T)
title('Step Response with Integral Controller')
```

Computer response:

```
 ans =

State-Space Representation of System with Integral Control


AI =

  1.0e+003 *

         0    0.0010         0         0         0
         0         0    0.0010         0         0
         0         0         0    0.0010         0
   -0.0148   -0.0213   -0.3700   -0.0390    2.5428
   -0.0001   -0.0001   -0.0017   -0.0001         0


BI =

     0
     0
     0
     0
```

```
     1


CI =

    0.0662    0.0857    1.6596    0.0720         0


DI =

     0


ans =

Integral Control Transfer Function


ans =

T(s)


Transfer function:
-1.421e-014 s^4 + 183.1 s^3 + 4220 s^2 + 217.9 s + 168.4
--------------------------------------------------------
s^5 + 39.05 s^4 + 553.1 s^3 + 4241 s^2 + 232.6 s + 168.4


ans =

Integral Control Transfer Function Zeros


ans =

  1.0e+016 *

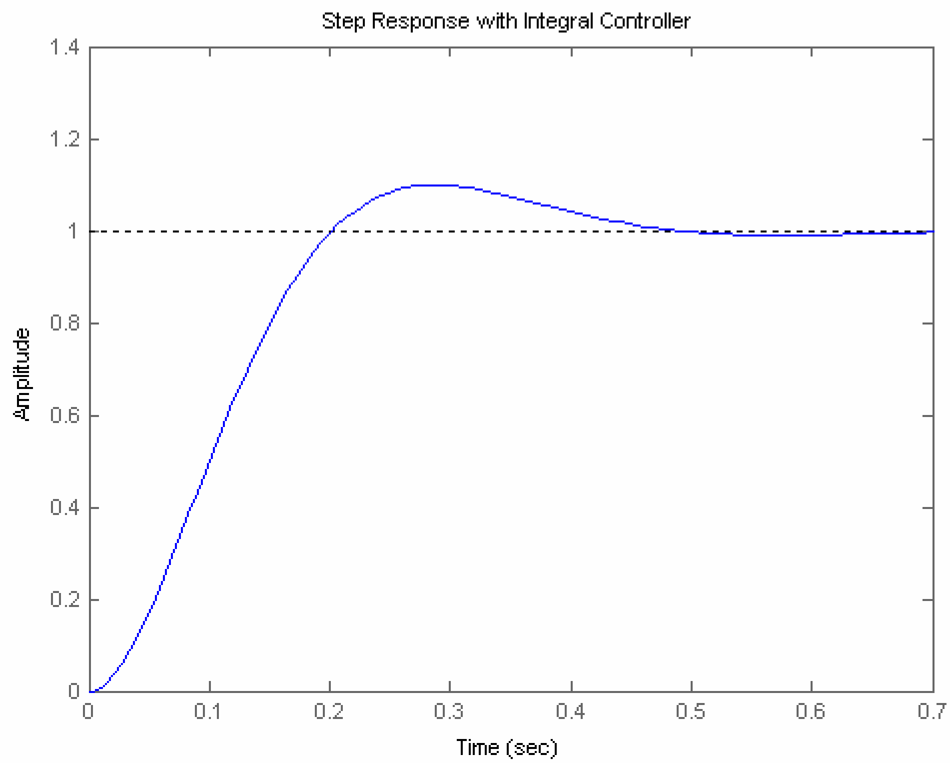
   1.2883
  -0.0000
  -0.0000 + 0.0000i
  -0.0000 - 0.0000i


ans =

Integral Control Transfer Function Poles


ans =

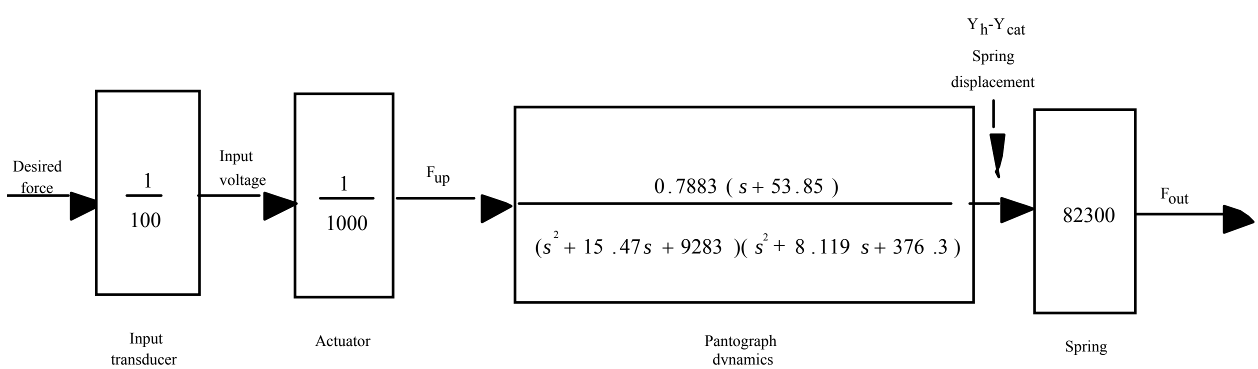
 -22.9998
  -8.0001 +10.9151i
  -8.0001 -10.9151i
  -0.0250 + 0.1984i
  -0.0250 - 0.1984i
```



38.

a.

The open-loop block diagram is



From Chapter 3, the state-space representation for $[Y_h(s) - Y_{cat}(s)]/F_{up}(s)$ is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9353 & -14.286 & 769.23 & 14.286 \\ 0 & 0 & 0 & 1 \\ 406.98 & 7.5581 & -406.98 & -9.3023 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0581 \end{bmatrix} f_{up}$$

$$y = \begin{bmatrix} 0.94911 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

where y = y_h - y_{cat} and $\mathbf{x} = \begin{bmatrix} y_h \\ \dot{y}_h \\ y_f \\ \dot{y}_f \end{bmatrix}$

Let v_i represent the input voltage shown on the diagram. Thus,

$$f_{up} = v_i/1000.$$

Also, $f_{out} = 82300(y_h - y_{cat})$.

Thus,

$$f_{out} = 82300y$$

Substituting f_{up} and f_{out} into the state-equations above yields

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9353 & -14.286 & 769.23 & 14.286 \\ 0 & 0 & 0 & 1 \\ 406.98 & 7.5581 & -406.98 & -9.3023 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0581\text{x}10^{-3} \end{bmatrix} v_i$$

$$f_{out} = \begin{bmatrix} 78,112 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

Thus,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9353 & -14.286 & 769.23 & 14.286 \\ 0 & 0 & 0 & 1 \\ 406.98 & 7.5581 & -406.98 & -9.3023 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0581\text{x}10^{-3} \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$$

Hence,

A-BK =

[0 , 1 , 0 , 0

[-9350 , -14.3 , 769 , 14.3

[0 , 0 , 0 , 1

[407 - 0.0000581 k1 , 7.56 - .0000581 k2 , -407 - 0.0000581 k3 , -9.30 - 0.0000581 k4]

and

$$|\mathbf{A\text{-}BK}| = s^4 + (0.0000581\ k4 + 23.60)\ s^3$$
$$+ (0.00083083\ k4 + 0.00083083\ k2 + 9781.882 + 0.0000581\ k3)\ s^2$$
$$+ (0.00083083\ k1 + 81141.36 + 0.543235\ k4 + 0.00083083\ k3 + 0.0446789\ k2)\ s$$
$$+(0\ .0446789\ k1 + 0.3492467\ 10^7 + 0.543235\ k3)$$

Input transient response specifications,

Po = 20

Ts = 1

yields poles at

-4.0000 + 7.8079i, -4.0000 - 7.8079i, -53.8500, -50.0000

Thus, the desired characteristic equation is

$$s^4 + 112s^3 + 3600s^2 + 29500s + 207000 = 0$$

We now equate the coefficients of |**A-BK**| to the coefficients of the desired characteristic equation. For compactness we solve for the coefficients, **K**, using the form **FK** = **G**, where

F =

0	0	0	0.0000581
0	0.00083083	0.0000581	0.00083083
0.00083083	0.0446789	0.00083083	0.543235
0 .0446789	0	0.543235	0

and

G =

88.4
–6181.882
-51641.36
-3285467

Solving for K using $K = F^{-1}G$

K =

-4.8225e8
-0.1131e8
0.3361e8
0.0152e8

b.

Integral Control Design

A =

1.0e+03 *

0	0.0010	0	0
-9.3530	-0.0143	0.7692	0.0143
0	0	0	0.0010
0.4070	0.0076	-0.4070	-0.0093

B =

1.0e-04 *

0
0
0
0.5810

C =

78112	0	0	0

Aaug = A-BK =

[0 , 1. , 0 , 0 , 0

[-9350. , -14.3 , 769. , 14.3 , 0

[0 , 0 , 0 , 1. , 0

[407. - 0.0000581 k1 , 7.56 - 0.0000581 k2 , -407. - 0.0000581 k3 ,

-9.30 - 0.0000581 k4 , 0.0000581 Ke

[-78100. , 0 , 0 , 0 , 0]

Desired poles

Po = 20

Ts = 1

Determine pole location

poles = -4.0000 + 7.8079i, -4.0000 - 7.8079i, -53.85, -50 , -50

Desired characteristic equation

$$s^5 + 162s^4 + 0.919e4s^3 + 0.210e6s^2 + 0.168e7s + 0.104e8$$

System characteristic equation

|sI-Aaug| =

$s^5 + (23.60 + 0.0000581\ k4)\ s^4$

$+ (0.00083083\ k4 + 0.00083083\ k2 + 0.0000581\ k3 + 9781.882)\ s^3$

$+ (0.00083083\ k1 + 81141.36 + 0.0446789\ k2 + 0.543235\ k4 + 0.00083083\ k3)\ s^2$

$+ (0.0446789\ k1 + 64.887823\ Ke + 0.543235\ k3 + 0.3492467\ 10^7)\ s$

$+ 3489.42209\ Ke$

Solving for Coefficients, **K**, using **FK** = **G** as in (a), where

F

0	0	0	5.8100e-05	0
0	8.3083e-04	5.8100e-05	8.3083e-04	0
8.3083e-04	4.4679e-02	8.3083e-04	5.4324e-01	0
4.4679e-02	0	5.4324e-01	0	6.4888e+01
0	0	0	0	3.4894e+03

G =

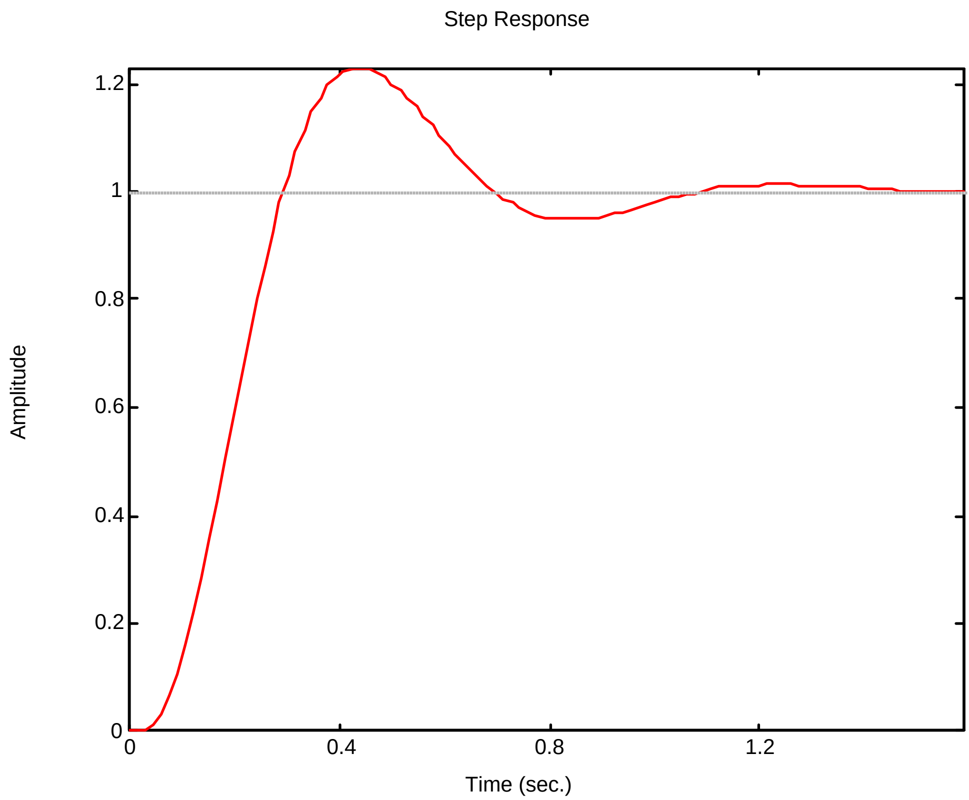
1.3840e+02
-5.9188e+02
1.2886e+05
-1.8125e+06
1.0800e+07

Thus,

K =

-1.0157e+09
-8.6768e+06
7.9827e+07
2.3821e+06

3.0951e+03



T H I R T E E N

Digital Control Systems

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Transient Design via Gain

a. From the answer to the antenna control challenge in Chapter 5, the equivalent forward transfer function found by neglecting the dynamics of the power amplifier, replacing the pots with unity gain, and including the integration in the sample-and-hold is

$$G_e(s) = \frac{0.16K}{s^2(s+1.32)}$$

But,

$$\frac{1}{s^2(s+1.32)} = -0.57392\frac{1}{s} + 0.57392\frac{1}{s+1.32} + 0.75758\frac{1}{s^2}$$

$$G_z = -0.57392\frac{z}{z-1} + 0.57392\frac{z}{z-e^{-1.32T}} + 0.75758\frac{Tz}{(z-1)^2}$$

$$T = 0.1$$

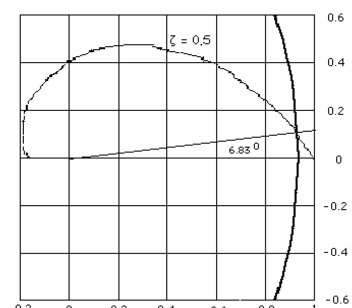
$$G_z = -0.57392\frac{z}{z-1} + 0.57392\frac{z}{z-e^{-0.132}} + 0.75758\frac{0.1z}{(z-1)^2}$$

$$G_z = 0.0047871\frac{(z+0.95696)z}{(z-1)^2(z-0.87634)}$$

Thus, $G_e(z) = 0.16K\frac{z-1}{z}G_z$, or,

$$G_e(z) = 7.659\text{x}10^{-4}K\frac{(z+0.95696)}{(z-1)(z-0.87634)}$$

b. Draw the root locus and overlay it over the ζ = 0.5 (i.e. 16.3% overshoot) curve.



We find that the root locus crosses at approximately 0.93 ± j0.11 with $7.659\text{x}10^{-4}\text{K} = 8.63\text{x}10^{-3}$.

Hence, K = 11.268.

c.

$$K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G_e(z) = \frac{(7.659\text{x}10^{-4}\text{K})(1.95696)}{0.12366} = 0.1366;$$

$$e(\infty) = \frac{1}{K_v} = 7.321$$

d.

Program:

```
T=0.1;                      %Input sampling time
numf=0.16;                  %Numerator of F(s)
denf=[1 1.32 0 0];          %Denominator of F(s)
'F(s)'                      %Display label
F=tf(numf,denf)             %Display F(s)
numc=conv([1 0],numf);      %Differentiate F(s) to compensate
                            %for c2dm which assumes series zoh
denc=denf;                  %Denominator of continuous system
                            %same as denominator of F(s)
C=tf(numc,denc);            %Form continuous system, C(s)
C=minreal(C,1e-10);         %Cancel common poles and zeros
D=c2d(C,T,'zoh');           %Convert to z assuming zoh
'F(z)'
D=minreal(D,1e-10)          %Cancel common poles and zeros and display
rlocus(D)
pos=(16.3);
z=-log(pos/100)/sqrt(pi^2+[log(pos/100)]^2);
zgrid(z,0)
title(['Root Locus with ' , num2str(pos), ' Percent Overshoot Line'])
[K,p]=rlocfind(D)           %Allows input by selecting point on
                            %graphic
```

Computer response:

```
ans =

F(s)

Transfer function:
     0.16
--------------
s^3 + 1.32 s^2

ans =

F(z)

Transfer function:
0.0007659 z + 0.000733
----------------------
z^2 - 1.876 z + 0.8763

Sampling time: 0.1
Select a point in the graphics window

selected_point =

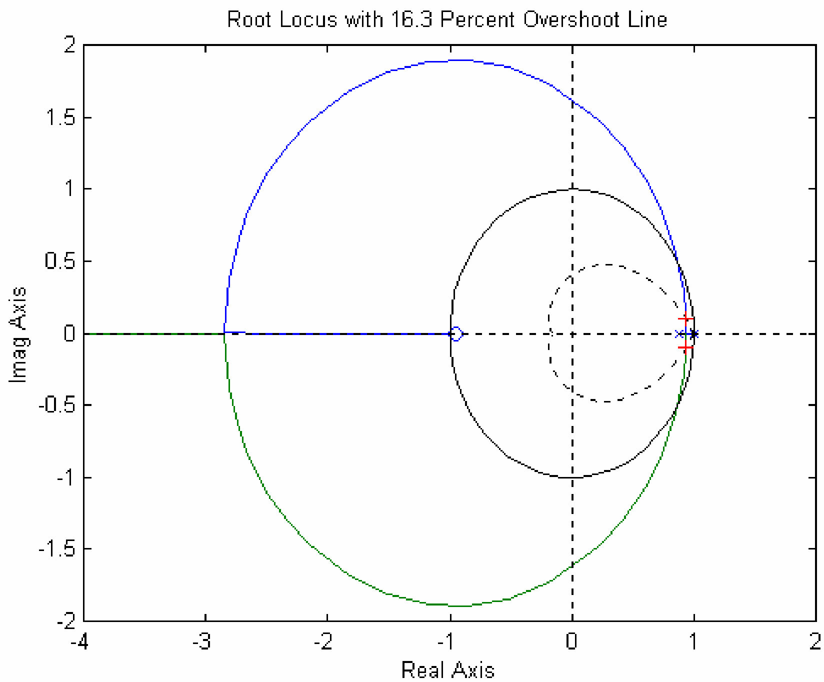
  9.2969e-001 +1.0219e-001i

K =

  9.8808e+000
```

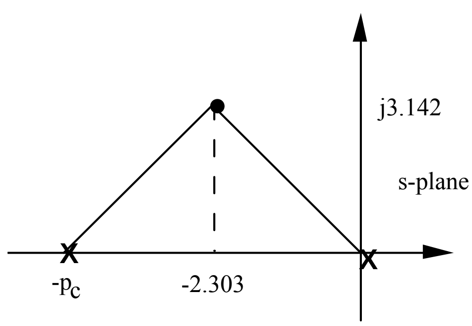
```
p =

  9.3439e-001 +1.0250e-001i
  9.3439e-001 -1.0250e-001i
```



Antenna Control: Digital Cascade Compensator Design

a. Let the compensator be $KG_c(s)$ and the plant be $G_p(s) = \dfrac{0.16}{s(s+1.32)}$. For 10% overshoot and a peak time of 1 second, $\zeta = 0.591$ and $\omega_n = 3.895$, which places the dominant poles at $-2.303 \pm j3.142$. If we place the compensator zero at -1.32 to cancel the plant's pole, then the following geometry results.



Hence, $p_c = 4.606$. Thus, $G_c(s) = \dfrac{K(s+1.32)}{(s+4.606)}$ and $G_c(s)G_p(s) = \dfrac{0.16K}{s(s+4.606)}$. Using the product of pole lengths to find the gain, $0.16K = (3.896)^2$, or $K = 94.87$. Hence,

$G_c(s) = \dfrac{94.87(s+1.32)}{(s+4.606)}$. Using a sampling interval of 0.01 s, the Tustin transformation of $G_c(s)$ is $G_c(z) = \dfrac{93.35(z-0.9869)}{(z-0.955)} = \dfrac{93.35z - 92.12}{z-0.955}$.

b. Cross multiplying**,**

$$(z - 0.955)X(z) = (93.35z - 92.12)E(z)$$

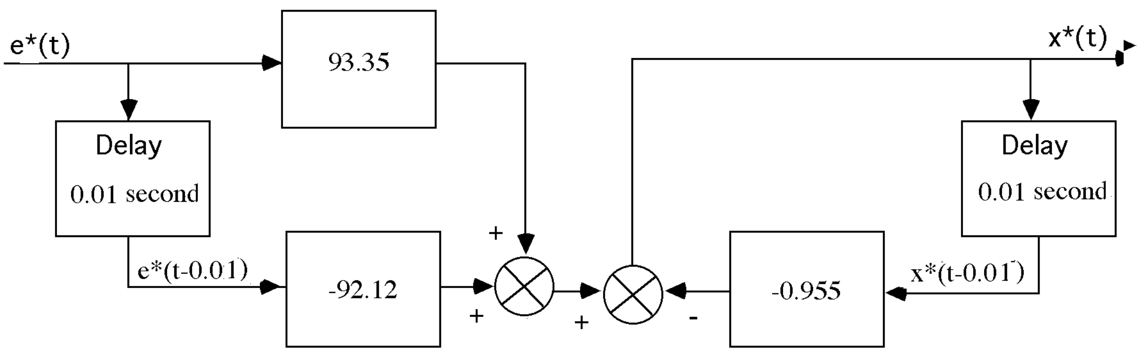
Solving for the highest power of z operating on *X(z)*,

$$zX(z) = (93.35z - 92.12)E(z) + 0.955X(z)$$

Solving for *X(z)*,

$$X(z) = (93.35 - 92.12z^{-1})E(z) + 0.955z^{-1}X(z)$$

Implementing this equation as a flowchart yields the following diagram



c.

Program:

```
's-plane lead design for Challenge - Lead Comp'
clf                             %Clear graph on screen.
'Uncompensated System'          %Display label.
numg=0.16;                      %Generate numerator of G(s).
deng=poly([0 -1.32]);           %Generate denominator of G(s).
'G(s)'                          %Display label.
G=tf(numg,deng);                %Create G(s).
Gzpk=zpk(G)                     %Display G(s).
pos=input('Type desired percent overshoot ');
                                %Input desired percent overshoot.
z=-log(pos/100)/sqrt(pi^2+[log(pos/100)]^2);
                                %Calculate damping ratio.
Tp=input('Type Desired Peak Time ');
                                %Input desired peak time.
wn=pi/(Tp*sqrt(1-z^2));         %Evaluate desired natural frequency.
b=input('Type Lead Compensator Zero, (s+b). b=  ');
                                %Input lead compensator zero.
done=1;                         %Set loop flag.

while done==1                   %Start loop for trying lead
                                %compensator pole.
a=input('Enter a Test Lead Compensator Pole, (s+a). a =     ');
                                %Enter test lead compensator pole.
numge=conv(numg,[1 b]);         %Generate numerator of Gc(s)G(s).
denge=conv([1 a],deng);         %Generate denominator of Gc(s)G(s).
Ge=tf(numge,denge);             %Create Ge(s)=Gc(s)G(s).
clf                             %Clear graph on screen.
```

```
rlocus(Ge)                      %Plot compensated root locus with
                                %test lead compensator pole.
axis([-5 2 -8 8]);              %Change axes ranges.
sgrid(z,wn)                     %Overlay grid on lead-compensated
                                %root locus.
title(['Lead-Compensated Root Locus with ' , num2str(pos),...
'%  Overshoot Line, Lead Pole at  ', num2str(-a),...
' and Required Wn'])            %Add title to lead-compensated root
                                %locus.
done=input('Are you done? (y=0,n=1)  ');
                                %Set loop flag.
end                             %End loop for trying compensator
                                %pole.
[K,p]=rlocfind(Ge);             %Generate gain, K, and closed-loop
                                %poles, p, for point selected
                                %interactively on the root locus.
'Gc(s)'                         %Display label.
Gc=K*tf([1 b],[1 a])            %Display lead compensator.
'Gc(s)G(s)'                     %Display label.
Ge                              %Display Gc(s)G(s).
'Closed-loop poles = '          %Display label.
p                               %Display lead-compensated system's
                                %closed-loop poles.
f=input('Give pole number that is operating point   ');
                                %Choose lead-compensated system
                                %dominant pole.
'Summary of estimated specifications for selected point on lead'
'compensated root locus'        %Display label.
operatingpoint=p(f)             %Display lead-compensated dominant
                                %pole.
gain=K                          %Display lead-compensated gain.
estimated_settling_time=4/abs(real(p(f)))
                                %Display lead-compensated settling
                                %time.
estimated_peak_time=pi/abs(imag(p(f)))
                                %Display lead-compensated peak time.
estimated_percent_overshoot=pos %Display lead-compensated percent
                                %overshoot.
estimated_damping_ratio=z       %Display lead-compensated damping
                                %ratio.
estimated_natural_frequency=sqrt(real(p(f))^2+imag(p(f))^2)
                                %Display lead-compensated natural
                                %frequency.
s=tf([1 0],1);                  %Create transfer function, "s".
sGe=s*Ge;                       %Create sGe(s) to evaluate Kv.
sGe=minreal(sGe);               %Cancel common poles and zeros.
Kv=dcgain(K*sGe)                %Display lead-compensated Kv.
ess=1/Kv                        %Display lead-compensated steady-
                                %state error for unit ramp input.
'T(s)'                          %Display label.
T=feedback(K*Ge,1)              %Create and display lead-compensated
                                %T(s).
'Press any key to continue and obtain the lead-compensated step'
'response'                      %Display label
pause
step(T)                         %Plot step response for lead
                                %compensated system.
title(['Lead-Compensated System with  ' ,num2str(pos),'% Overshoot'])
                                %Add title to step response of PD
                                %compensated system.
pause

'z-plane conversion for Challenge - Lead Comp'
clf                             %Clear graph.
'Gc(s) in polynomial form'      %Print label.
Gcs=Gc                          %Create Gc(s) in polynomial form.
'Gc(s) in polynomial form'      %Print label.
```

```
Gcszpk=zpk(Gcs)                  %Create Gc(s) in factored form.
'Gc(z) in polynomial form via Tustin Transformation'
                                 %Print label.
Gcz=c2d(Gcs,1/100,'tustin')      %Form Gc(z) via Tustin
                                 %transformation.
'Gc(z) in factored form via Tustin Transformation'
                                 %Print label.
Gczzpk=zpk(Gcz)                  %Show Gc(z) in factored form.
'Gp(s) in polynomial form'       %Print label.
Gps=G                            %Create Gp(s) in polynomial form.
'Gp(s) in factored form'         %Print label.
Gpszpk=zpk(Gps)                  %Create Gp(s) in factored form.
'Gp(z) in polynomial form'       %Print label.
Gpz=c2d(Gps,1/100,'zoh')         %Form Gp(z) via zoh transformation.
'Gp(z) in factored form'         %Print label.
Gpzzpk=zpk(Gpz)                  %Form Gp(z) in factored form.
pole(Gpz)                        %Find poles.
Gez=Gcz*Gpz;                     %Form Ge(z) = Gc(z)Gp(z).
'Ge(z) = Gc(z)Gp(z) in factored form'
                                 %Print label.
Gezzpk=zpk(Gez)                  %Form Ge(z) in factored form.
'z-1'                            %Print label.
zm1=tf([1 -1],1,1/100)           %Form z-1.
zm1Gez=minreal(zm1*Gez,.00001);
'(z-1)Ge(z)'                     %Print label.
zm1Gezzpk=zpk(zm1Gez)
pole(zm1Gez)
Kv=300*dcgain(zm1Gez)
Tz=feedback(Gez,1)
step(Tz)
title('Closed-Loop Digital Step Response')
                                 %Add title to step response.
```

Computer response:

```
ans =

s-plane lead design for Challenge - Lead Comp


ans =

Uncompensated System


ans =

G(s)


Zero/pole/gain:
   0.16
----------
s (s+1.32)

Type desired percent overshoot 10
Type Desired Peak Time 1
Type Lead Compensator Zero, (s+b). b=  1.32
Enter a Test Lead Compensator Pole, (s+a). a =     4.606
Are you done? (y=0,n=1)  0
Select a point in the graphics window

selected_point =

  -2.3045 + 3.1056i


ans =
```

```
Gc(s)


Transfer function:
93.43 s + 123.3
---------------
   s + 4.606


ans =

Gc(s)G(s)


Transfer function:
    0.16 s + 0.2112
------------------------
s^3 + 5.926 s^2 + 6.08 s


ans =

Closed-loop poles =


p =

  -2.3030 + 3.1056i
  -2.3030 - 3.1056i
  -1.3200

Give pole number that is operating point   1

ans =

Summary of estimated specifications for selected point on lead


ans =

compensated root locus


operatingpoint =

  -2.3030 + 3.1056i


gain =

   93.4281


estimated_settling_time =

    1.7369


estimated_peak_time =

    1.0116


estimated_percent_overshoot =

    10
```

```
estimated_damping_ratio =

    0.5912

estimated_natural_frequency =

    3.8663

Kv =

    3.2454

ess =

    0.3081

ans =

T(s)

Transfer function:
         14.95 s + 19.73
---------------------------------
s^3 + 5.926 s^2 + 21.03 s + 19.73

ans =

Press any key to continue and obtain the lead-compensated step

ans =

response

ans =

z-plane conversion for Challenge - Lead Comp

ans =

Gc(s) in polynomial form

Transfer function:
93.43 s + 123.3
---------------
   s + 4.606

ans =

Gc(s) in polynomial form

Zero/pole/gain:
93.4281 (s+1.32)
----------------
```

```
   (s+4.606)


ans =

Gc(z) in polynomial form via Tustin Transformation


Transfer function:
91.93 z - 90.72
---------------
   z - 0.955

Sampling time: 0.01

ans =

Gc(z) in factored form via Tustin Transformation


Zero/pole/gain:
91.9277 (z-0.9869)
------------------
    (z-0.955)

Sampling time: 0.01

ans =

Gp(s) in polynomial form


Transfer function:
    0.16
------------
s^2 + 1.32 s


ans =

Gp(s) in factored form


Zero/pole/gain:
   0.16
----------
s (s+1.32)


ans =

Gp(z) in polynomial form


Transfer function:
7.965e-006 z + 7.93e-006
------------------------
 z^2 - 1.987 z + 0.9869

Sampling time: 0.01

ans =

Gp(z) in factored form


Zero/pole/gain:
```

```
7.9649e-006 (z+0.9956)
----------------------
   (z-1) (z-0.9869)

Sampling time: 0.01

ans =

    1.0000
    0.9869


ans =

Ge(z) = Gc(z)Gp(z) in factored form


Zero/pole/gain:
0.0007322 (z+0.9956) (z-0.9869)
-------------------------------
  (z-1) (z-0.9869) (z-0.955)

Sampling time: 0.01

ans =

z-1


Transfer function:
z - 1

Sampling time: 0.01

ans =

(z-1)Ge(z)


Zero/pole/gain:
0.0007322 (z+0.9956)
--------------------
     (z-0.955)

Sampling time: 0.01

ans =

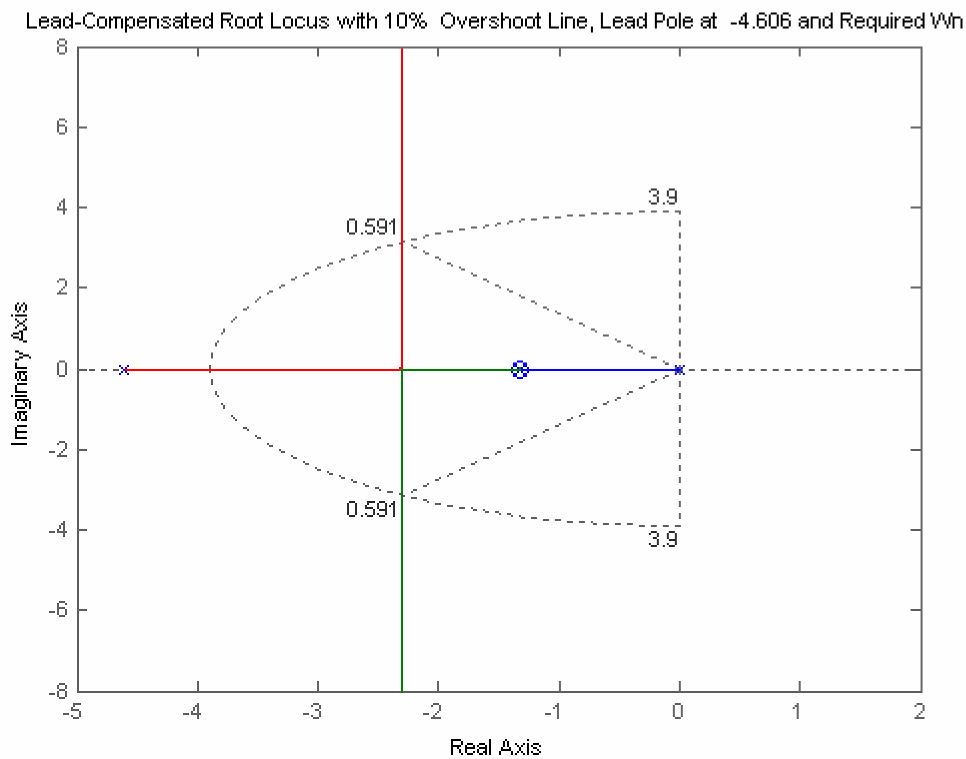
    0.9550

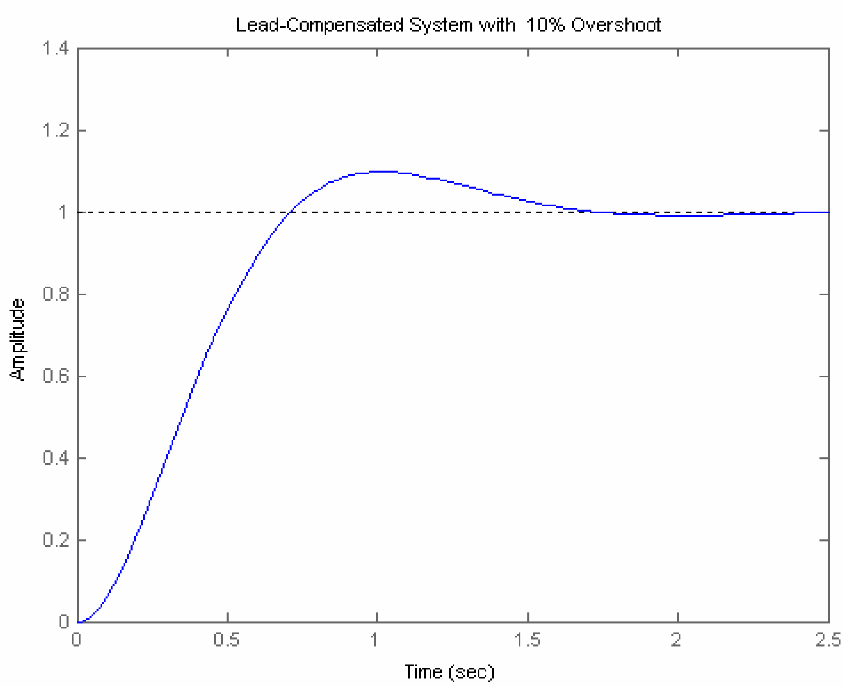

Kv =

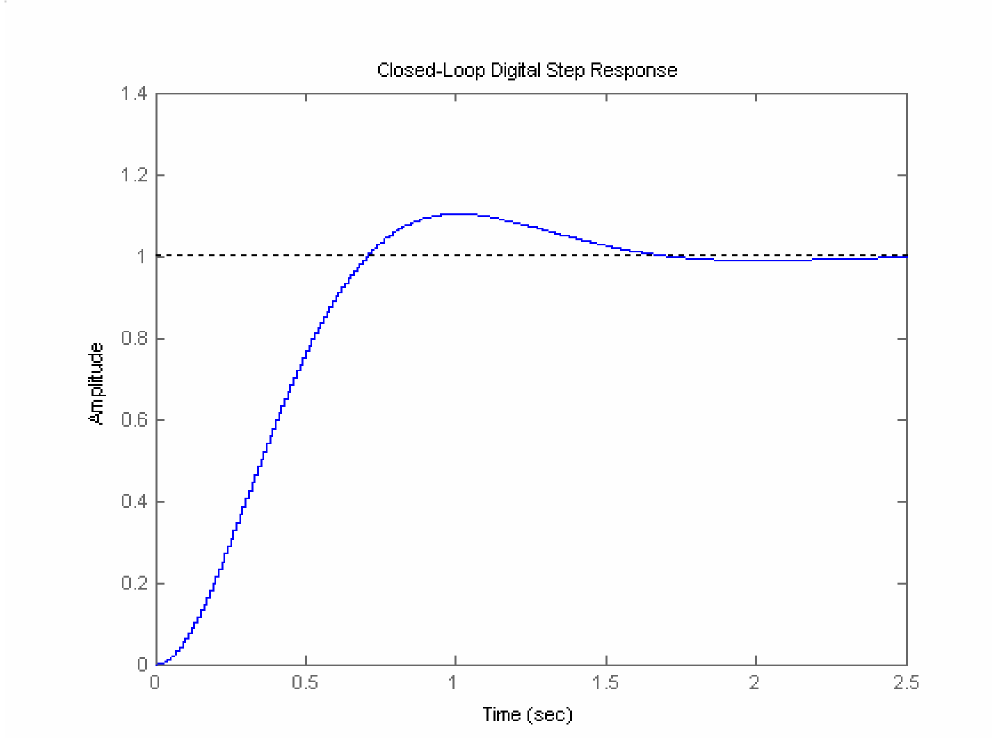
    9.7362


Transfer function:
0.0007322 z^2 + 6.387e-006 z - 0.0007194
----------------------------------------
   z^3 - 2.941 z^2 + 2.884 z - 0.9432

Sampling time: 0.01
```


Lead-Compensated Root Locus with 10% Overshoot Line, Lead Pole at -4.606 and Required Wn
Imaginary Axis
Real Axis
0.591
0.591
3.9
3.9
8
6
4
2
0
-2
-4
-6
-8
-5
-4
-3
-2
-1
0
1
2


Lead-Compensated System with 10% Overshoot
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.5
1
1.5
2
2.5


Closed-Loop Digital Step Response
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5

ANSWERS TO REVIEW QUESTIONS

1. (1) Supervisory functions external to the loop; (2) controller functions in the loop

2. (1) Control of multiple loops by the same hardware; (2) modifications made with software, not hardware; (3) more noise immunity (4) large gains usually not required

3. Quantization error; conversion time

4. An ideal sampler followed by a sample-and-hold

5. $z = e^{sT}$

6. The value of the time waveform only at the sampling instants

7. Partial fraction expansion; division to yield power series

8. Partial fraction

9. Division to yield power series

10. The input must be sampled; the output must be either sampled or thought of as sampled.

11. c(t) is $c^*(t) = c(kT)$, i.e. the output only at the sampling instants.

12. No; the waveform is only valid at the sampling instants. Instability may be apparent if one could only see between the sampling instants. The roots of the denominator of G(z) must be checked to see that they are within the unit circle.

13. A sample-and-hold must be present between the cascaded systems.

14. Inside the unit circle

15. Raible table; Jury's stability test

16. z=+1

17. There is no difference.

18. Map the point back to the s-plane. Since $z = e^{sT}$, $s = (1/T) \ln z$. Thus, $\sigma = (1/T) \ln (\text{Re } z)$, and $\omega = (1/T) \ln (\text{Im } z)$.

19. Determine the point on the s-plane and use $z = e^{sT}$. Thus, $\text{Re } z = e^{\sigma T} \cos \omega$, and $\text{Im } z = e^{\sigma T} \sin \omega$.

20. Use the techniques described in Chapters 9 and 11 and then convert the design to a digital compensator using the Tustin transformation.

21. Both compensators yield the same output at the sampling instants.

SOLUTIONS TO PROBLEMS

1.

a. $f(t) = e^{-at}$; $f^*(t) = \sum_{k=0}^{\infty} e^{-akT}\delta(t-kT)$; $F^*(s) = \sum_{k=0}^{\infty} e^{-akT} e^{-kTs} = 1 + e^{-aT} e^{-Ts} + e^{-a2T} e^{-2Ts} + \ldots$ Thus,

$F(z) = 1 + e^{-aT} z^{-1} + e^{-a2T} z^{-2} + \ldots = 1 + x^{-1} + x^{-2} + \ldots$ where $x = e^{-aT} z^{-1}$.

But, $F(z) = \dfrac{1}{1 - x^{-1}} = \dfrac{1}{1 - e^{-aT} z^{-1}} = \dfrac{z}{z - e^{-aT}}$.

b. $f(t) = u(t)$; $f^*(t) = \sum_{k=0}^{\infty} \delta(t-kT)$; $F^*(s) = \sum_{k=0}^{\infty} e^{-kT s} = 1 + e^{-Ts} + e^{-2Ts} + \ldots$

Thus, $F(z) = 1 + z^{-1} + z^{-2} + \ldots$ Since $\dfrac{1}{1 - z^{-1}} = 1 + z^{-1} + z^{-2} + z^{-3}$, $F(z) = \sum_{k=0}^{\infty} z^{-k} = \dfrac{1}{1 - z^{-1}} =$

$\dfrac{z}{z - 1}$.

c. $f(t) = t^2 e^{-at}$; $f^*(t) = \sum_{k=0}^{\infty} (kT)^2 e^{-akT} \delta(t-kT)$; $F^*(s) = T^2 \sum_{k=0}^{\infty} k^2 e^{-akT} e^{-kTs}$

$= T^2 \sum_{k=0}^{\infty} k^2 (e^{-(s+a)T})^k = T^2 \sum_{k=0}^{\infty} k^2 x^k = T^2(x + 4x^2 + 9x^3 + 16x^4 + \ldots)$, where $x = e^{-(s+a)T}$.

Let $s_1 = x + 4x^2 + 9x^3 + 16x^4 + \ldots$ Thus, $xs_1 = x^2 + 4x^3 + 9x^4 + 16x^5 + \ldots$

Let $s_2 = s_1 - xs_1 = x + 3x^2 + 5x^3 + 7x^4 + \ldots$ Thus, $xs_2 = x^2 + 4x^3 + 9x^4 + 16x^3 + \ldots$

Let $s_3 = s_2 - xs_2 = x + 2x^2 + 2x^3 + 2x^4 + \ldots$ Thus $xs_3 = x^2 + 2x^3 + 2x^4 + 2x^3 + \ldots$

Let $s_4 = s_3 - xs_3 = x + x^2$.

Solving for s_3,

$$s_3 = \frac{x + x^2}{1 - x}$$

and

$$s_2 = \frac{s_3}{1 - x} = \frac{x + x^2}{(1 - x)^2}$$

and

$$s_1 = \frac{s_2}{1 - x} = \frac{x + x^2}{(1 - x)^3}$$

Thus

$$F^*(s) = T^2 s_1 = T^2 \frac{x + x^2}{(1 - x)^3} = T^2 \frac{(e^{-(s+a)T} + e^{-2(s+a)T})}{(1 - e^{-(s+a)T})^3} =$$

$$\frac{T^2[z^{-1}e^{-aT} + z^{-2}e^{-2aT}]}{z^{-3}(z - e^{-aT})^3} = \frac{T^2 z e^{-aT}[z + e^{-aT}]}{(z - e^{-aT})^3}$$

d. $f(t) = \cos(\omega kT)$; $f^*(t) = \sum_{k=0}^{\infty} \cos(\omega kT)\delta(t - kT)$; $F^*(s) = \sum_{k=0}^{\infty} \cos(\omega kT) e^{-kTs}$

$$= \sum_{k=0}^{\infty} \frac{(e^{j\omega kT} + e^{-j\omega kT})e^{-kTs}}{2} = \frac{1}{2} \sum_{k=0}^{\infty} (e^{T(s-j\omega)})^{-k} + (e^{T(s+j\omega)})^{-k}$$

But,

$$\sum_{k=0}^{\infty} x^{-k} = \frac{1}{1 - x^{-1}} .$$

Thus,

$$F^*(s) = \frac{1}{2}\left[\frac{1}{1-e^{-T(s-j\omega)}} + \frac{1}{1-e^{-T(s+j\omega)}}\right] = \frac{1}{2}\left[\frac{2-e^{-Ts}(e^{j\omega T}+e^{-j\omega T})}{1-e^{-T(s-j\omega)}-e^{-T(s+j\omega)})+e^{-T(s-j\omega)}e^{-T(s+j\omega)}}\right]$$

$$= \frac{1}{2}\left[\frac{2-e^{-Ts}(2\cos(\omega T)}{1-e^{-Ts}(e^{j\omega T}+e^{-j\omega T})+e^{-2Ts}}\right] = \frac{1-z^{-1}\cos(\omega T)}{1-2z^{-1}\cos(\omega T)+z^{-2}}$$

Therefore,

$$F(z) = \frac{z(z-\cos(\omega T))}{z^2-2z\cos(\omega T)+1}$$

2.

Program:

```
syms T a w n                  %Construct symbolic objects for
                              %'T', 'a','w', and 'n'.
'(a)'                         %Display label.
'f(kT)'                       %Display label.
f=exp(-a*n*T);                %Define f(kT).
pretty(f)                     %Pretty print f(kT)
'F(z)'                        %Display label.
F=ztrans(f);                  %Find z-transform, F(z).
pretty(F)                     %Pretty print F(z).

'(b)'                         %Display label.
'f(kT)'                       %Display label.
f=exp(-0*n*T);                %Define f(kT)
pretty(f)                     %Pretty print f(kT)
'F(z)'                        %Display label.
F=ztrans(f);                  %Find z-transform, F(z).
pretty(F)                     %Pretty print F(z).

'(c)'                         %Display label.
'f(kT)'                       %Display label.
f=(n*T)^2*exp(-a*n*T);        %Define f(kT)
pretty(f)                     %Pretty print f(kT)
'F(z)'                        %Display label.
F=ztrans(f);                  %Find z-transform, F(z).
pretty(F)                     %Pretty print F(z).

'(d)'                         %Display label.
'f(kT)'                       %Display label.
f=cos(w*n*T);                 %Define f(kT)
pretty(f)                     %Pretty print f(kT)
'F(z)'                        %Display label.
F=ztrans(f);                  %Find z-transform, F(z).
pretty(F)                     %Pretty print F(z).
```

Computer response:

```
ans =

(a)

ans =

f(kT)


                              exp(-a n T)
```

```
ans =
F(z)
                           z
              -------------------------
                        /     z         \
              exp(-a T) |---------- - 1|
                        \exp(-a T)      /
ans =
(b)
ans =
f(kT)
                           1
ans =



F(z)
                           z
                         -----
                         z - 1
ans =
(c)
ans =
f(kT)
                        2  2
                       n  T  exp(-a n T)
ans =
F(z)
                 2
                T  z exp(-a T) (z + exp(-a T))
                ------------------------------
                                        3
                       (z - exp(-a T))
ans =
(d)
ans =
f(kT)
                          cos(w n T)
ans =
F(z)
                       (z - cos(w T)) z
                     ---------------------
                      2
                     z  - 2 z cos(w T) + 1
```

3.

a.

$$F(z)=\frac{z(z+3)(z+5)}{(z-0.4)(z-0.6)(z-0.8)}$$

$$\frac{F(z)}{z}=\frac{229.5}{z-0.4}-\frac{504}{z-0.6}+\frac{275.5}{z-0.8}$$

$$F(z)=\frac{229.5z}{z-0.4}-\frac{504z}{z-0.6}+\frac{275.5z}{z-0.8}$$

$$f(kT)=229.5(0.4)^k-504(0.6)^k+275.5(0.8)^k,\ k=0,\ 1,\ 2,\ 3,...$$

b.

$$F(z)=\frac{(z+0.2)(z+0.4)}{(z-0.1)(z-0.5)(z-0.9)}$$

$$\frac{F(z)}{z}=-\frac{1.778}{z}+\frac{4.6875}{z-0.1}-\frac{7.875}{z-0.5}+\frac{4.9653}{z-0.9}$$

$$F(z)=-1.778+\frac{4.6875z}{z-0.1}-\frac{7.875z}{z-0.5}+\frac{4.9653z}{z-0.9}$$

$$f(kT)=4.6875(0.1)^k-7.875(0.5)^k+4.9653(0.9)^k,\ k=1,2,\ 3,...$$

c.

$$F(z)=\frac{(z+1)(z+0.3)(z+0.4)}{z(z-0.2)(z-0.5)(z-0.7)}$$

$$\frac{F(z)}{z}=\frac{(z+1)(z+0.3)(z+0.4)}{z^2(z-0.2)(z-0.5)(z-0.7)}$$

$$=\frac{38.1633}{z-0.7}-\frac{72}{z-0.5}+\frac{60}{z-0.2}-\frac{26.1633}{z}-\frac{1.7143}{z^2}$$

$$F(z)=\frac{38.1633z}{z-0.7}-\frac{72z}{z-0.5}+\frac{60z}{z-0.2}-26.1633-\frac{1.7143}{z}$$

$$F=38.1633(0.7)^k-72(0.5)^k+60(0.2)^k \quad \text{for } k=2,3,4,\ldots$$

$$=1 \text{ for } k=1$$

$$=0 \text{ for } k=0$$

4.

Program:

```
'(a)'
syms z k
F=vpa(z*(z+3)*(z+5)/((z-0.4)*(z-0.6)*(z-0.8)),4);
pretty(F)
f=vpa(iztrans(F),4);
pretty(f)
'(b)'
syms z k
F=vpa((z+0.2)*(z+0.4)/((z-0.1)*(z-0.5)*(z-0.9)),4);
pretty(F)
```

```
f=vpa(iztrans(F),4);
pretty(f)
'(c)'
syms z k
F=vpa((z+1)*(z+0.3)*(z+0.4)/(z*(z-0.2)*(z-0.5)*(z-0.7)),4);
pretty(F)
f=vpa(iztrans(F),4);
pretty(f)
```

Computer response:

```
ans =

(a)

                       z (z + 3.) (z + 5.)
              ------------------------------------
              (z - .4000) (z - .6000) (z - .8000)

                      n               n               n
          275.5 .8000  - 504.0 .6000  + 229.5 .4000

ans =

(b)

                     (z + .2000) (z + .4000)
              ------------------------------------
              (z - .1000) (z - .5000) (z - .9000)

                                          n               n               n
    -1.778 charfcn[0](n) + 4.965 .9000  - 7.875 .5000  + 4.688 .1000

ans =

(c)

                 (z + 1.) (z + .3000) (z + .4000)
             --------------------------------------
             z (z - .2000) (z - .5000) (z - .7000)

                                                                n               n
  -1.714 charfcn[1](n) - 26.16 charfcn[0](n) + 38.16 .7000  - 72.00 .5000

                       n
         + 60.00 .2000
```

5.

a.

	By division	By Formula	
Instant	**Value**	**k**	**Value**
0	1	0	1
1	9.8	1	9.8
2	31.6	2	31.6
3	46.88	3	46.88
4	53.4016	4	53.4016
5	53.43488	5	53.43488
6	49.64608	6	49.64608
7	44.043776	7	44.043776
8	37.90637056	8	37.90637056
9	31.95798733	9	31.95798733
10	26.5581568	10	26.5581568
11	21.84639857	11	21.84639857
12	17.83896791	12	17.83896791
13	14.48905384	13	14.48905384
14	11.72227881	14	11.72227881
15	9.456567702	15	9.456567702
16	7.612550239	16	7.612550239
17	6.118437551	17	6.118437551
18	4.911796342	18	4.911796342
19	3.939668009	19	3.939668009
20	3.15787423	20	3.15787423
21	2.529983782	21	2.529983782
22	2.026197867	22	2.026197867
23	1.622284879	23	1.622284879
24	1.298623886	24	1.298623886
25	1.039376712	25	1.039376712
26	0.831787937	26	0.831787937
27	0.665602292	27	0.665602292
28	0.532584999	28	0.532584999
29	0.4261299	29	0.4261299
30	0.34094106	30	0.34094106

b.

	By division	By Formula	
Instant	**Value**	**k**	**Value**
1	1	1	1.00002
2	2.1	2	2.100018
3	2.64	3	2.6400162
4	2.766	4	2.76601458
5	2.6859	5	2.685913122
6	2.51571	6	2.51572181
7	2.313354	7	2.313364629
8	2.1066276	8	2.106637166
9	1.90826949	9	1.908278099
10	1.723594881	10	1.723602629
11	1.554311564	11	1.554318538
12	1.400418494	12	1.40042477
13	1.261145687	13	1.261151336
14	1.13541564	14	1.135420724
15	1.022066337	15	1.022070912
16	0.919955834	16	0.919959951
17	0.828008315	17	0.828012021
18	0.745231516	18	0.745234852
19	0.670720381	19	0.670723383
20	0.603654351	20	0.603657053
21	0.54329192	21	0.543294352
22	0.48896423	22	0.488966419
23	0.440068558	23	0.440070528
24	0.396062078	24	0.39606385
25	0.356456058	25	0.356457653
26	0.320810546	26	0.320811982
27	0.288729538	27	0.28873083
28	0.259856608	28	0.259857771
29	0.233870959	29	0.233872006
30	0.210483869	30	0.210484811
31	0.189435485	31	0.189436333

c.

Instant	via Division	via Closed Form Expression
0		0
1	1	1
2	3.1	3.100017
3	4.57	4.5700119
4	4.759	4.75900833
5	4.1833	4.183305831
6	3.36871	3.368714082
7	2.581177	2.581179857
8	1.9189399	1.9189419
9	1.39943113	1.39943253
10	1.007711431	1.007712411
11	0.71945743	0.719458116
12	0.510650836	0.510651317
13	0.360971088	0.360971424
14	0.254437549	0.254437785
15	0.178985186	0.178985351
16	0.125729082	0.125729198
17	0.088230084	0.088230165
18	0.061870922	0.061870978
19	0.043364577	0.043364617
20	0.03038267	0.030382697
21	0.021281602	0.021281621
22	0.014903988	0.014904001
23	0.010436225	0.010436234
24	0.007307074	0.00730708

6.

a.

$$G(s)=\frac{(s+4)}{(s+2)(s+5)}=\frac{0.6667}{s+2}+\frac{0.3333}{s+5}$$

$$G(z)=\frac{0.6667z}{z-e^{-2T}}+\frac{0.3333z}{z-e^{-5T}}$$

For $T = 0.5$ s,

$$G(z)=\frac{0.6667z}{z-0.3679}+\frac{0.3333z}{z-0.082085}=\frac{z(z-0.1774)}{(z-0.3679)(z-0.082085)}$$

b.

$$G(s)=\frac{(s+1)(s+2)}{s(s+3)(s+4)}=\frac{0.1667}{s}-\frac{0.6667}{s+3}+\frac{1.5}{s+4}$$

$$G(z)=\frac{0.1667z}{z-1}-\frac{0.6667z}{z-e^{-3T}}+\frac{1.5z}{z-e^{-4T}}$$

For $T = 0.5$ s,

$$G(z)=\frac{0.1667z}{z-1}-\frac{0.6667z}{z-0.22313}+\frac{1.5z}{z-0.13534}=\frac{z(z-0.29675)(z-0.8408)}{(z-1)(z-0.22313)(z-0.13534)}$$

c.

$$G(s)=\frac{20}{(s+3)(s^2+6s+25)}=\frac{1.25}{s+3}-\frac{1.25s+3.57}{s^2+6s+25}=\frac{1.25}{s+3}-\frac{1.25(s+3)}{(s+3)^2+4^2}$$

$$G(z)=-1.25\frac{z}{z-e^{-aT}}-1.25\frac{z^2-zae^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$$

For $a=3;\ \omega=4;\ T=0.5$,

$$G(z)=-1.25\frac{z}{z-0.2231}-1.25\frac{z^2+0.0929z}{z^2+0.1857z+0.0498}$$
$$=0.395\frac{z(z+0.2232)}{(z-0.2231)(z^2+0.1857z+0.0498)}$$

d.

$$G(s)=\frac{15}{s(s+1)(s^2+10s+81)}=\frac{0.1852}{s}-\frac{0.2083}{s+1}+0.02314\frac{s+0.9978}{s^2+10s+81}$$
$$=\frac{0.1852}{s}-\frac{0.2083}{s+1}+0.02314\frac{(s+5)-0.5348\sqrt{56}}{(s+5)^2+56}$$

$$G(z)=0.1852\frac{z}{z-1}-0.2083\frac{z}{z-e^{\beta T}}+0.02314\frac{z^2-zae^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}-0.0124\frac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$$

For $a=5;\ \beta=1;\ \omega=\sqrt{56};\ T=0.5$,

$$G(z)=0.1852\frac{z}{z-1}-0.2083\frac{z}{z-0.6065}+0.02314\frac{z^2+0.0678z}{z^2+0.1355z+0.006738}+0.0005748\frac{z}{z^2+0.1355z+0.006738}$$

$$=\frac{0.00004z^4+0.05781z^3+0.02344z^2+0.001946z}{(z-1)(z-0.6065)\left(z^2+0.1355z+0.006738\right)}\approx\frac{0.05781z^3+0.02344z^2+0.001946z}{(z-1)(z-0.6065)\left(z^2+0.1355z+0.006738\right)}$$

$$=0.05781\frac{z^3+0.4055z^2+0.0337z}{(z-1)(z-0.6065)\left(z^2+0.1355z+0.006738\right)}=0.05781\frac{z(z+0.2888)(z+0.1167)}{(z-1)(z-0.6065)\left(z^2+0.1355z+0.006738\right)}$$

7.

Program:

```
'(a)'
syms s z n T                  %Construct symbolic objects for
                              %'s', 'z', 'n',and 'T'.
Gs=(s+4)/((s+2)*(s+5));       %Form G(s).
'G(s)'                        %Display label.
pretty(Gs)                    %Pretty print G(s).
%'g(t)'                       %Display label.
gt=ilaplace(Gs);              %Find g(t).
%pretty(gt)                   %Pretty print g(t).
gnT=compose(gt,n*T);          %Find g(nT).
%'g(kT)'                      %Display label.
%pretty(gnT)                  %Pretty print g(nT).
Gz=ztrans(gnT);               %Find G(z).
Gz=simplify(Gz);              %Simplify G(z).
```

```
%'G(z)'                         %Display label.
%pretty(Gz)                     %Pretty print G(z).
Gz=subs(Gz,T,0.5);              %Let T = 0.5 in G(z).
Gz=vpa(simplify(Gz),6);         %Simplify G(z) and evaluate numerical
                                %values to 6 places.
Gz=vpa(factor(Gz),6);           %Factor G(z).

'G(z) evaluated for T=0.5'      %Display label.
pretty(Gz)                      %Pretty print G(z) with numerical
                                %values.
'(b)'
Gs=(s+1)*(s+2)/(s*(s+3)*(s+4));
                                %Form G(s) = G(s).
'G(s)'                          %Display label.
pretty(Gs)                      %Pretty print G(s).
%'g(t)'                         %Display label.
gt=ilaplace(Gs);                %Find g(t).
%pretty(gt)                     %Pretty print g(t).
gnT=compose(gt,n*T);            %Find g(nT).
%'g(kT)'                        %Display label.
%pretty(gnT)                    %Pretty print g(nT).
Gz=ztrans(gnT);                 %Find G(z).
Gz=simplify(Gz);                %Simplify G(z).
%'G(z)'                         %Display label.
%pretty(Gz)                     %Pretty print G(z).
Gz=subs(Gz,T,0.5);              %Let T = 0.5 in G(z).
Gz=vpa(simplify(Gz),6);         %Simplify G(z) and evaluate numerical
                                %values to 6 places.
Gz=vpa(factor(Gz),6);           %Factor G(z).
'G(z) evaluated for T=0.5'      %Display label.
pretty(Gz)                      %Pretty print G(z) with numerical
                                %values.
'(c)'
Gs=20/((s+3)*(s^2+6*s+25));     %Form G(s) = G(s).
'G(s)'                          %Display label.
pretty(Gs)                      %Pretty print G(s).
%'g(t)'                         %Display label.
gt=ilaplace(Gs);                %Find g(t).
%pretty(gt)                     %Pretty print g(t).
gnT=compose(gt,n*T);            %Find g(nT).
%'g(kT)'                        %Display label.
%pretty(gnT)                    %Pretty print g(nT).
Gz=ztrans(gnT);                 %Find G(z).
Gz=simplify(Gz);                %Simplify G(z).
%'G(z)'                         %Display label.
%pretty(Gz)                     %Pretty print G(z).
Gz=subs(Gz,T,0.5);              %Let T = 0.5 in G(z).
Gz=vpa(simplify(Gz),6);         %Simplify G(z) and evaluate numerical
                                %values to 6 places.
Gz=vpa(factor(Gz),6);           %Factor G(z).
'G(z) evaluated for T=0.5'      %Display label.
pretty(Gz)                      %Pretty print G(z) with numerical
                                %values.
'(d)'
Gs=15/(s*(s+1)*(s^2+10*s+81));
                                %Form G(s) = G(s).
'G(s)'                          %Display label.
pretty(Gs)                      %Pretty print G(s).
%'g(t)'                         %Display label.
gt=ilaplace(Gs);                %Find g(t).
%pretty(gt)                     %Pretty print g(t).
gnT=compose(gt,n*T);            %Find g(nT).
%'g(kT)'                        %Display label.
%pretty(gnT)                    %Pretty print g(nT).
Gz=ztrans(gnT);                 %Find G(z).
Gz=simplify(Gz);                %Simplify G(z).
%'G(z)'                         %Display label.
```

```
%pretty(Gz)                 %Pretty print G(z).
Gz=subs(Gz,T,0.5);          %Let T = 0.5 in G(z).
Gz=vpa(simplify(Gz),6);     %Simplify G(z) and evaluate numerical
                            %values to 6 places.
Gz=vpa(factor(Gz),6);       %Factor G(z).
'G(z) evaluated for T=0.5'  %Display label.
pretty(Gz)                  %Pretty print G(z) with numerical
                            %values.
```

Computer response:

```
ans =

(a)

ans =

G(s)

                                  s + 4
                             ---------------
                             (s + 2) (s + 5)
ans =

G(z) evaluated for T=0.5

                                z (z - .177350)
                  1.00000 ----------------------------
                          (z - .0820850) (z - .367880)
ans =

(b)

ans =

G(s)

                             (s + 1) (s + 2)
                            -----------------
                            s (s + 3) (s + 4)
ans =

G(z) evaluated for T=0.5

                       z (z - .296742) (z - .840812)
              1.00000 ------------------------------------
                      (z - .135335) (z - .223130) (z - 1.)
ans =

ans =

(c)


ans =

G(s)


                                  20
                        -----------------------
                                  2
                        (s + 3) (s  + 6 s + 25)

ans =

G(z) evaluated for T=0.5
```

```
                          (z + .223130) z
.394980 -----------------------------------------
                            2
        (z - .223135) (z  + .185705 z + .0497861)
```

```
ans =

(d)


ans =

G(s)


                      15
          --------------------------
                      2
          s (s + 1) (s  + 10 s + 81)

ans =

G(z) evaluated for T=0.5

                         (z + .289175) (z + .116364) z
.0578297 -----------------------------------------------------------
                                                2
         (z - .606535) (z - .999995) (z  + .135489 z + .00673794)
```

8.

a.

```
                            20          4                      1
G2(s) = G(s)/s = ----------  =  ---- - 4/5 1/s + 4/5 -----
                        2           2                  s + 5
                       s  (s + 5)    s
```

Thus,

```
g2t = 4 k T - 4/5 + 4/5 exp(-5 k T)
```

Hence,

```
                   /    T z            z                      z            \
G(z) =  (1 - 1/z) |4 -------- - 4/5 ----- + 4/5 -------------------------|
                   |         2       z - 1                  /    z        \|
                   |  (z - 1)                    exp(-5 T) |--------- - 1||
                   \                                        \exp(-5 T)    //
```

Letting $T = 0.3$,

```
                  6.482 z + 3.964
G(z) = 4000 ------------------------
            (z - 1.) (4.482 z - 1.)
```

b.

$$G2(s) = G(s)/s = \frac{20}{s^2\,(s+5)\,(s+3)} = 4/3\,\frac{1}{s^2} - \frac{32}{45}\,1/s + 10/9\,\frac{1}{s+3} - 2/5\,\frac{1}{s+5}$$

Thus,

$$g2t = 4/3\,k\,T - \frac{32}{45} + 10/9\,\exp(-3\,k\,T) - 2/5\,\exp(-5\,k\,T)$$

Hence,

$$G(z) = (1 - 1/z)\left(4/3\,\frac{T\,z}{(z-1)^2} - \frac{32}{45}\,\frac{z}{z-1} + 10/9\,\frac{z}{\exp(-3\,T)\left(\dfrac{z}{\exp(-3\,T)} - 1\right)} - 2/5\,\frac{z}{\exp(-5\,T)\left(\dfrac{z}{\exp(-5\,T)} - 1\right)}\right)$$

Letting $T = 0.3$,

$$G(z) = .04444\,\frac{12.82\,z^2 + 29.1\,z + 3.84}{(z - 1.)\,(2.460\,z - 1.)\,(4.482\,z - 1.)}$$

c.

$$G_e(z) = G_a(z)G(z)$$

where $G_a(z)$ is the answer to part (a) and $G(z)$, the pulse transfer function for $\frac{1}{s+3}$ in cascade with a zero-order-hold will now be found:

$$G2(s) = G(s)/s = \frac{1}{s\,(s+3)} = 1/s - 1/3\,\frac{1}{s+3}$$

Thus,

$$g2t = 1/3 - 1/3\,\exp(-3\,k\,T)$$

Hence,

```
                   /       z                          z             \
G(z) = (1 - 1/z)  |1/3 ----- - 1/3 -------------------------|
                  |     z - 1                  /     z        \||
                  |                exp(-3 T)  |--------- - 1||
                   \                           \exp(-3 T)     //
```

Letting $T = 0.3$,

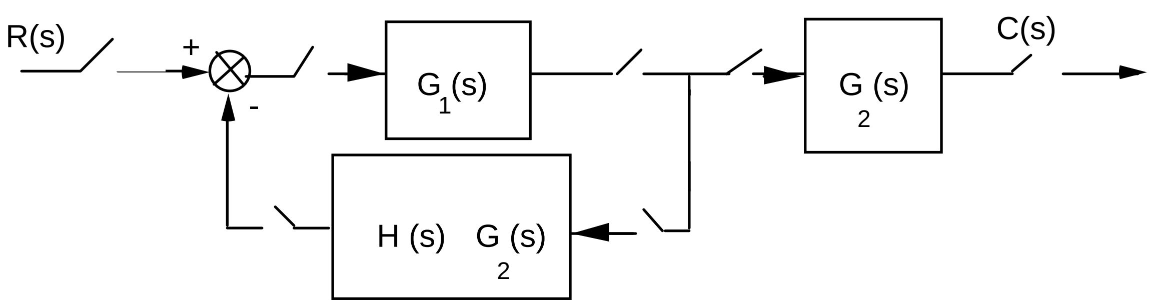
```
            .4866
G(z) =  ------------
        2.460 z - 1.
```

Thus,

$$G_e(z) = G_a(z)G(z) = 0.19464\frac{6.482z + 3.964}{(z-1)(4.482z-1)(2.46z-1)}$$

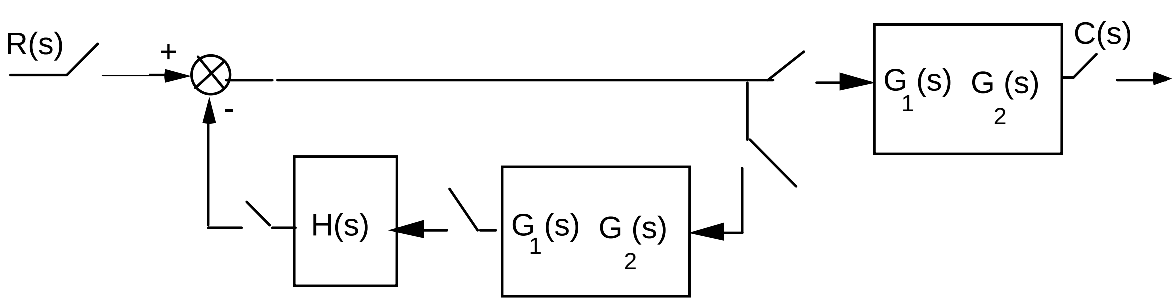
9.

a. Add phantom samplers at the input, output, and feedback path after H(s). Push G_2 (s) and its input sampler to the right past the pickoff point. Add a phantom sampler after $G_1(s)$. Hence,



From this block diagram, $T(z) = \frac{G_1(z)G_2(z)}{1 + G_1(z)HG_2(z)}$.

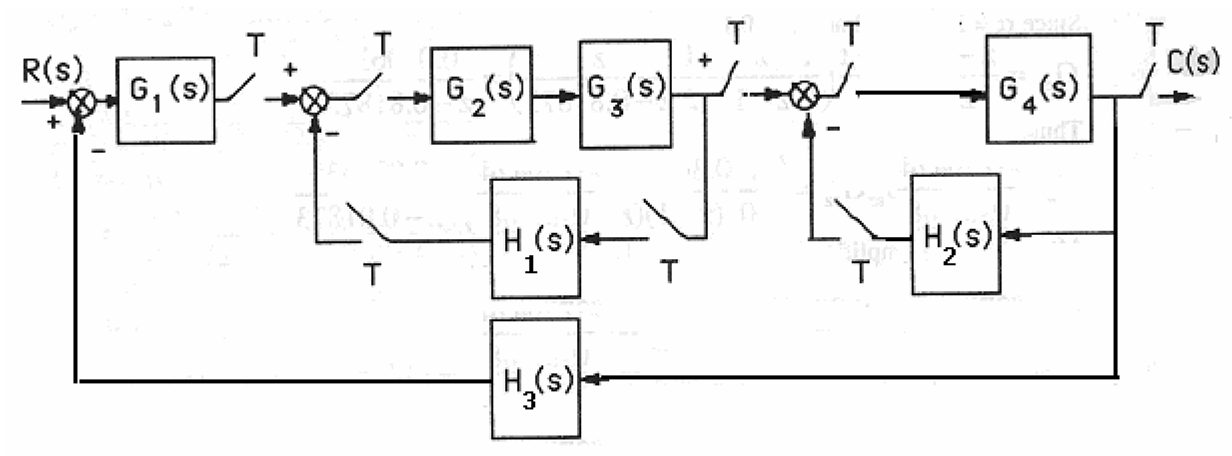
b. Add phantom samplers to the input, output, and the output of H(s). Push $G_1(s)G_2(s)$ and its input sampler to the right past the pickoff point. Add a phantom sampler at the output.



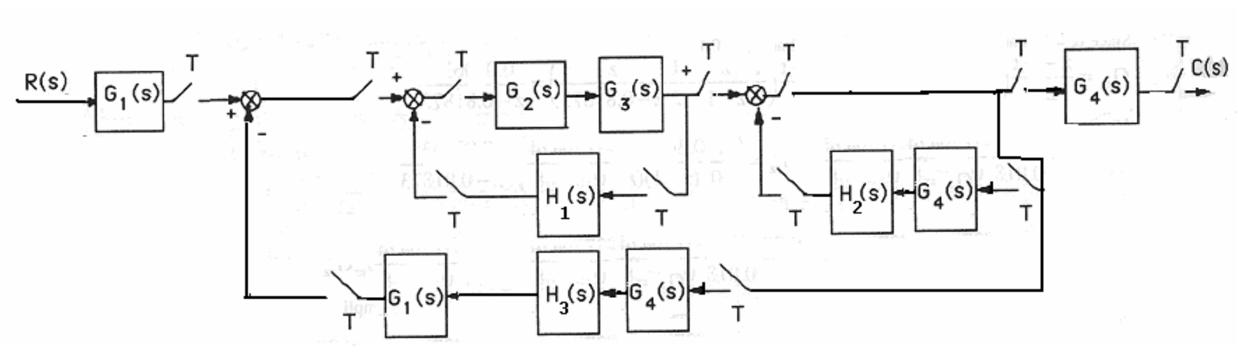
From this block diagram, $T(z) = \dfrac{G_1G_2(z)}{1 + G_1G_2(z)H(z)}$.

10.

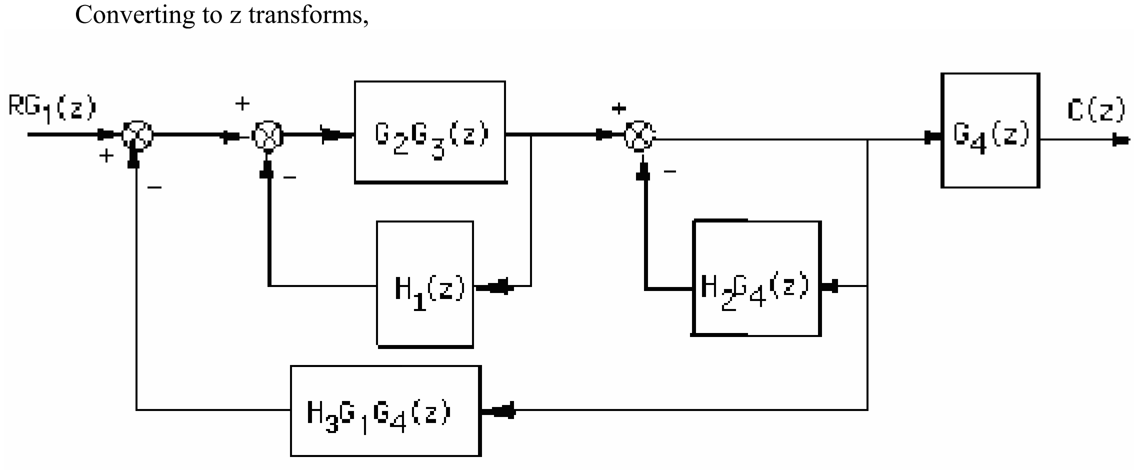
Add phantom samplers after $G_1(s)$, $G_3(s)$, $G_4(s)$, $H_1(s)$, and $H_2(s)$.



Push $G_1(s)$ and its sampler to the left past the summing junction. Also, push $G_4(s)$ and its input sampler to the right past the pickoff point. The resulting block diagram is,



Converting to z transforms,

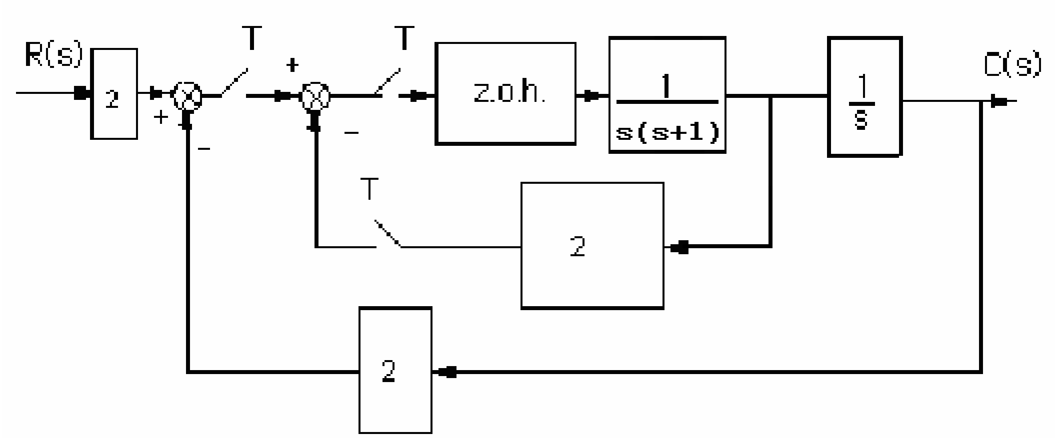


$$C(s) = RG_1(z)G_4(z)\left[\frac{\dfrac{G_2G_3(z)}{(1+G_2G_3(z)H_1(z))} * \dfrac{1}{(1+H_2G_4(z))}}{1+\dfrac{G_2G_3(z)}{(1+G_2G_3(z)H_1(z))} * \dfrac{1}{(1+H_2G_4(z))} H_3G_1G_4(z)}\right]$$

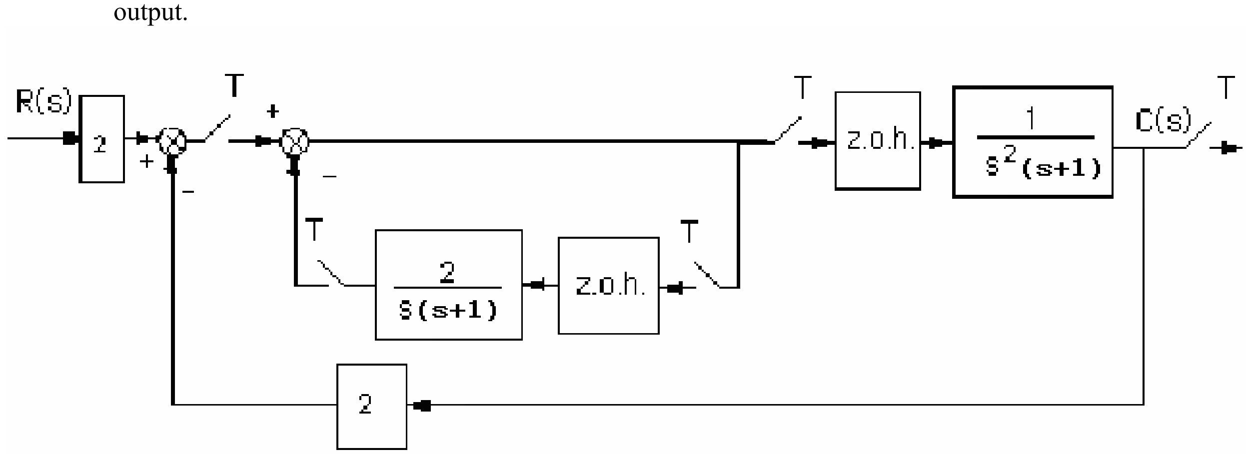
$$= \frac{RG_1(z)G_4(z)G_2G_3(z)}{(1+G_2G_3(z)H_1(z))(1+H_2G_4(z))+G_2G_3(z)H_3G_1G_4(z)}$$

11.

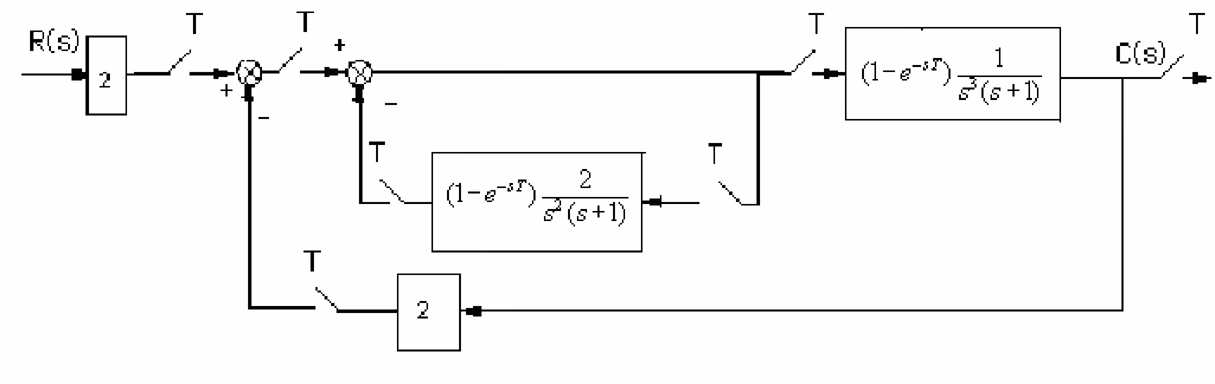
Push gain of 2 to the left past the summing junction and add phantom samplers as shown.



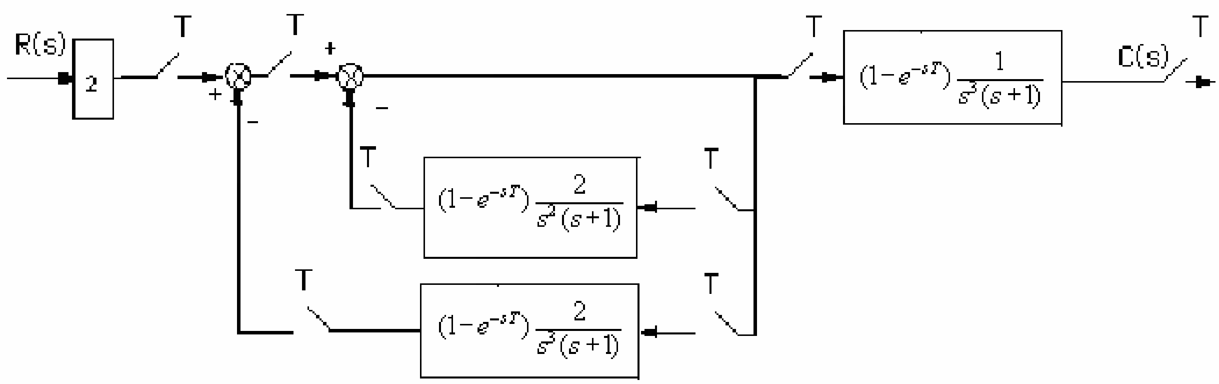
Push the z.o.h. and $\dfrac{1}{s(s+1)}$ to the right past the pickoff point. Also, add a phantom sampler at the output.



Add phantom samplers after the gain of 2 at the input and in the feedback. Also, represent the z.o.h. as Laplace transforms.



Push the last block to the right past the pickoff point and get,



Find the z transform for each transfer function.

$$G_1(s) = 2$$

transforms into

$$G_1(z) = 2\,.$$

$$H_1(s) = (1-e^{-sT})\frac{2}{s^2(s+1)} = (1-e^{-sT})\left[\frac{2}{s^2} - \frac{2}{s} + \frac{2}{s+1}\right]$$

transforms into

$$H_1(z) = \frac{z-1}{z}\left[2\frac{Tz}{(z-1)^2} - 2\frac{z}{z-1} + 2\frac{z}{z-e^{-T}}\right] = 2\frac{Tz - Te^{-T} + ze^{-T} - z - e^{-T} + 1}{(z-1)(z-e^{-T})}$$

$$H_2(s) = (1-e^{-sT})\frac{2}{s^3(s+1)} = (1-e^{-sT})\left[\frac{2}{s} - \frac{2}{s+1} - \frac{2}{s^2} + \frac{2}{s^3}\right]$$

transforms into

$$H_2(z) = \frac{z-1}{z}\left[\frac{2z}{z-1} - \frac{2z}{z-e^{-T}} - \frac{2Tz}{(z-1)^2} + \frac{T^2 z(z+1)}{(z-1)^3}\right]$$

$$= \frac{(T^2 - 2e^{-T} + 2 - 2T)z^2 + (4e^{-T} - 4 + 2Te^{-T} + 2T + T^2 - T^2e^{-T})z + (2 - 2e^{-T} - 2Te^{-T} - T^2e^{-T})}{(z-1)^2(z-e^{-T})}$$

$$G_2(s) = (1-e^{-sT})\frac{1}{s^3(s+1)}$$

transforms into

$$\frac{1}{2}H_2(z)$$

Thus, the closed-loop transfer function is

$$T(z) = G_1(z)G_2(z)\left[\frac{1}{1+H_1(z)+H_2(z)}\right]$$

12.

$$G(z) = \frac{z-1}{z} \; z\left\{\frac{1}{s^2(s+1)}\right\}.$$

Using Eq. (13.49)

$$G(z) = \frac{T}{z-1} - \frac{(1-e^{-T})}{z-e^{-T}} = \frac{(T-1+e^{-T})z + (1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})}$$

But,

$$T(z) = \frac{G(z)}{1+G(z)} = \frac{(T-1+e^{-T})z + (1-e^{-T}-Te^{-T})}{z^2 + (T-2)z + (1-Te^{-T})}$$

The roots of the denominator are inside the unit circle for $0<T<3.923$.

13.

Program:

```
numg1=10*[1 7];
deng1=poly([-1 -3 -4 -5]);
G1=tf(numg1,deng1);
for T=5:-.01:0;
Gz=c2d(G1,T,'zoh');
Tz=feedback(Gz,1);
r=pole(Tz);
rm=max(abs(r));
if rm<=1;
break;
end;
end;
T
r
rm
```

Computer response:

```
T =

    3.3600


r =

   -0.9990
   -0.0461
   -0.0001
```

```
   -0.0000


rm =

    0.9990

>>
T =

    3.3600


r =

   -0.9990
   -0.0461
   -0.0001
   -0.0000


rm =

    0.9990
```

14.

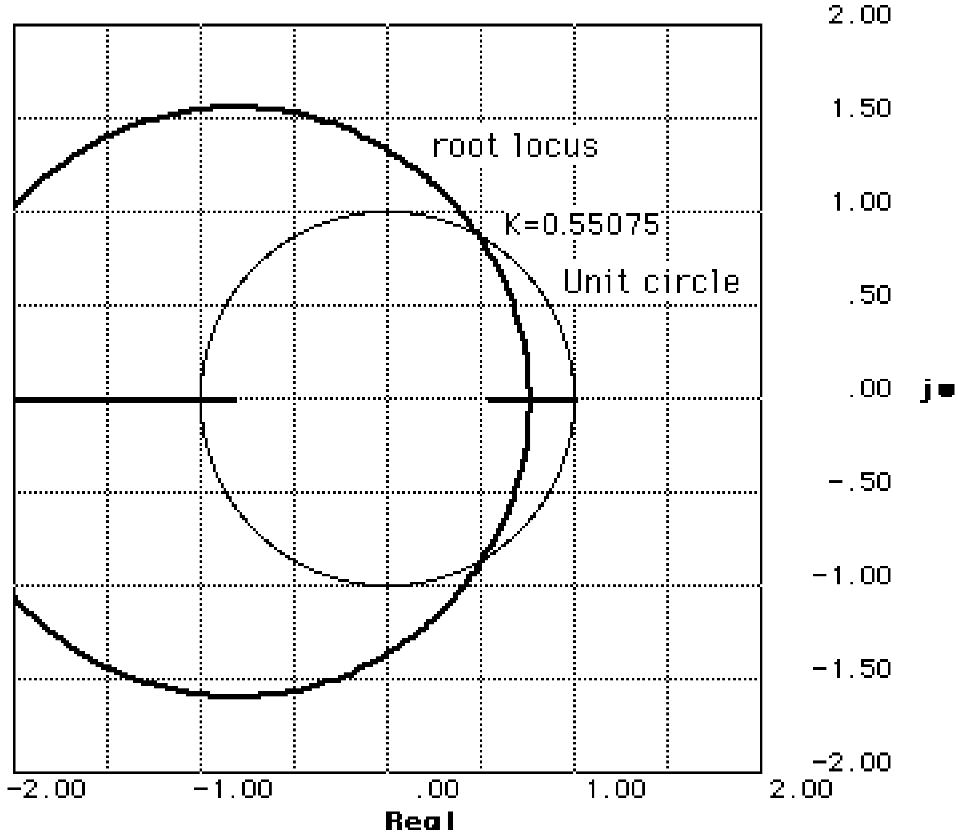
$$G_s = K\left(1-e^{-sT}\right)\frac{3}{s^2(s+3)}$$

$$G_s = \left(1-e^{-sT}\right)K\left(\frac{1}{3}\frac{1}{s+3}-\frac{1}{3}\frac{1}{s}+\frac{1}{s^2}\right)$$

$$G_z = K\left(\frac{z-1}{z}\left[\frac{1}{3}\frac{z}{z-e^{-3\cdot 0.2}}-\frac{1}{3}\frac{z}{z-1}+\frac{0.2z}{(z-1)^2}\right]\right)$$

$$G_z = K\left(0.049604\frac{z+0.81917}{[z-1][z-0.54881]}\right)$$

The root locus for this function shows it crossing the unit circle at 60.06 degrees at a gain of .55078. Thus, K= 0.55078/0.049604 = 11.104 and 0<K<11.104.



15.

a.

$$G_s = \left(1 - e^{-Ts}\right) \frac{1}{s(s+\alpha)}$$

$$G_s = \left(1 - e^{-Ts}\right)\left(-\frac{1}{\alpha[s+\alpha]} + \frac{1}{\alpha s}\right)$$

$$G_z = \frac{z-1}{z}\left(-\frac{1}{\alpha}\frac{z}{z - e^{-\alpha T}} + \frac{1}{\alpha}\frac{z}{z-1}\right)$$

$$G_z = \frac{-\frac{1}{e^{T\alpha}} + 1}{\alpha\left(z - \frac{1}{e^{T\alpha}}\right)}$$

$$\alpha = 2$$

$$T = 0.5$$

$$G_z = 0.31606 \frac{1}{z - 0.36788}$$

First, check to see that the system is stable.

$$T_z = \frac{G_z}{1 + G_z}$$

$$T_z = 0.31606 \frac{1}{z - 0.051819}$$

Since the closed-loop poles are inside the unit circle, the system is stable. Next, evaluate the static error constants and the steady-state error.

$$K_p = \lim_{z\to 1} G(z) = 0.5 \qquad e^*(\infty) = \frac{1}{1+K_p} = \frac{2}{3}$$

$$K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z) = 0 \qquad e^*(\infty) = \frac{1}{K_v} = \infty$$

$$K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z) = 0 \qquad e^*(\infty) = \frac{1}{K_a} = \infty$$

b.

$$G_s = \left(1 - e^{-Ts}\right)\frac{K\alpha}{s^2(s+\alpha)}$$

From Equation 13.48

$$G_z = K\frac{\alpha T\left(z - e^{-\alpha T}\right) - (z-1)\left(1 - e^{-\alpha T}\right)}{\alpha(z-1)\left(z - e^{-\alpha T}\right)}$$

$K = 10$

$\alpha = 2$

$T = 0.1$

$$G_z = 5\,\frac{0.018731\,(z + 0.93553)}{(z-1)\,(z - 0.81873)}$$

First, test stability.

$$T_z = \frac{G_z}{1 + G_z}$$

$$T_z = 0.093654\,\frac{z + 0.93553}{(z - 0.86254 + 0.40296i)\,(z - 0.86254 - 0.40296i)}$$

The system is stable. The closed-loop poles are inside the unit circle. Now find the static error constants and the steady-state errors.

$$K_p = \lim_{z\to 1} G(z) = \infty \qquad e^*(\infty) = \frac{1}{1+K_p} = 0$$

$$K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z) = 10 \qquad e^*(\infty) = \frac{1}{K_v} = 0.1$$

$$K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z) = 0 \qquad e^*(\infty) = \frac{1}{K_a} = \infty$$

c.

$$G_z = \frac{1.28}{z-0.37}$$

First, test stability.

$$T_z = \frac{G_z}{1+G_z}$$

$$T_z = 1.28\frac{1}{z+0.91}$$

The system is stable. The closed-loop pole is inside the unit circle. Now find the static error constants and the steady-state errors.

$$K_p = \lim_{z\to 1} G(z) = 2.03 \qquad e^*(\infty) = \frac{1}{1+K_p} = 0.33$$

$$K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z) = 0 \qquad e^*(\infty) = \frac{1}{K_v} = \infty$$

$$K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z) = 0 \qquad e^*(\infty) = \frac{1}{K_a} = \infty$$

d.

$$G_z = \frac{0.13(z+1)}{(z-1)(z-0.74)}$$

First, test stability.

$$T_z = \frac{G_z}{1+G_z}$$

$$T_z = 0.13\frac{z+1}{z^2-1.61z+0.87}$$

$$T_z = 0.13\frac{z+1}{(z+[-0.805+0.47114i])(z+[-0.805-0.47114i])}$$

The system is stable. The closed-loop pole is inside the unit circle. Now find the static error constants and the steady-state errors.

$$K_p = \lim_{z\to 1} G(z) = \infty \qquad e^*(\infty) = \frac{1}{1+K_p} = 0$$

$$K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z) = 10 \qquad e^*(\infty) = \frac{1}{K_v} = 0.1$$

$$K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z) = 0 \qquad e^*(\infty) = \frac{1}{K_a} = \infty$$

16.

Program:

```
T=0.1;
numgz=[0.04406 -0.03624 -0.03284 0.02857];
dengz=[1 -3.394 +4.29 -2.393 +0.4966];
'G(z)'
Gz=tf(numgz,dengz,0.1)
'Zeros of G(z)'
zeros=roots(numgz)
'Poles of G(z)'
```

```
poles=roots(dengz)
%Check stability
Tz=feedback(Gz,1);
'Closed-Loop Poles'
r=pole(Tz)
M=abs(r)
pause
'Find Kp'
Gz=minreal(Gz,.00001);
Kp=dcgain(Gz)
'Find Kv'
factorkv=tf([1 -1],[1 0],0.1);    %Makes transfer function
                                  %proper and yields same Kv
Gzkv=factorkv*Gz;

Gzkv=minreal(Gzkv,.00001);        %Cancel common poles and
                                  %zeros
Kv=(1/T)*dcgain(Gzkv)
'Find Ka'
factorka=tf([1 -2 1],[1 0 0],0.1);%Makes transfer function
                                  %proper and yields same Ka
Gzka=factorka*Gz;

Gzka=minreal(Gzka,.00001);        %Cancel common poles and
                                  %zeros
Ka=(1/T)^2*dcgain(Gzka)
```

Computer response:

```
ans =

G(z)


Transfer function:
0.04406 z^3 - 0.03624 z^2 - 0.03284 z + 0.02857
-----------------------------------------------
 z^4 - 3.394 z^3 + 4.29 z^2 - 2.393 z + 0.4966

Sampling time: 0.1

ans =

Zeros of G(z)


zeros =

  -0.8753
   0.8489 + 0.1419i
   0.8489 - 0.1419i


ans =

Poles of G(z)


poles =

   1.0392
   0.8496 + 0.0839i
   0.8496 - 0.0839i
   0.6557


ans =
```

```
Closed-Loop Poles

r =

   0.9176 + 0.1699i
   0.9176 - 0.1699i
   0.7573 + 0.1716i
   0.7573 - 0.1716i

M =

    0.9332
    0.9332
    0.7765
    0.7765

ans =

Find Kp

Kp =

   -8.8750

ans =

Find Kv

Kv =

     0

ans =

Find Ka

Ka =

     0
```

17.

First find G(z)

$$G_s = \left(K\left[1-e^{-sT}\right]\right)\frac{1}{s\,(s+1)\,(s+3)}$$

$$G_s = \left(K\left[1-e^{-sT}\right]\right)\left(\frac{1}{6}\,\frac{1}{s+3}-\frac{1}{2}\,\frac{1}{s+1}+\frac{1}{3}\,\frac{1}{s}\right)$$

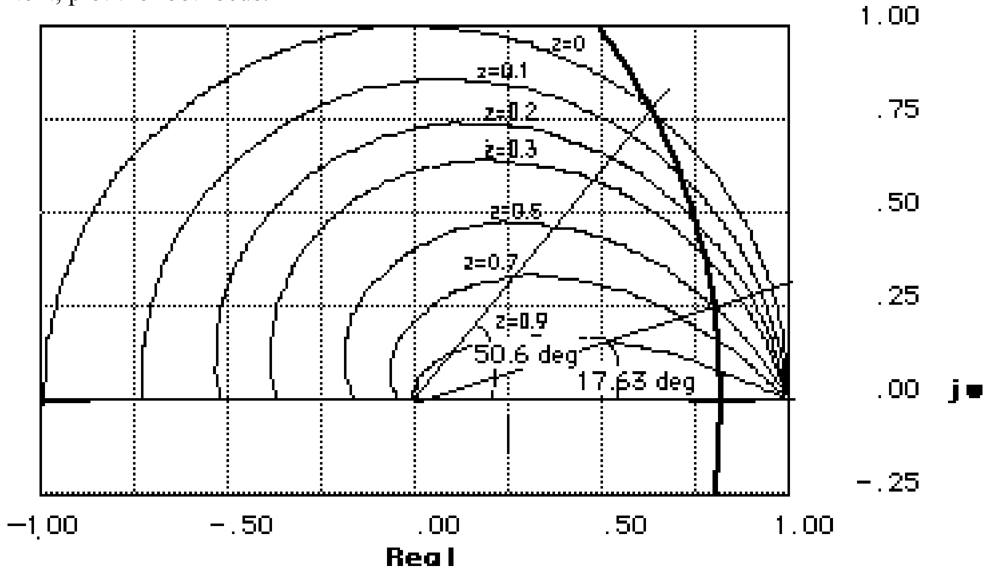
$$T = 0.1$$

$$G_{\mathbf{z}} = K\,\frac{z-1}{z}\left(\frac{1}{6}\,\frac{z}{z-e^{-3T}}-\frac{1}{2}\,\frac{z}{z-e^{-T}}+\frac{1}{3}\,\frac{z}{z-1}\right)$$

$$G_{\mathbf{z}} = \frac{K\,(z-1)\left(\frac{1}{3}\,\frac{z}{z-1}-\frac{1}{2}\,\frac{z}{z-0.90484}+\frac{1}{6}\,\frac{z}{z-0.74082}\right)}{z}$$

$$G_{\mathbf{z}} = 0.0043843\,\frac{K\,(z+0.87519)}{(z-0.74082)\,(z-0.90484)}$$

Next, plot the root locus.



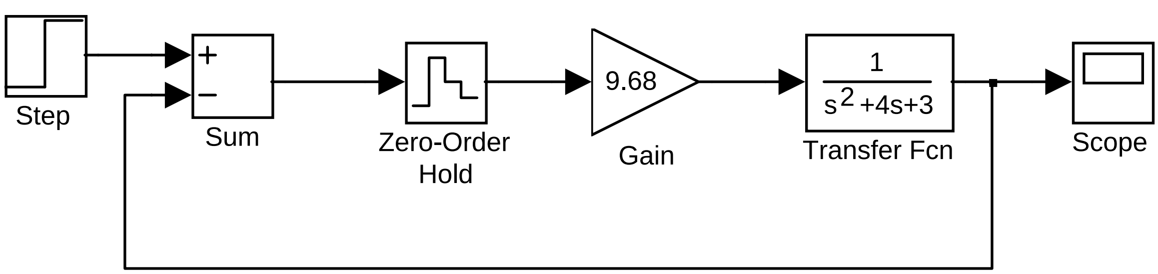
Root locus intersects 0.5 damping ratio for
0.0043843K=0.042444. Thus, K=9.68 for 16.3% overshoot.

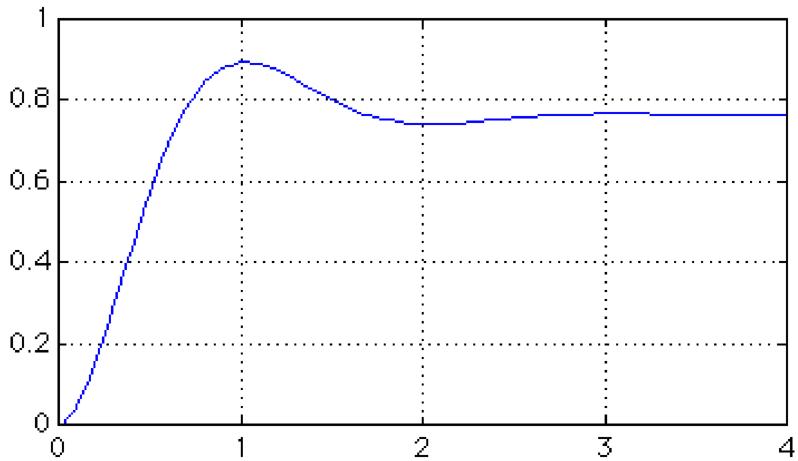
$K = 9.68$

$$G_z = 0.04244\,\frac{z + 0.87519}{z^2 - 1.6457\,z + 0.67032}$$

Root locus intersects 0 damping ratio for
0.0043843K=0.37642. Thus, 0<K<85.86 for stability.

18.





19.

Program:

```
numgz=0.13*[1 1];
dengz=poly([1 0.74]);
Gz=tf(numgz,dengz,0.1)
Gzpkz=zpk(Gz)
Tz=feedback(Gz,1)
Ltiview
```

Computer response:

```
Transfer function:
   0.13 z + 0.13
-------------------
z^2 - 1.74 z + 0.74

Sampling time: 0.1

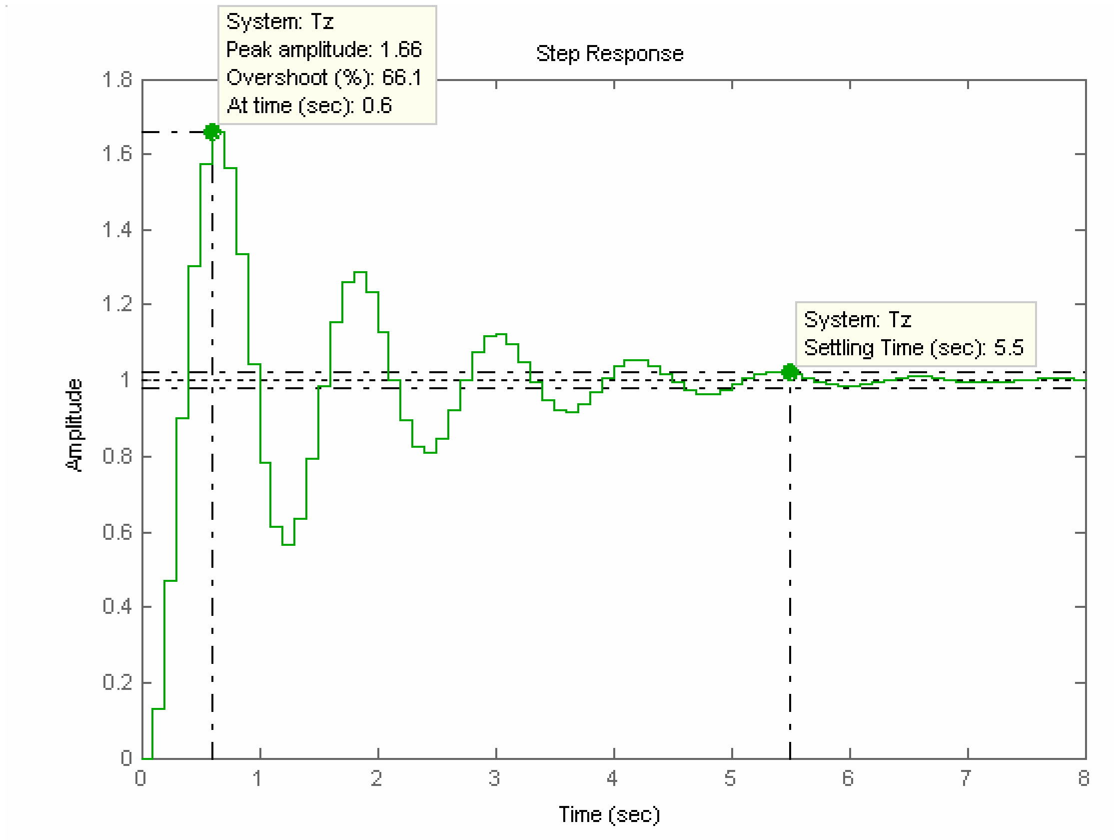


Zero/pole/gain:
  0.13 (z+1)
--------------
(z-1) (z-0.74)

Sampling time: 0.1

Transfer function:
   0.13 z + 0.13
-------------------
z^2 - 1.61 z + 0.87

Sampling time: 0.1
```



20.

Program:

```
%Digitize G1(s) preceded by a sample and hold
numg1=1;
deng1=poly([-1  -3]);
'G1(s)'
G1s=tf(numg1,deng1)
'G(z)'
Gz=c2d(G1s,0.1,'zoh')
%Input transient response specifications
Po=input('Type %OS  ');
%Determine damping ratio
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2));
%Plot root locus
rlocus(Gz)
zgrid(z,0)
title(['Root Locus'])
[K,p]=rlocfind(Gz)  %Allows input by selecting point on graphic
pause
'T(z)'
Tz=feedback(K*Gz,1)
step(Tz)
```

Computer response:

```
ans =

G1(s)


Transfer function:
```

```
      1
-------------
s^2 + 4 s + 3


ans =

G(z)


Transfer function:
0.004384 z + 0.003837
----------------------
z^2 - 1.646 z + 0.6703

Sampling time: 0.1
Type %OS  16.3
Select a point in the graphics window

selected_point =

   0.8016 + 0.2553i


K =

    9.7200


p =

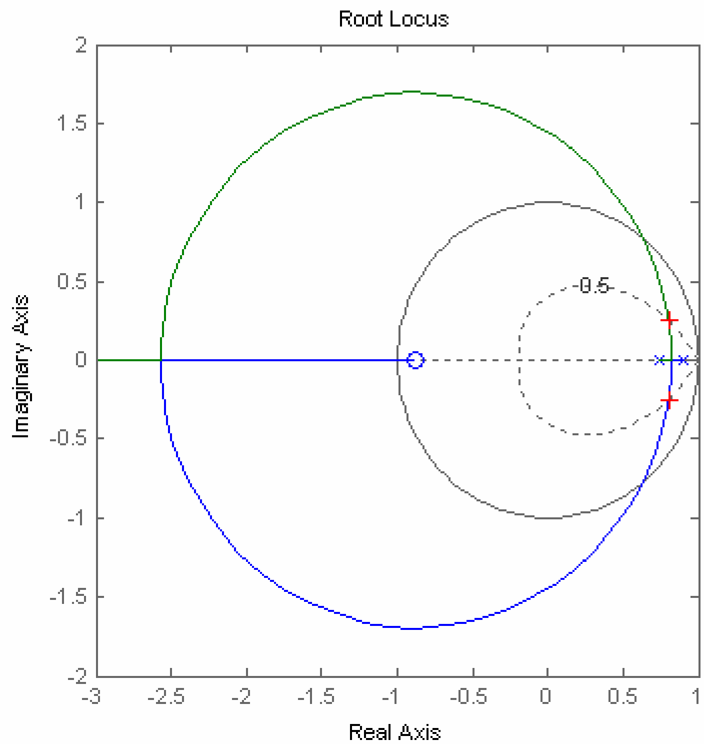
   0.8015 + 0.2553i
   0.8015 - 0.2553i

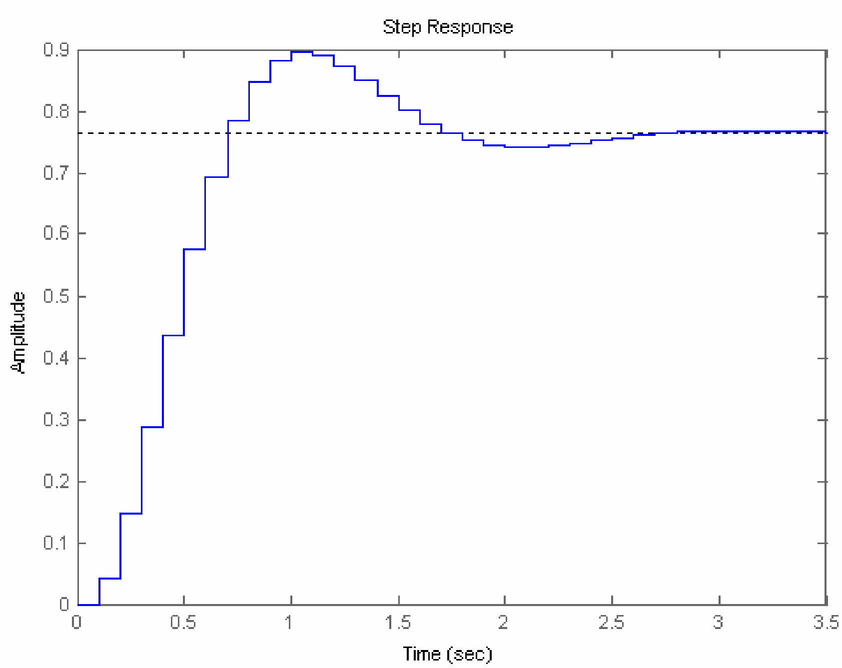

ans =

T(z)


Transfer function:
  0.04262 z + 0.0373
----------------------
z^2 - 1.603 z + 0.7076

Sampling time: 0.1
```


Root Locus
Imaginary Axis
Real Axis
0.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1


Step Response
Amplitude
Time (sec)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.5
1
1.5
2
2.5
3
3.5

21.

Using the result from Problem 13.12

$$G_z = \frac{\left(T-1+e^{-T}\right)z+\left(1-e^{-T}-Te^{-T}\right)}{(z-1)\left(z-e^{-T}\right)}K$$

Letting T=0.1,

$$G_z = \frac{(0.0048374\,(z+0.96722))\,K}{(z-1)\,(z-0.90484)}$$

For Tp = 2 seconds,

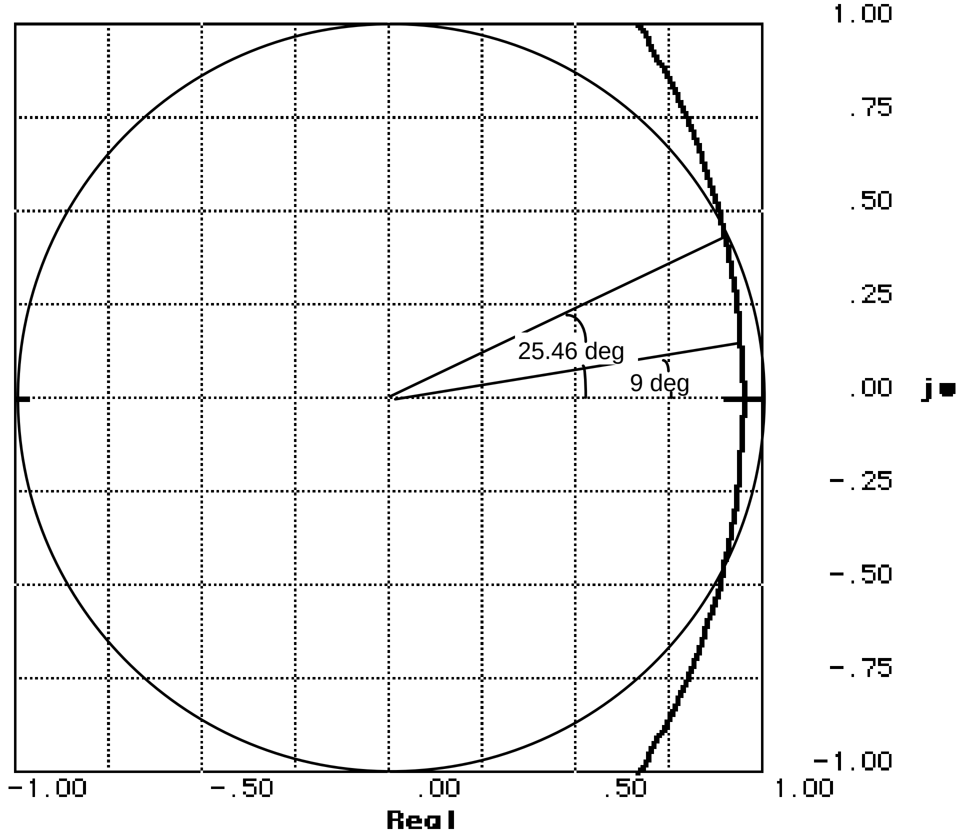
$$\frac{T_p}{T} = 20$$

Hence,

$$\frac{\pi}{\theta_1} = 20$$

Or,

$$\theta_1 = 9^0$$



The root locus intersects the Tp/T curve at 0.958<9 deg. with a gain of 0.0129. Hence, 4.837E-3 K = 0.0129, or K=2.67.

To determine stability, we see that the root locus intersects the 0 damping ratio curve at 1<25.4 deg. with a gain of 0.0983. Hence, 4.837E-3 K = 0.0983, or K=20.32.

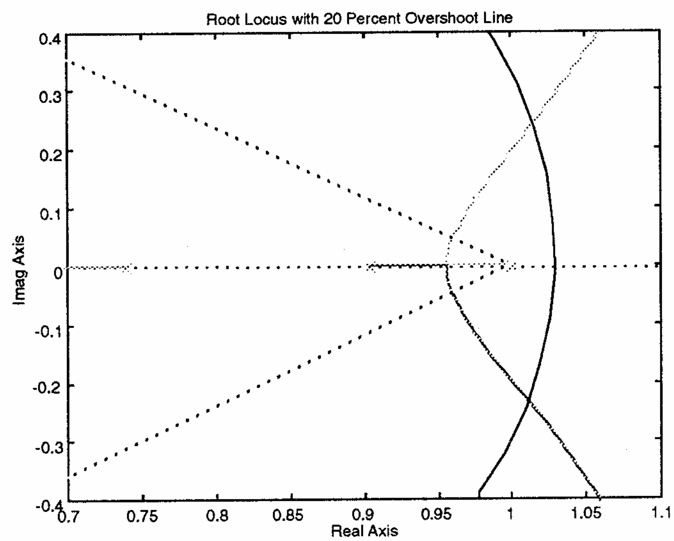
22.

First find G(z). $G(z) = K\frac{z-1}{z}\ z\{\frac{1}{s^2(s+1)(s+3)}\} = K\frac{z-1}{z}$

$z\{-\frac{1}{18}\frac{1}{s+3}+\frac{1}{2}\frac{1}{s+1}-\frac{4}{9}\frac{1}{s}+\frac{1}{3}\frac{1}{s^2}\}$

For T=0.1, $G(z) = K\frac{z-1}{z}\left(\frac{-\frac{1}{18}z}{z-0.74082}+\frac{\frac{1}{2}z}{z-0.90484}-\frac{\frac{4}{9}z}{z-1}+\frac{\frac{1}{30}z}{[z-1]^2}\right)$

$= 0.00015103K\frac{(z+0.24204)(z+3.3828)}{(z-1)(z-0.74082)(z-0.90484)}$. Plotting the root locus and overlaying the 20% overshoot curve, we select the point of intersection and calculate the gain: 0.00015103K = 1.789. Thus, K = 11845.33. Finding the intersection with the unit circle yields 0.00015103K = 9.85. Thus, 0 < K < 65218.83 for stability.

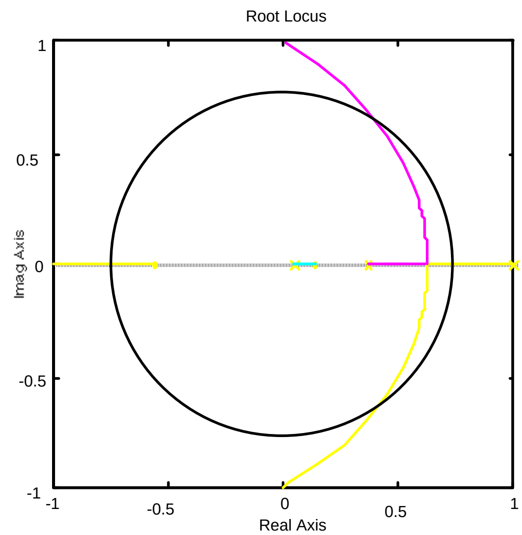


23.

First find G(z). $G(z) = K\frac{z-1}{z}\ z\{\frac{(s+2)}{s^2(s+1)(s+3)}\} = K\frac{z-1}{z}\ z\{\frac{1}{18}\frac{1}{s+3}+\frac{1}{2}\frac{1}{s+1}-\frac{5}{9}\frac{1}{s}+\frac{2}{3}\frac{1}{s^2}\} =$

For T=1, $G(z) = K\frac{z-1}{z}\left(\frac{\frac{1}{18}z}{z-0.049787}+\frac{\frac{1}{2}z}{z-0.36788}-\frac{\frac{5}{9}z}{z-1}+\frac{\frac{2}{3}z}{[z-1]^2}\right)$

$= 0.29782K\frac{(z-0.13774)(z+0.55935)}{(z-1)(z-0.049787)(z-0.36788)}$. Plotting the root locus and overlaying the $T_s = 15$ second circle, we select the point of intersection (0.4 + j0.63) and calculate the gain: 0.29782K = 1.6881. Thus, K = 5.668. Finding the intersection with the unit circle yields 0.29782K = 4.4. Thus, 0 < K < 14.77 for stability.



24.

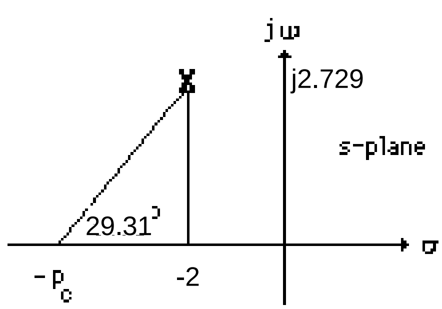
Substituting Eq. (13.88) into $G_c(s)$ and letting T = 0.01 yields

$$G_c(z) = \frac{1180z^2 - 1839z + 661.1}{z^2 - 1} = 1180\frac{(z-0.9959)(z-0.5625)}{(z+1)(z-1)}$$

25.

Since %OS = 10%, $\zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} = 0.591$. Since $T_s = \frac{4}{\zeta\omega_n} = 2$ seconds,

$\omega_n = 3.383$ rad/s. Hence, the location of the closed-loop poles must be $-2 \pm j2.729$. The summation of angles from open-loop poles to $-2 \pm j2.729$ is -192.99^o. Therefore, the design point is not on the root locus. A compensator whose angular contribution is $192.99^o - 180^o = 12.99^o$ is required. Assume a compensator zero at -5 canceling the pole at -5. Adding the compensator zero at –5 to the plant's poles yields -150.69^0 at to $-2 \pm j2.729$. Thus, the compensator's pole must contribute $180^0 - 150.69 = 29.31^0$. The following geometry results.



Thus,

$$\frac{2.729}{p_c - 2} = \tan(29.31^0)$$

Hence, p_c = 6.86.Adding the compensator pole and zero to the system poles, the gain at the design point is found to be 124.3. Summarizing the results: $G_c(s) = \frac{124.3(s+5)}{(s+6.86)}$. Substituting Eq. (13.88) into $G_c(s)$ and letting T = 0.01 yields

$$G_c(z) = \frac{123.2z - 117.2}{z - 0.9337} = \frac{123.2(z - 0.9512)}{(z - 0.9337)}$$

26.

Program:

```
'Design of digital lead compensation'
clf                                 %Clear graph on screen.
'Uncompensated System'              %Display label.
numg=1;                             %Generate numerator of G(s).
deng=poly([0 -5 -8]);               %Generate denominator of G(s).
'G(s)'                              %Display label.
G=tf(numg,deng)                     %Create and display G(s).
pos=input('Type desired percent overshoot ');
                                    %Input desired percent overshoot.
z=-log(pos/100)/sqrt(pi^2+[log(pos/100)]^2);
                                    %Calculate damping ratio.
rlocus(G)                           %Plot uncompensated root locus.
sgrid(z,0)                          %Overlay desired percent overshoot
                                    %line.
title(['Uncompensated Root Locus with ' , num2str(pos),...
'% Overshoot Line'])                %Title uncompensated root locus.
[K,p]=rlocfind(G);                  %Generate gain, K, and closed-loop
                                    %poles, p, for point selected
                                    %interactively on the root locus.
'Closed-loop poles = '              %Display label.
p                                   %Display closed-loop poles.
f=input('Give pole number that is operating point   ');
                                    %Choose uncompensated system
                                    %dominant pole.
'Summary of estimated specifications for selected point on'
'uncompensated root locus'          %Display label.
operatingpoint=p(f)                 %Display uncompensated dominant
                                    %pole.
gain=K                              %Display uncompensated gain.
estimated_settling_time=4/abs(real(p(f)))
                                    %Display uncompensated settling
                                    %time.
estimated_peak_time=pi/abs(imag(p(f)))
                                    %Display uncompensated peak time.
estimated_percent_overshoot=pos     %Display uncompensated percent
                                    %overshoot.
estimated_damping_ratio=z           %Display uncompensated damping
                                    %ratio.
estimated_natural_frequency=sqrt(real(p(f))^2+imag(p(f))^2)
                                    %Display uncompensated natural
                                    %frequency.
numkv=conv([1 0],numg);             %Set up numerator to evaluate Kv.
denkv=deng;                         %Set up denominator to evaluate Kv.
sG=tf(numkv,denkv);                 %Create sG(s).
sG=minreal(sG);                     %Cancel common poles and zeros.
Kv=dcgain(K*sG)                     %Display uncompensated Kv.
ess=1/Kv                            %Display uncompensated steady-state
                                    %error for unit ramp input.
'T(s)'                              %Display label.
T=feedback(K*G,1)                   %Create and display T(s).
step(T)                             %Plot step response of uncompensated
                                    %system.
title(['Uncompensated System with  ' ,num2str(pos),'% Overshoot'])
                                    %Add title to uncompensated step
                                    %response.
```

```
'Press any key to go to lead compensation'
                                    %Display label.
pause
Ts=input('Type Desired Settling Time ');
                                    %Input desired settling time.
b=input('Type Lead Compensator Zero, (s+b). b=  ');
                                    %Input lead compensator zero.
done=1;                             %Set loop flag.
while done==1                       %Start loop for trying lead
                                    %compensator pole.
a=input('Enter a Test Lead Compensator Pole, (s+a). a =     ');
                                    %Enter test lead compensator pole.
numge=conv(numg,[1 b]);             %Generate numerator of Gc(s)G(s).
denge=conv([1 a],deng);             %Generate denominator of Gc(s)G(s).
Ge=tf(numge,denge);                 %Create Ge(s)=Gc(s)G(s).
wn=4/(Ts*z);                        %Evaluate desired natural frequency.
clf                                 %Clear graph on screen.
rlocus(Ge)                          %Plot compensated root locus with
                                    %test lead compensator pole.
axis([-10,10,-10,10])               %Change lead-compensated root locus
                                    %axes.
sgrid(z,wn)                         %Overlay grid on lead-compensated
                                    %root locus.
title(['Lead-Compensated Root Locus with ' , num2str(pos),...
'%  Overshoot Line, Lead Pole at  ', num2str(-a),...
' and Required Wn'])                %Add title to lead-compensated root
                                    %locus.
done=input('Are you done? (y=0,n=1)  ');
                                    %Set loop flag.
end                                 %End loop for trying compensator
                                    %pole.
[K,p]=rlocfind(Ge);                 %Generate gain, K, and closed-loop
                                    %poles, p, for point selected
                                    %interactively on the root locus.
'Gc(s)'                             %Display label.
Gc=tf([1 b],[1 a])                  %Display lead compensator.
'Gc(s)G(s)'                         %Display label.
Ge                                  %Display Gc(s)G(s).
'Closed-loop poles = '              %Display label.
p                                   %Display lead-compensated system's
                                    %closed-loop poles.
f=input('Give pole number that is operating point   ');
                                    %Choose lead-compensated system
                                    %dominant pole.
'Summary of estimated specifications for selected point on lead'
'compensated root locus'            %Display label.
operatingpoint=p(f)                 %Display lead-compensated dominant
                                    %pole.
gain=K                              %Display lead-compensated gain.
estimated_settling_time=4/abs(real(p(f)))
                                    %Display lead-compensated settling
                                    %time.
estimated_peak_time=pi/abs(imag(p(f)))
                                    %Display lead-compensated peak time.
estimated_percent_overshoot=pos     %Display lead-compensated percent
                                    %overshoot.
estimated_damping_ratio=z           %Display lead-compensated damping
                                    %ratio.
estimated_natural_frequency=sqrt(real(p(f))^2+imag(p(f))^2)
                                    %Display lead-compensated natural
                                    %frequency.
s=tf([1 0],1);                      %Create transfer function, "s".
sGe=s*Ge;                           %Create sGe(s) to evaluate Kv.
sGe=minreal(sGe);                   %Cancel common poles and zeros.
Kv=dcgain(K*sGe)                    %Display lead-compensated Kv.
ess=1/Kv                            %Display lead-compensated steady-
                                    %state error for unit ramp input.
```

```
'T(s)'                                 %Display label.
T=feedback(K*Ge,1)                     %Create and display lead-compensated
                                       %T(s).
'Press any key to continue and obtain the lead-compensated step'
'response'                             %Display label
pause
step(T)                                %Plot step response for lead
                                       %compensated system.
title(['Lead-Compensated System with  ' ,num2str(pos),'% Overshoot'])
                                       %Add title to step response of PD
                                       %compensated system.
pause
'Digital design'                       %Print label.
T=0.01                                 %Define sampling interval.
clf                                    %Clear graph.
'Gc(s) in polynomial form'             %Print label.
Gcs=K*Gc                               %Create Gc(s) in polynomial form.
'Gc(s) in polynomial form'             %Print label.
Gcszpk=zpk(Gcs)                        %Create Gc(s) in factored form.
'Gc(z) in polynomial form via Tustin Transformation'
                                       %Print label.
Gcz=c2d(Gcs,T,'tustin')                %Form Gc(z) via Tustin transformation.
'Gc(z) in factored form via Tustin Transformation'
                                       %Print label.
Gczzpk=zpk(Gcz)                        %Show Gc(z) in factored form.
'Gp(s) in polynomial form'             %Print label.
Gps=G                                  %Create Gp(s) in polynomial form.
'Gp(s) in factored form'               %Print label.
Gpszpk=zpk(Gps)                        %Create Gp(s) in factored form.
'Gp(z) in polynomial form'             %Print label.
Gpz=c2d(Gps,T,'zoh')                   %Form Gp(z) via zoh transformation.
'Gp(z) in factored form'               %Print label.
Gpzzpk=zpk(Gpz)                        %Form Gp(z) in faactored form.
pole(Gpz)                              %Find poles of Gp(z).
Gez=Gcz*Gpz;                           %Form Ge(z) = Gc(z)Gp(z).
'Ge(z) = Gc(z)Gp(z) in factored form'
                                       %Print label.
Gezzpk=zpk(Gez)                        %Form Ge(z) in factored form.
'z-1'                                  %Print label.
zm1=tf([1 -1],1,T)                     %Form z-1.
zm1Gez=minreal(zm1*Gez,.00001);        %Cancel common factors.
'(z-1)Ge(z)'                           %Print label.
zm1Gezzpk=zpk(zm1Gez)                  %Form & display (z-1)Ge(z) in
                                       %factored form.
pole(zm1Gez)                           %Find poles of (z-1)Ge(z).
Kv=10*dcgain(zm1Gez)                   %Find Kv.
Tz=feedback(Gez,1)                     %Find closed-loop
                                       %transfer function, T(z)
step(Tz)                               %Find step reponse.
title('Closed-Loop Digital Step Response')
                                       %Add title to step response.
```

Computer response:

```
ans =

Design of digital lead compensation


ans =

Uncompensated System


ans =

G(s)
```

```
Transfer function:
         1
-------------------
s^3 + 13 s^2 + 40 s

Type desired percent overshoot 10
Select a point in the graphics window

selected_point =

  -1.6435 + 2.2437i


ans =

Closed-loop poles =


p =

  -9.6740
  -1.6630 + 2.2492i
  -1.6630 - 2.2492i

Give pole number that is operating point   2

ans =

Summary of estimated specifications for selected point on


ans =

uncompensated root locus


operatingpoint =

  -1.6630 + 2.2492i


gain =

   75.6925


estimated_settling_time =

    2.4053


estimated_peak_time =

    1.3968


estimated_percent_overshoot =

    10


estimated_damping_ratio =

    0.5912


estimated_natural_frequency =
```

```
    2.7972


Kv =

    1.8923


ess =

    0.5285


ans =

T(s)


Transfer function:
           75.69
---------------------------
s^3 + 13 s^2 + 40 s + 75.69


ans =

Press any key to go to lead compensation

Type Desired Settling Time 2
Type Lead Compensator Zero, (s+b). b=  5
Enter a Test Lead Compensator Pole, (s+a). a =     6.8
Are you done? (y=0,n=1)  0
Select a point in the graphics window

selected_point =

  -1.9709 + 2.6692i


ans =

Gc(s)


Transfer function:
 s + 5
-------
s + 6.8


ans =

Gc(s)G(s)


Transfer function:
              s + 5
----------------------------------
s^4 + 19.8 s^3 + 128.4 s^2 + 272 s


ans =

Closed-loop poles =
```

```
p =

 -10.7971
  -5.0000
  -2.0015 + 2.6785i
  -2.0015 - 2.6785i

Give pole number that is operating point   3

ans =

Summary of estimated specifications for selected point on lead


ans =

compensated root locus


operatingpoint =

  -2.0015 + 2.6785i


gain =

  120.7142


estimated_settling_time =

    1.9985


estimated_peak_time =

    1.1729


estimated_percent_overshoot =

    10


estimated_damping_ratio =

    0.5912


estimated_natural_frequency =

    3.3437


Kv =

    2.2190


ess =

    0.4507


ans =

T(s)
```

```
Transfer function:
              120.7 s + 603.6
--------------------------------------------
s^4 + 19.8 s^3 + 128.4 s^2 + 392.7 s + 603.6


ans =

Press any key to continue and obtain the lead-compensated step


ans =

response


ans =

Digital design


T =

    0.0100


ans =

Gc(s) in polynomial form


Transfer function:
120.7 s + 603.6
---------------
    s + 6.8


ans =

Gc(s) in polynomial form


Zero/pole/gain:
120.7142 (s+5)
--------------
   (s+6.8)


ans =

Gc(z) in polynomial form via Tustin Transformation


Transfer function:
119.7 z - 113.8
---------------
  z - 0.9342

Sampling time: 0.01

ans =

Gc(z) in factored form via Tustin Transformation
```

```
Zero/pole/gain:
119.6635 (z-0.9512)
-------------------
    (z-0.9342)

Sampling time: 0.01

ans =

Gp(s) in polynomial form


Transfer function:
         1
-------------------
s^3 + 13 s^2 + 40 s


ans =

Gp(s) in factored form


Zero/pole/gain:
      1
-------------
s (s+8) (s+5)


ans =

Gp(z) in polynomial form


Transfer function:
1.614e-007 z^2 + 6.249e-007 z + 1.512e-007
------------------------------------------
    z^3 - 2.874 z^2 + 2.752 z - 0.8781

Sampling time: 0.01

ans =

Gp(z) in factored form


Zero/pole/gain:
1.6136e-007 (z+3.613) (z+0.2593)
--------------------------------
  (z-1) (z-0.9512) (z-0.9231)

Sampling time: 0.01

ans =

    1.0000
    0.9512
    0.9231


ans =

Ge(z) = Gc(z)Gp(z) in factored form


Zero/pole/gain
```

```
 1.9308e-005 (z+3.613) (z-0.9512) (z+0.2593)
 ------------------------------------------
  (z-1) (z-0.9512) (z-0.9342) (z-0.9231)

Sampling time: 0.01

ans =

z-1


Transfer function:
z - 1

Sampling time: 0.01

ans =

(z-1)Ge(z)


Zero/pole/gain:
1.9308e-005 (z+3.613) (z+0.2593)
-------------------------------
     (z-0.9342) (z-0.9231)

Sampling time: 0.01

ans =

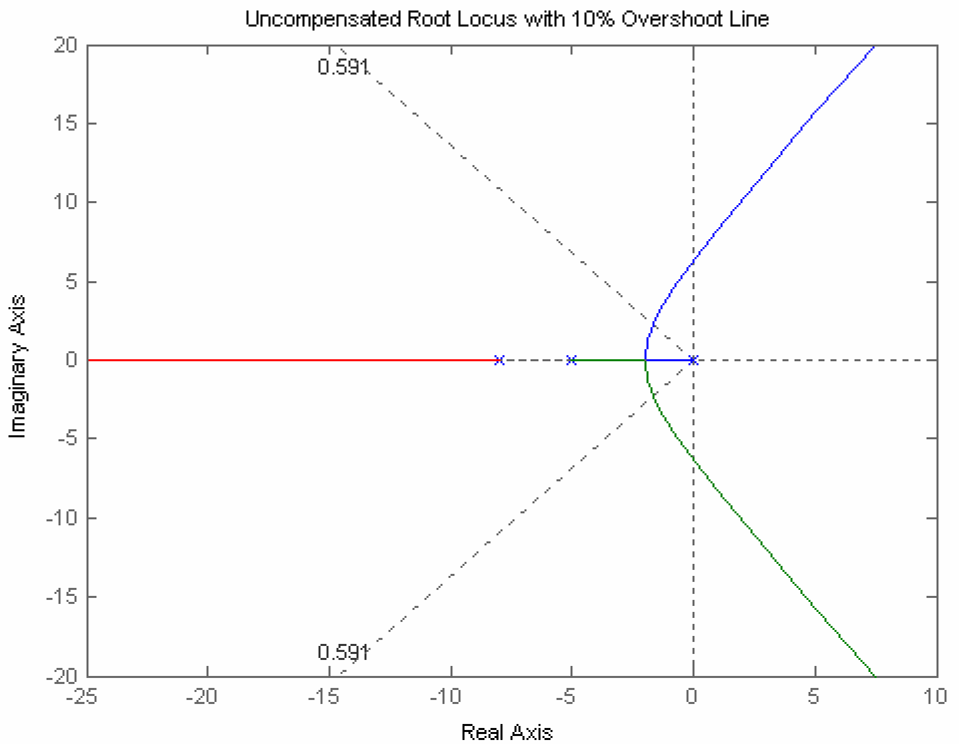
    0.9342
    0.9231

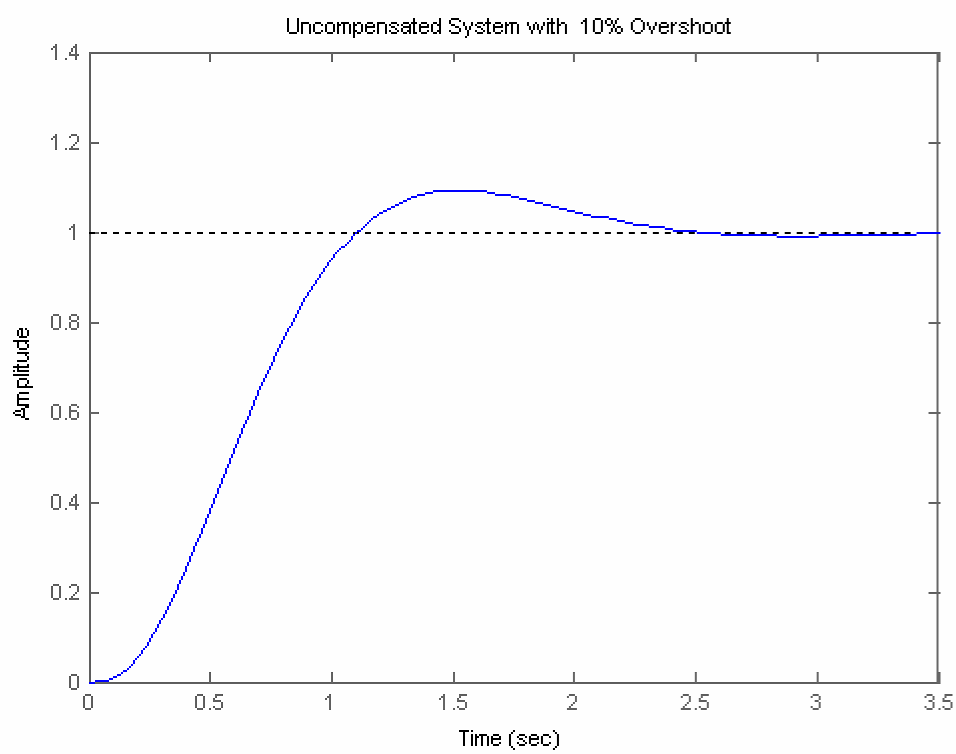

Kv =

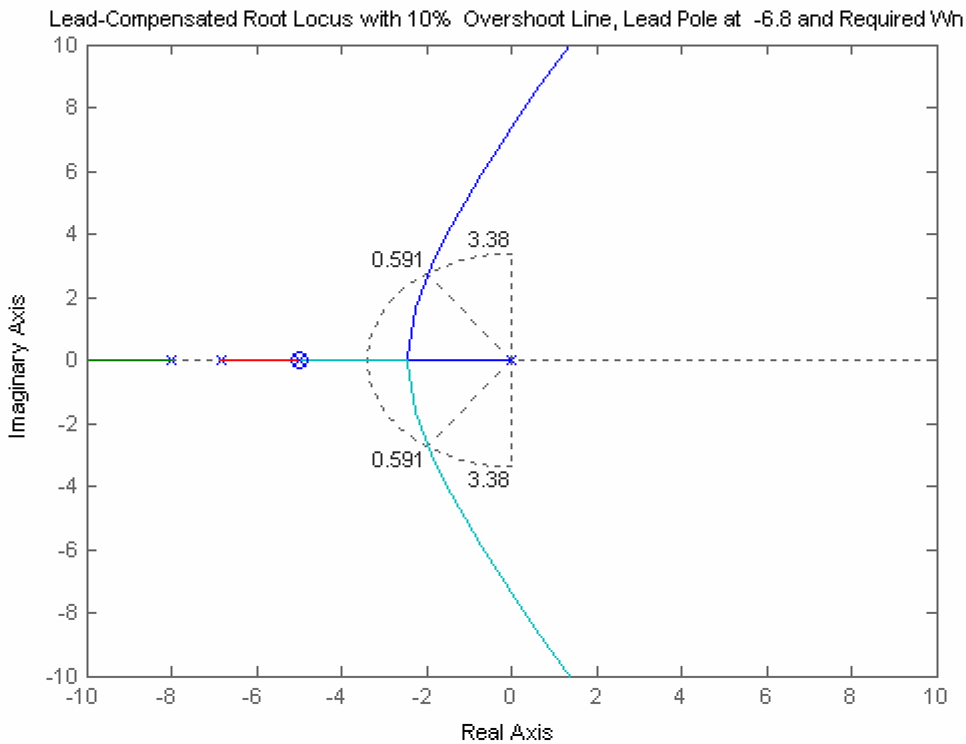
    0.2219

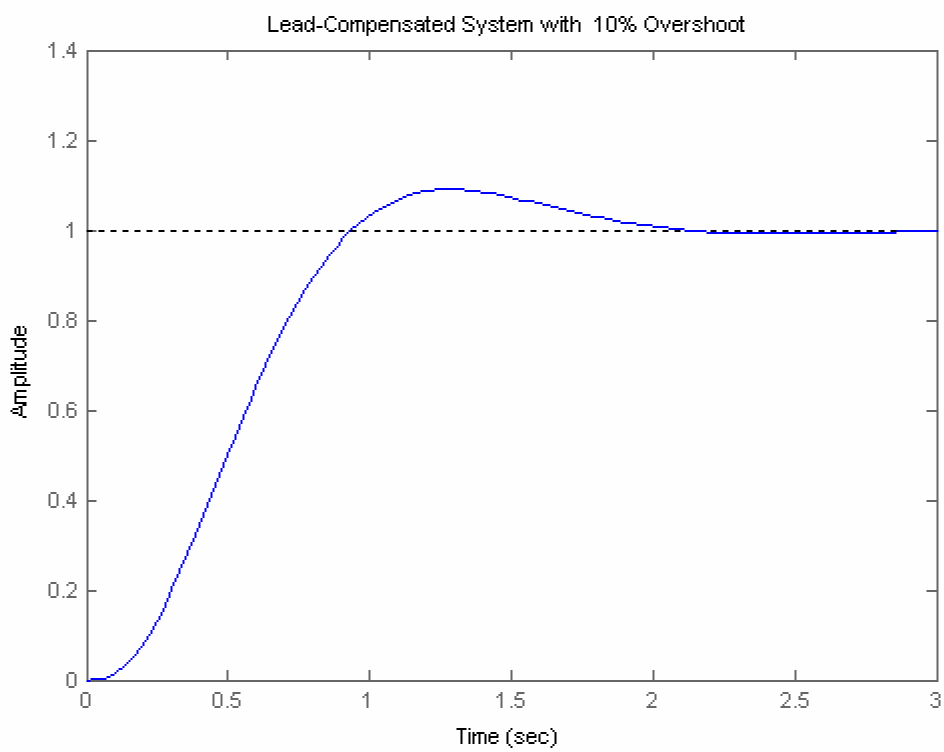

Transfer function:
1.931e-005 z^3 + 5.641e-005 z^2 - 5.303e-005 z - 1.721e-005
-----------------------------------------------------------
       z^4 - 3.809 z^3 + 5.438 z^2 - 3.45 z + 0.8203

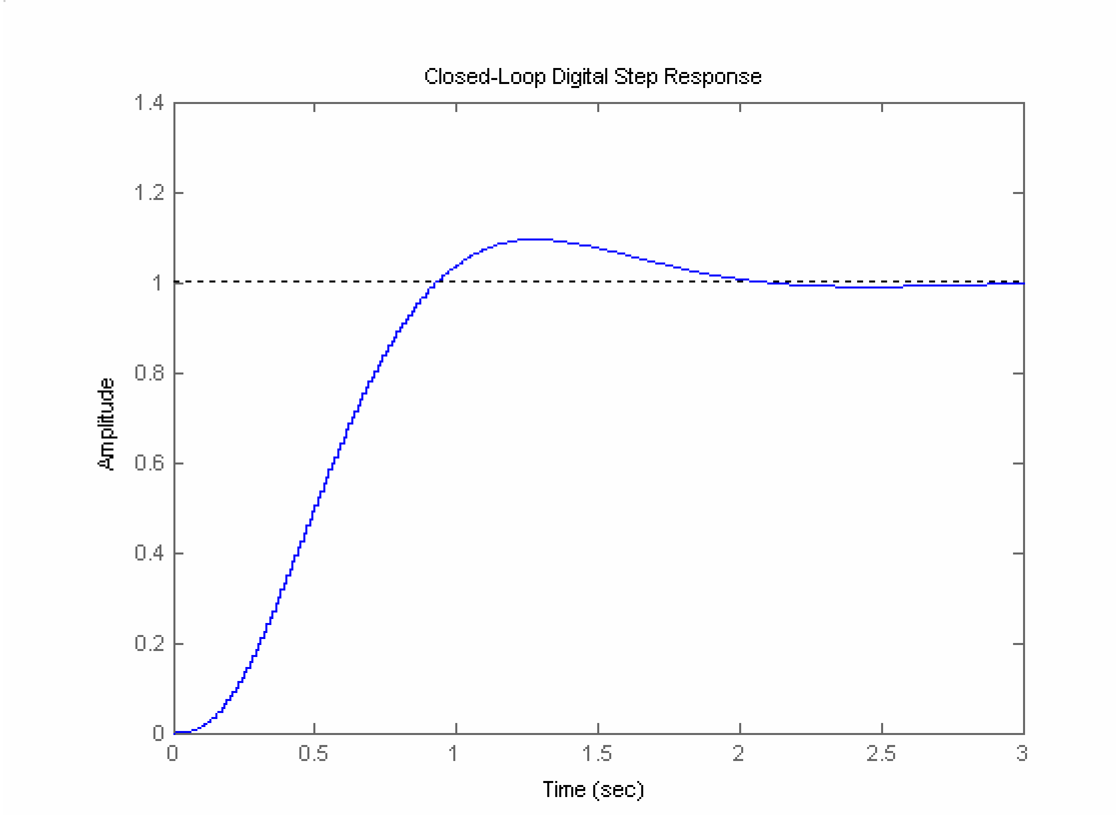
Sampling time: 0.01
```


Uncompensated Root Locus with 10% Overshoot Line
0.591
0.591
Imaginary Axis
Real Axis
20
15
10
5
0
-5
-10
-15
-20
-25
-20
-15
-10
-5
0
5
10


Uncompensated System with 10% Overshoot
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5


Lead-Compensated Root Locus with 10% Overshoot Line, Lead Pole at -6.8 and Required Wn
Imaginary Axis
Real Axis
0.591
3.38
0.591
3.38
10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10

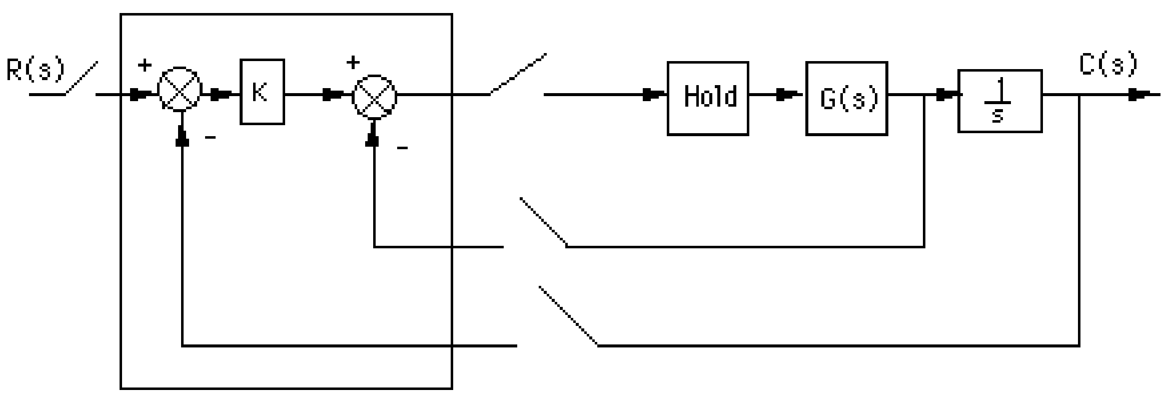

Lead-Compensated System with 10% Overshoot
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3



SOLUTIONS TO DESIGN PROBLEMS

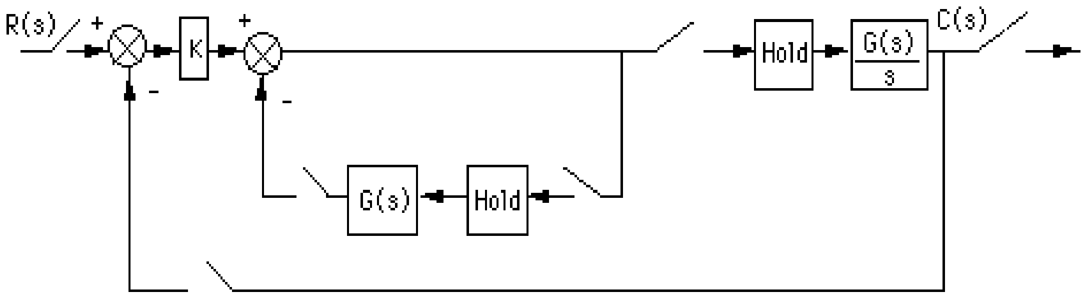
27.

a. Push negative sign from vehicle dynamics to the left past the summing junction. The computer will be the area inside the large box with the inputs and outputs shown sampled. G(s) is the combined rudder actuator and vehicle dynamics. Also, the yaw rate sensor is shown equivalently before the integrator with unity feedback.

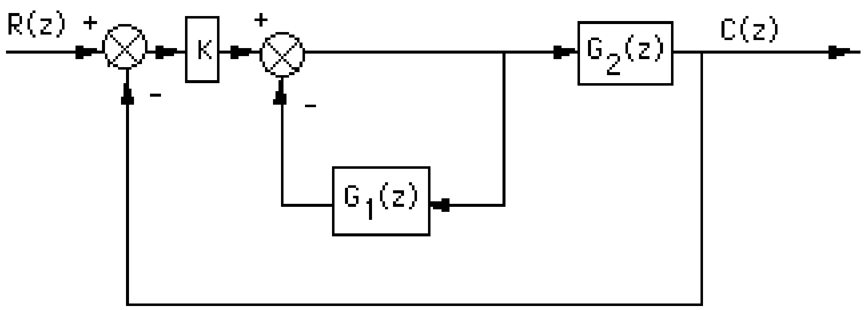


where $G(s) = \dfrac{0.25(s+0.437)}{(s+2)(s+1.29)(s+0.193)}$.

b. Add a phantom sampler at the output and push G(s) along with its sample and hold to the right past the pickoff point.



Move the outer-loop sampler to the output of $\frac{G(s)}{s}$ and write the z transforms of the transfer functions.



where

$$G_1(s) = (1 - e^{-Ts})\ \frac{0.25(s+0.437)}{s(s+2)(s+1.29)(s+0.193)}$$

and

$$G_2(s) = (1 - e^{-Ts})\ \frac{0.25(s+0.437)}{s^2(s+2)(s+1.29)(s+0.193)} \quad .$$

Now find the z transforms of $G_1(s)$ and $G_2(s)$. For $G_1(z)$.

Since

$$\frac{0.25(s+0.437)}{s(s+2)(s+1.29)(s+0.193)} = 0.15228\frac{1}{s+2} - 0.15944\frac{1}{s+0.193} - 0.21224\frac{1}{s+1.29} + 0.2194\frac{1}{s}$$

$$G_1(z) = \frac{z-1}{z}\left(0.15228\frac{z}{z-e^{-2T}} - 0.15944\frac{z}{z-e^{-0.193T}} - 0.21224\frac{z}{z-e^{-1.29T}} + 0.2194\frac{z}{z-1}\right)$$

$$T = 0.1$$

$$G_1(z) = \frac{0.0011305z^2 - 6.0812\times10^{-5}z - 0.00097764}{(z-0.81873)(z-0.87897)(z-0.98089)}$$

For $G_2(z)$:

Since

$$\frac{0.25(s+0.437)}{s^2(s+2)(s+1.29)(s+0.193)} = -0.076142\frac{1}{s+2} + 0.82613\frac{1}{s+0.193} + 0.16453\frac{1}{s+1.29} - 0.91452\frac{1}{s} + 0.2194\frac{1}{s^2}$$

$$G_2(z) = \frac{z-1}{z}\left(-0.076142\frac{z}{z-e^{-2T}} + 0.82613\frac{z}{z-e^{-0.193T}} + 0.16453\frac{z}{z-e^{-1.29T}} - 0.91452\frac{z}{z-1} + 0.2194\frac{Tz}{[z-1]^2}\right)$$

$T = 0.1$

$$G_2(z) = \frac{3.8642\times10^{-5}z^3 + 0.00010636z^2 - 0.00010404z - 3.1765\times10^{-5}}{(z-1)(z-0.81873)(z-0.87897)(z-0.98089)}$$

Now find the closed-loop transfer function. First find the equivalent forward transfer function.

$$G_e(z) = K\frac{G_2(z)}{1+G_1(z)}$$

$$G_e = 3.8642\times10^{-5}\frac{(z+0.24807)(z-0.95724)(z+3.4616)K}{(z-1)(z-0.75327)(z^2-1.9253z+0.93574)}$$

Thus,

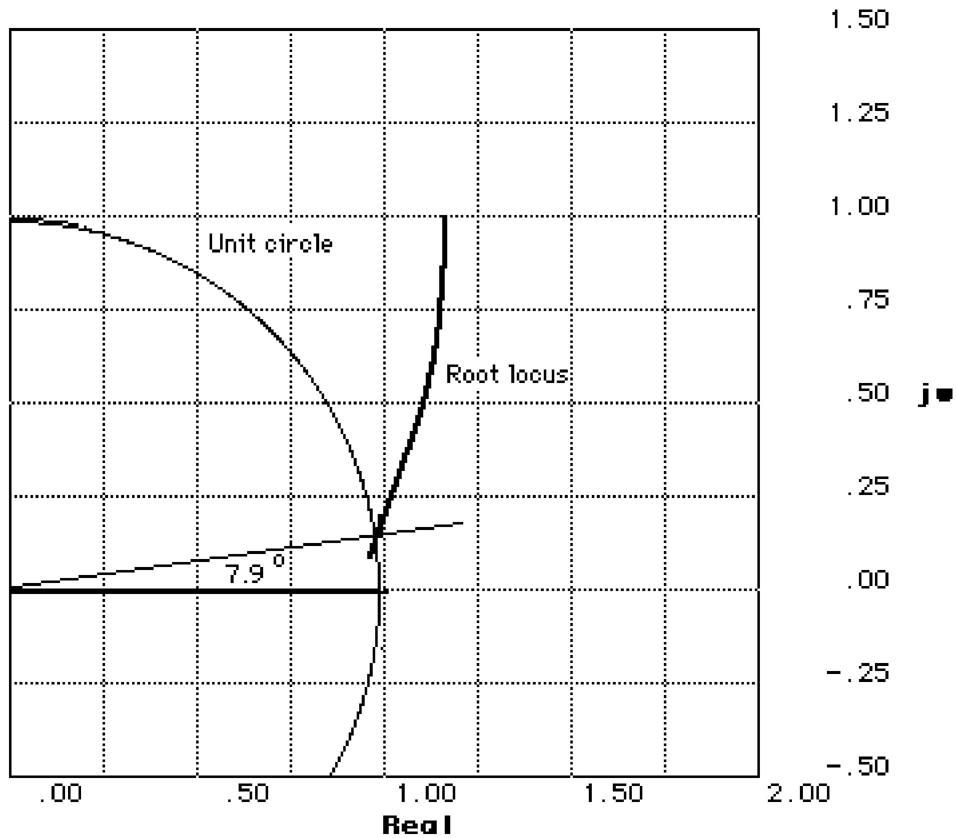
$$T(z) = \frac{G_e(z)}{1+G_e(z)}$$

Substituting values,

$$T = 3.8642\times10^{-5}\frac{(z+0.24807)(z-0.95724)(z+3.4616)K}{z^4 + (3.8642\times10^{-5}K - 3.6786)z^3 + (0.00010636K + 5.0646)z^2 - (0.00010404K + 3.0909)z + (-3.1765\times10^{-5}K + 0.70487)}$$

c. Using $G_e(z)$, plot the root locus and see where it crosses the unit circle.

$$G_e = 3.8642\times10^{-5}\frac{(z+0.24807)(z-0.95724)(z+3.4616)K}{(z-1)([z-0.75327][z-0.96266+0.095008i][z-0.96266-0.095008i])}$$



The root locus crosses the unit circle when $3.8642\text{x}10^{-5}K = 5.797\text{x}10^{-4}$, or K = 15.

28.

a. First find G(z).

$$G(z) = K\frac{z-1}{z}\ z\{\frac{1}{s^2(s^2+7s+1220)}\}$$

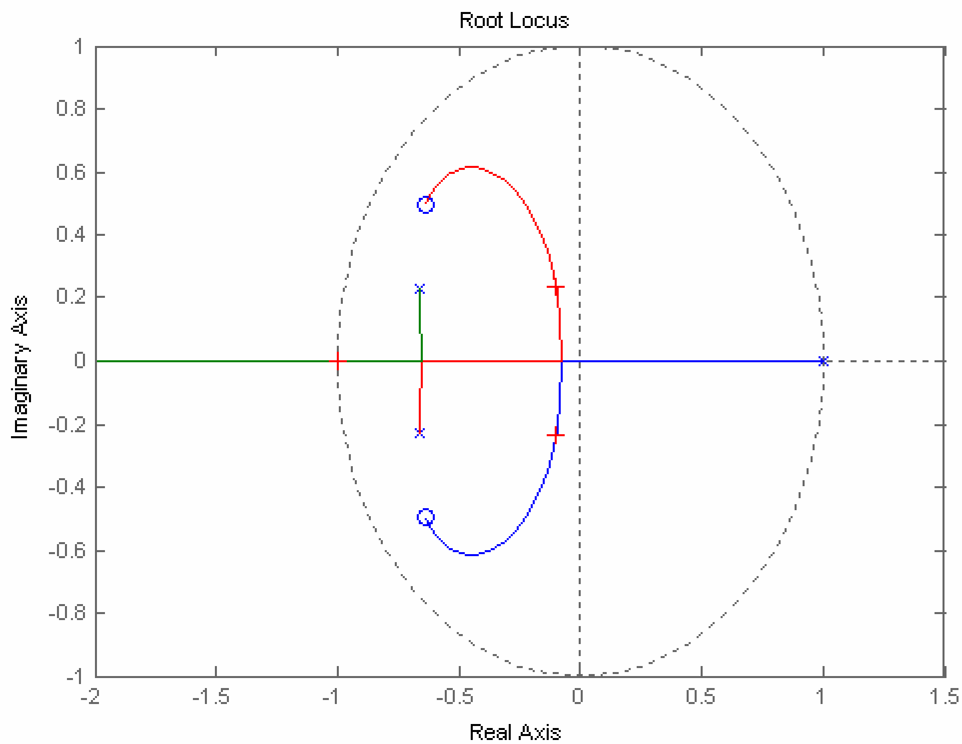
$$= K\frac{z-1}{z}\ z\{6.7186\text{x}10^{-7}(\frac{7(s+3.5)-34.4\sqrt{1207.8}}{(s+3.5)^2+1207.8} - 7\frac{1}{s} + 1220\frac{1}{s^2})\}$$

For T = 0.1,

$$= K\frac{z-1}{z}\ \{6.7186\text{x}10^{7}(7\frac{z^2+0.66582z}{z^2+1.3316z+0.49659} + 7.8472\frac{z}{z^2+1.3316z+0.49659} - 7\frac{z}{z-1} + 122\frac{z}{(z-1)^2})\}$$

$$G(z) = K\ 7.9405\times10^{-5}\ \frac{(z+0.63582+0.49355i)(z+0.63582-0.49355i)}{(z-1)([z+0.66582+0.2308i][z+0.66582-0.2308i])}$$

b.



c. The root locus intersects the unit circle at -1 with a gain, $7.9405 \times 10^{-5} K = 10866$, or

$0 < K < 136.84 \times 10^{6}$.

d.

Program:

```
%Digitize G1(s) preceded by a sample and hold
numg1=1;
deng1=[1 7 1220 0];
'G1(s)'
G1s=tf(numg1,deng1)
'G(z)'
Gz=c2d(G1s,0.1,'zoh')
[numgz,dengz]=tfdata(Gz,'v');
'Zeros of G(z)'
roots(numgz)
'Poles of G(z)'
roots(dengz)
%Plot root locus
rlocus(Gz)
title(['Root Locus'])
[K,p]=rlocfind(Gz)
```

Computer response:

```
ans =

G1(s)
```

```
Transfer function:
         1
--------------------
s^3 + 7 s^2 + 1220 s


ans =

G(z)


Transfer function:
7.947e-005 z^2 + 0.0001008 z + 5.15e-005
----------------------------------------
  z^3 + 0.3316 z^2 - 0.8351 z - 0.4966

Sampling time: 0.1

ans =

Zeros of G(z)


ans =

  -0.6345 + 0.4955i
  -0.6345 - 0.4955i


ans =

Poles of G(z)


ans =

   1.0000
  -0.6658 + 0.2308i
  -0.6658 - 0.2308i

Select a point in the graphics window

selected_point =

   -0.9977


K =

  1.0885e+004


p =

  -0.9977
  -0.0995 + 0.2330i
  -0.0995 - 0.2330i
```

See part (b) for root locus plot.

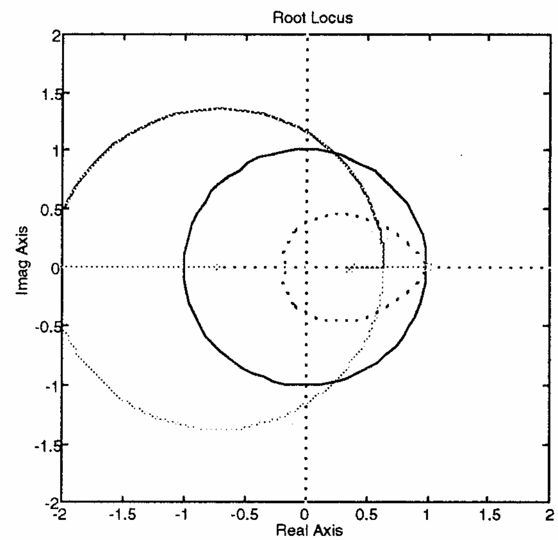
29.

a. First find G(z). $G(z) = K\frac{z-1}{z}\ z\{\frac{20000}{s^2(s+100)}\} = K\frac{z-1}{z}\ z\{2\frac{1}{s+100} - 2\frac{1}{s} + 200\frac{1}{s^2}\}$

For T = 0.01, $G(z) = K\frac{z-1}{z}\left(-2\frac{z}{z-1} + 2\frac{z}{[z-1]^2} + 2\frac{z}{z-0.36788}\right)$

$$= 0.73576\text{K}\,\frac{z + 0.71828}{(z - 1)(z - 0.36788)}.$$

b. Plotting the root locus. Finding the intersection with the unit circle yields 0.73576K = 1.178. Thus, 0 < K < 1.601 for stability.



c. Using the root locus, we find the intersection with the 15% overshoot curve (ζ = 0.517) at 0.5955 + j0.3747 with 0.73576K = 0.24. Thus K = 0.326.

d.
Program:

```
%Digitize G1(s) preceded by a sample and hold
numg1=20000;
deng1=[1 100 0];
'G1(s)'
G1s=tf(numg1,deng1)
'G(z)'
Gz=c2d(G1s,0.01,'zoh')
[numgz,dengz]=tfdata(Gz,'v');
'Zeros of G(z)'
roots(numgz)
'Poles of G(z)'
roots(dengz)
%Input transient response specifications
Po=input('Type %OS  ');
%Determine damping ratio
z=(-log(Po/100))/(sqrt(pi^2+log(Po/100)^2))
%Plot root locus
rlocus(Gz)
zgrid(z,0)
title(['Root Locus'])
[K,p]=rlocfind(Gz)  %Allows input by selecting point on graphic.
```

Computer response:

```
ans =

G1(s)
```

```
Transfer function:
   20000
-----------
s^2 + 100 s


ans =

G(z)


Transfer function:
  0.7358 z + 0.5285
----------------------
z^2 - 1.368 z + 0.3679

Sampling time: 0.01

ans =

Zeros of G(z)


ans =

   -0.7183


ans =

Poles of G(z)


ans =

    1.0000
    0.3679

Type %OS  15

z =

    0.5169

Select a point in the graphics window

selected_point =

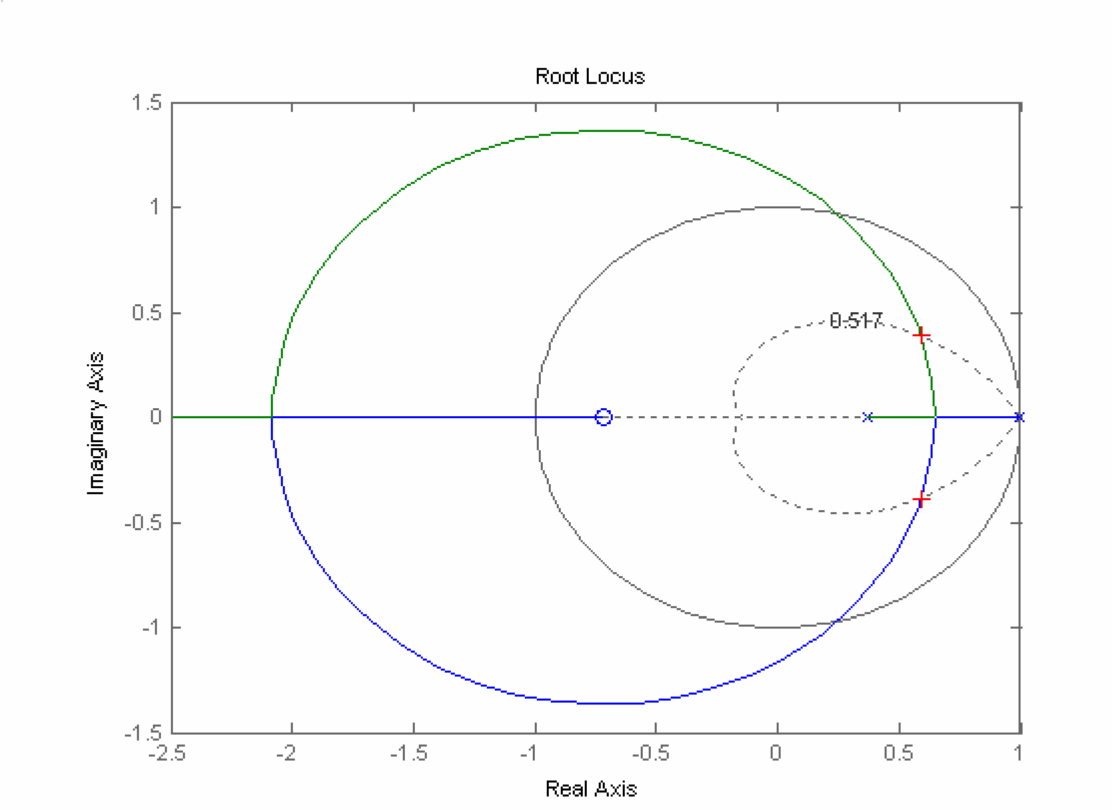
   0.5949 + 0.3888i


K =

    0.2509


p =

   0.5917 + 0.3878i
   0.5917 - 0.3878
```



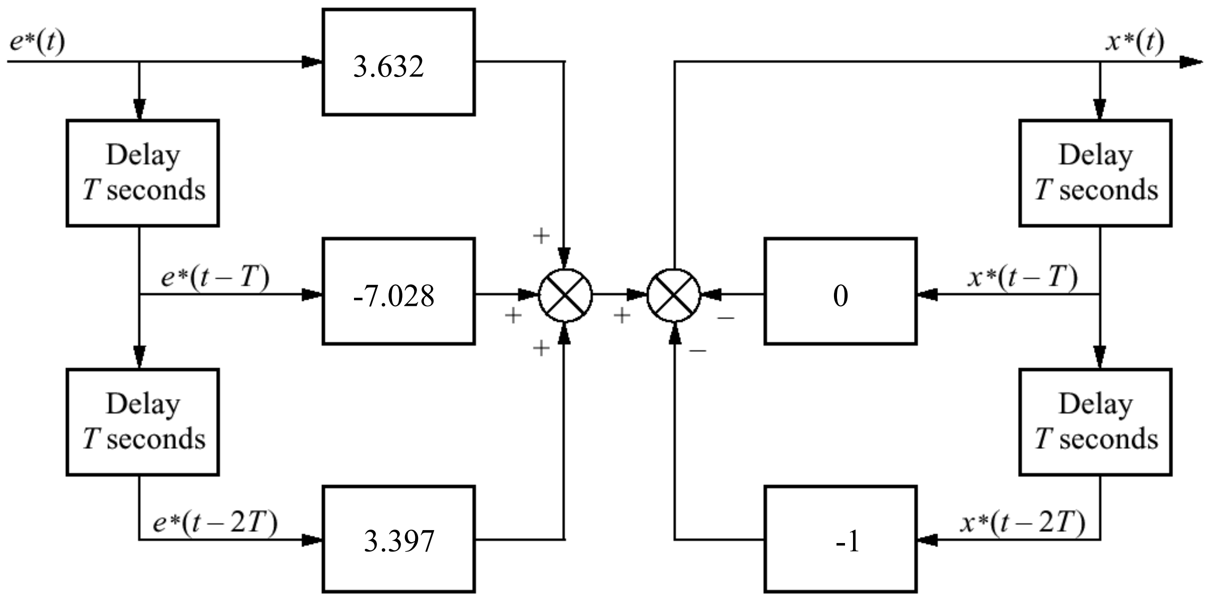
30.

$$G_{PID}(s) = \frac{0.5857(s+0.19)(s+0.01)}{s}.$$

Substituting Eq. (13.88) with T = 1/3 second,

$$G_c(z) = \frac{3.632z^2 - 7.028z + 3.397}{z^2 - 1} = \frac{3.632(z-0.9967)(z-0.9386)}{(z+1)(z-1)}$$

Drawing the flow diagram yields



T = 1/3 second

31.

a. From Chapter 9, the plant without the pots and unity gain power amplifier is

$$G_p(s) = \frac{64.88\,(s+53.85)}{(s^2 + 8.119s + 376.3)\,(s^2 + 15.47s + 9283)}$$

The PID controller and notch filter with gain adjusted for replacement of pots (i.e. divided by 100) was

$$G_c(s) = \frac{26.82\,(s+24.1)\,(s+0.1)\,(s^2 + 16.s + 9200)}{s\,(s+60)^2}$$

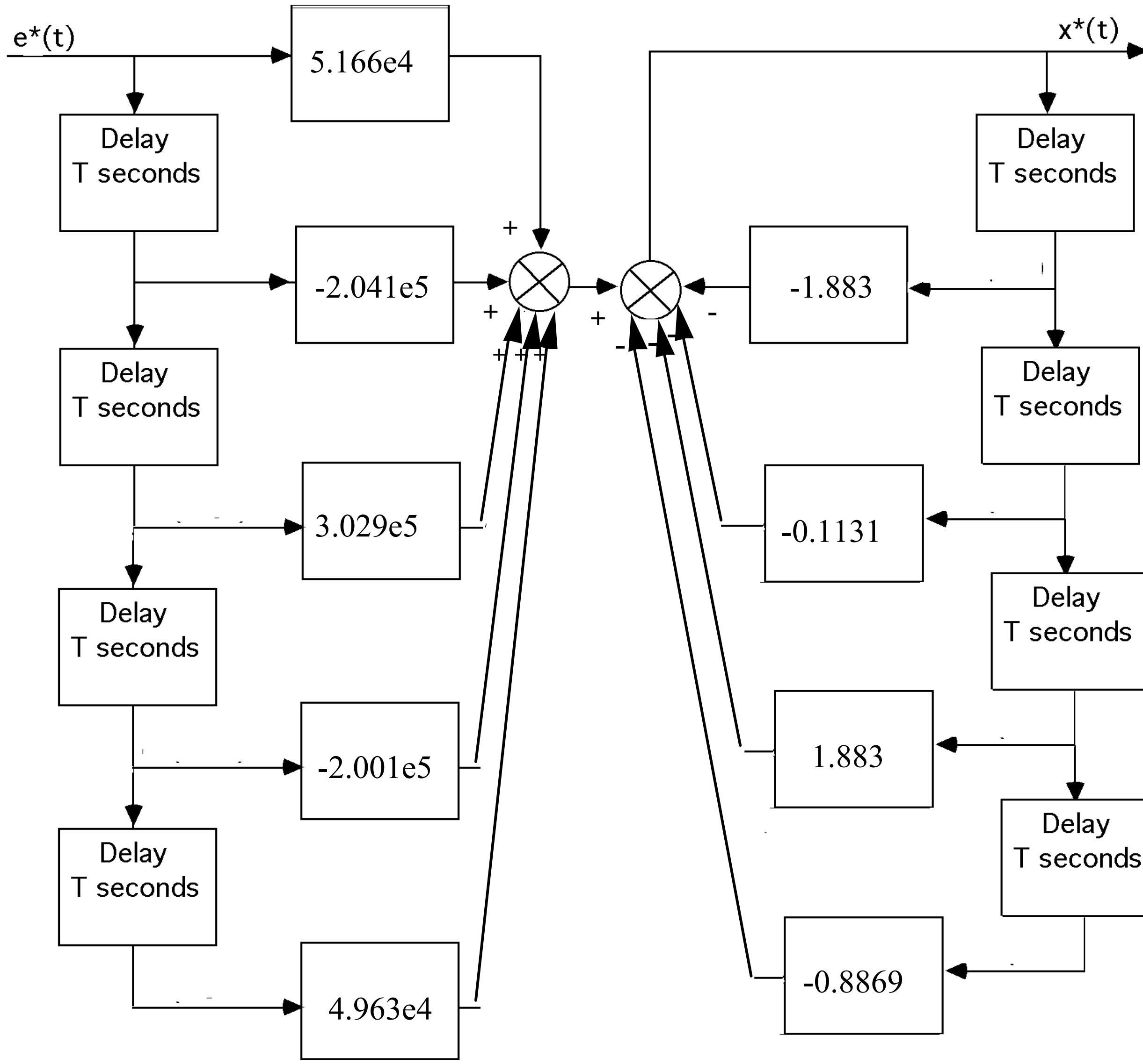
Thus, Ge(s) = Gp(s)G_c(s) is

$$G_{et}(s) = \frac{1740.0816\,(s+53.85)(s^2 + 16s + 9200)(s+24.09)(s+0.1)}{s\,(s^2 + 8.119s + 376.3)\,(s^2 + 15.47s + 9283)(s+60)^2}$$

A Bode magnitude plot of $G_e(s)$ shows $\omega_c = 36.375$ rad/s. Thus, the maximum T should be in the range $0.15/\omega_c$ to $0.5/\omega_c$ or 4.1237e-03 to 1.3746e-02. Let us select T = 0.001.
Performing a Tustin transformation on $G_c(s)$ yields

$$G_c(z) = \frac{5.166e04\, z^4 - 2.041e05\, z^3 + 3.029e05\, z^2 - 2.001e05\, z + 4.963e04}{Z^4 - 1.883\, z^3 - 0.1131\, z^2 + 1.883\, z - 0.8869}$$

b. Drawing the flowchart



T = 0.001

c.

Program:

```
syms s
'Compensator from Chapter 9'
T=.001
Gc=26.82*(s^2+16*s+9200)*(s+24.09)*(s+.1)/(s*((s+60)^2));
Gc=vpa(Gc,4);
[numgc,dengc]=numden(Gc);
numgc=sym2poly(numgc);
dengc=sym2poly(dengc);
Gc=tf(numgc,dengc);
'Gc(s)'
Gczpk=zpk(Gc)
'Gc(z)'
Gcz=c2d(Gc,T,'tustin')
'Gc(z)'
Gczzpk=zpk(Gcz)
'Plant from Chapter 9'
```

```
Gp=64.88*(s+53.85)/[(s^2+15.47*s+9283)*(s^2+8.119*s+376.3)];
Gp=vpa(Gp,4);
[numgp,dengp]=numden(Gp);
numgp=sym2poly(numgp);
dengp=sym2poly(dengp);
'Gp(s)'
Gp=tf(numgp,dengp)
'Gp(s)'
Gpzpk=zpk(Gp)
'Gp(z)'
Gpz=c2d(Gp,T,'zoh')
'Gez=Gcz*Gpz'
Gez=Gcz*Gpz
Tz=feedback(Gez,1);
t=0:T:1;
step(Tz,t)
pause
t=0:T:50;
step(Tz,t)
```

Computer response:

```
ans =

Compensator from Chapter 9


T =

    0.0010


ans =

Gc(s)


Zero/pole/gain:
26.82 (s+24.09) (s+0.1) (s^2  + 16s + 9198)
-------------------------------------------
                s (s+60)^2


ans =

Gc(z)


Transfer function:
5.17e004 z^4 - 2.043e005 z^3 + 3.031e005 z^2 - 2.002e005 z + 4.966e004
----------------------------------------------------------------------
           z^4 - 1.883 z^3 - 0.1131 z^2 + 1.883 z - 0.8869

Sampling time: 0.001

ans =

Gc(z)


Zero/pole/gain:
51699.4442 (z-1) (z-0.9762) (z^2  - 1.975z + 0.9842)
----------------------------------------------------
              (z+1) (z-1) (z-0.9417)^2

Sampling time: 0.001

ans =
```

```
Plant from Chapter 9

ans =

Gp(s)

Transfer function:
                   64.88 s + 3494
----------------------------------------------------
s^4 + 23.59 s^3 + 9785 s^2 + 8.119e004 s + 3.493e006


ans =

Gp(s)

Zero/pole/gain:
               64.88 (s+53.85)
----------------------------------------------
(s^2  + 8.119s + 376.3) (s^2  + 15.47s + 9283)


ans =

Gp(z)

Transfer function:
1.089e-008 z^3 + 3.355e-008 z^2 - 3.051e-008 z - 1.048e-008
-----------------------------------------------------------
       z^4 - 3.967 z^3 + 5.911 z^2 - 3.92 z + 0.9767

Sampling time: 0.001

ans =

Gez=Gcz*Gpz

Transfer function:

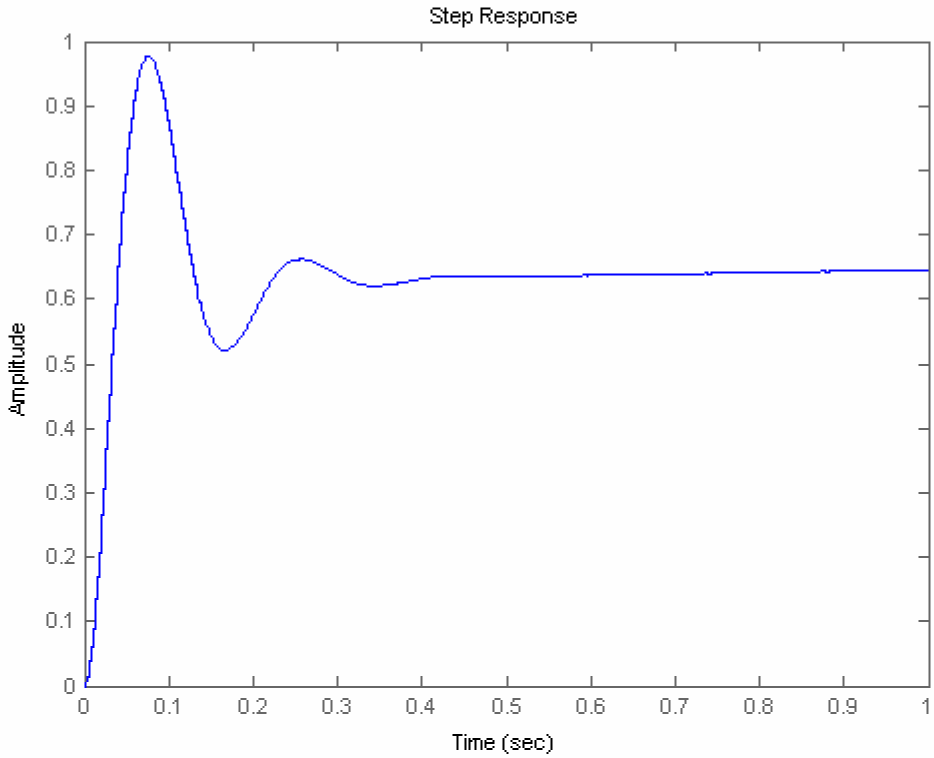
0.000563 z^7 - 0.0004901 z^6 - 0.005129 z^5 + 0.01368 z^4 - 0.01328 z^3

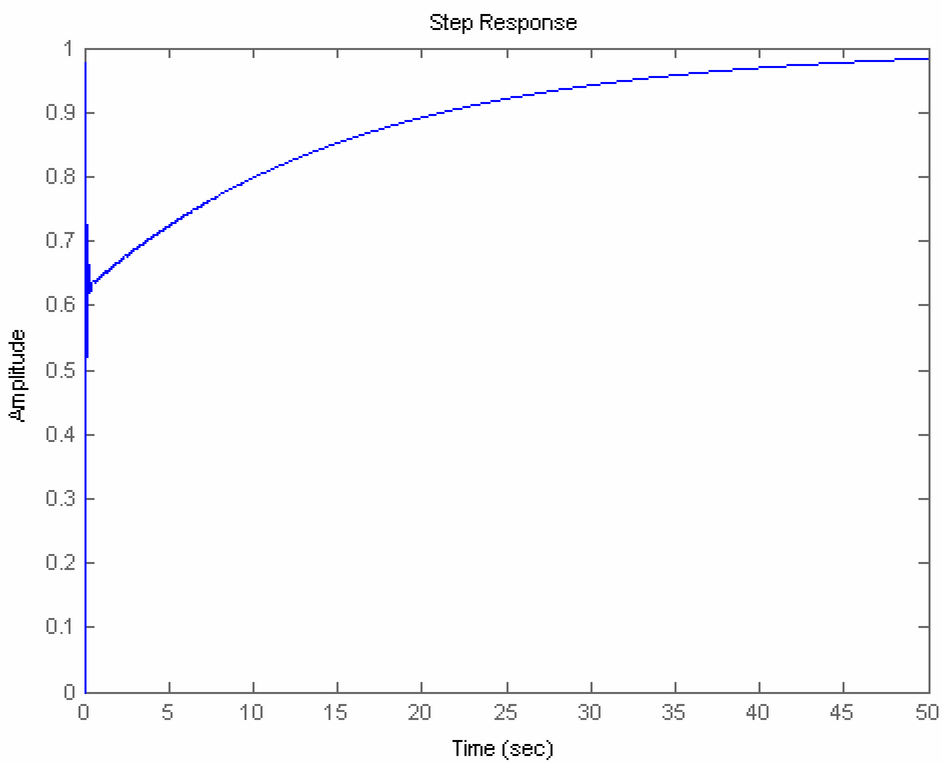
                                  + 0.004599 z^2 + 0.0005822 z - 0.0005203

--------------------------------------------------------------------------
z^8 - 5.85 z^7 + 13.27 z^6 - 12.72 z^5 - 0.6664 z^4 + 13.25 z^3

                                            - 12.74 z^2 + 5.317 z - 0.8662


Sampling time: 0.001
```


Step Response
Amplitude
Time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Step Response
Amplitude
Time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
5
10
15
20
25
30
35
40
45
50